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On a shape characterization of some two-polyhedra

by

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Abstract. The main purpose of this paper is to give a shape characterization of surfaces.

Introduction. K. Borsuk has formulated the following problem: give a shape characterization of manifolds. We solve this problem in a very special case: for surfaces (i.e. closed 2-dimensional manifolds). We will prove (Corollary (2.11) and Theorem (3.23)) the following:

THEOREM. *A continuum (metric) X has the shape of a surface if and only if X is pointed movable, the shape dimension* $\mathrm{Fd}\,X$ *is* 2, *the second Čech homology group* $\check{H}_2(X, Z_2) \cong Z_2$ *and the first shape group* $\check{\pi}_1(X)$ *is isomorphic to a fundamental group of a surface.*

In fact, in § 2, we give a shape characterization of two polyhedra of a class which contains all surfaces with the trivial second homotopy group (Theorem (2.9)).

In § 3 we give a shape characterization of a bouquet (one point union) of the projective plane and 2-spheres.

If X is a connected compact FANR with vanishing "Wall obstruction", then X has the shape of a pointed finite simplicial complex with dimension $\max(3, \mathrm{Fd}\,X)$, see [8]. There is no shape characterization of the class of all (finite) two-polyhedra. If X is a connected compact FANR with vanishing "Wall obstruction", $\mathrm{Fd}\,X = 2$ and $\check{\pi}_1(X) \cong Z_p$, then X has the shape of a (finite) 2-polyhedron; X has the shape of a bouquet of the pseudoprojective plane of order p and 2-spheres.

We assume that the reader is familiar with the basic notions of shape theory for metric compacta (see [2], [4] or [17]).

1. Shape of pointed movable continua with fundamental dimension 2 and with finitely presented 1-shape group. J. Krasinkiewicz has proved ([12], Theorem 3.1, p. 151 and Theorem 4.2, p. 152) that if (X, x) is a pointed 1-movable continuum then there exists a pointed ANR-sequence $(\boldsymbol{X}, \boldsymbol{x}) = \{(X_n, x_n), p_n^m\}$ associated with (X, x) (i.e. $\varprojlim(\boldsymbol{X}, \boldsymbol{x}) = (X, x)$) such that the corresponding sequence of fundamental groups $\pi_1(\boldsymbol{X}, \boldsymbol{x})$ is an epi-sequence; if $(\boldsymbol{X}', \boldsymbol{x}') = \{(X', x_n'), p_n'^m\}$ is any ANR-sequence associated with (X, x) then X_n can be obtained from X_n' by attaching to X_n' a finite number of 2-cells. It is easy to see ([3], proof of Theorem 2, p. 616) that if $\boldsymbol{G} = \{G_n, q_n^m\}$ is an epi-sequence of groups such that the inverse limit $\varprojlim \boldsymbol{G}$ is countable, then q_n^m is an isomorphism for sufficiently large n (otherwise $\underleftarrow{\lim} G$ contains the Cantor set). Thus we have the following

(1.1) PROPOSITION. *Let (X, x) be a pointed 1-movable continuum with a countable 1-shape group $\check{\pi}_1(X, x)$. Then there exists an inverse sequence $(\mathbf{X}, \mathbf{x}) = \{(X_n, x_n), p_n^m\}$ of pointed connected polyhedra associated with (X, x) such that*

(1.2) $\pi_1(\mathbf{X}, \mathbf{x})$ *is an iso-sequence and* $\dim X_n \leqslant \max(2, \operatorname{Fd} X)$ *for every* n.

Let (X, x) be a pointed 1-movable continuum with $\operatorname{Fd} X = 2$ and with $\check{\pi}_1(X) \cong \pi_1(Q)$, where Q is a connected 2-polyhedron. Then there exists an inverse sequence of connected 2-polyhedra $(\mathbf{X}, \mathbf{x}) = \{(X_n, x_n), p_n^m\}$ associated with (X, x) satisfying condition (1.2). Since $\pi_1(Q) = \check{\pi}_1(X) = \pi_1(X_n)$ for every n, by a theorem of Whitehead [24] there exist integers k_n, l_n such that $X_n \vee k_n S^2$ and $Q \vee l_n S^2$ have the same homotopy type (here $Y \vee kS^2$ denotes a one-point union of a space Y an k-copies of 2-spheres S^2). Thus it is easy to obtain the following

(1.3) PROPOSITION. *Let (X, x) be a pointed 1-movable continuum with $\operatorname{Fd} X = 2$ and $\check{\pi}_1(X) = \pi_1(Q)$, where Q is a 2-polyhedron. Then there exists an inverse sequence $(\mathbf{X}, \mathbf{x}) = \{(X_n, x_n), p_n^m\}$ of connected 2-polyhedra such that*

(1.4) $\operatorname{Sh}(X, x) = \operatorname{Sh}(\underleftarrow{\lim}(\mathbf{X}, \mathbf{x}), \pi_1(\mathbf{X}, \mathbf{x})$ *is an iso-sequence and* $X_n = Q \vee l_n S^2$ *for every* n.

2. A shape chracterization of surfaces with trivial 2-homotopy groups. Let ZG denote the integral group ring over a group G, and let $\Delta_Z(G)$ denote the fundamental ideal, i.e. $(n_1 g_1 + n_2 g_2 + \ldots + n_k g_k) \in \Delta_Z(G)$ (where $n_i \in Z$ and $g_i \in G$) iff $(n_1 + n_2 + \ldots + n_k) = 0$. We say that $\Delta_Z(G)$ is residually nilpotent iff $\bigcap_{n=1}^{\infty} \Delta_Z^n(G) = 0$. Let us formulate the following

(2.1) LEMMA. *Let $f\colon \bigoplus_n ZG \to \bigoplus_k ZG$ and $g\colon \bigoplus_n ZG \to \bigoplus_n ZG$ be ZG-homomorphisms of ZG-modules such that $\operatorname{im} f = \operatorname{im}(f \circ g)$ and that $\operatorname{im} g \subset \bigoplus_n \Delta_Z(G) \subset \bigoplus_n ZG$. If $\Delta_Z(G)$ is residually nilpotent, then f is trivial.*

Proof. Observe that $\operatorname{im} f = \operatorname{im}(f \circ g) \subset \bigoplus_k \Delta_Z(G)$ and if $\operatorname{im} f \subset \bigoplus_k \Delta_Z^l(G)$ then $\operatorname{im} f = \operatorname{im}(f \circ g) \subset \bigoplus_k \Delta_Z^{l+1}(G)$. Thus $\operatorname{im} f \subset \bigcap_{l=1}^{\infty} \Delta_Z^l(G) = 0$, and so f is trivial.

Now we will prove the following

(2.2) LEMMA. *Let R be a principal entire ring and let E be a finitely generated R-module. If $\{E_n, f_n^m\}$ is an inverse sequence of R-modules such that*

(2.3) $E_n = E \oplus F_n$, *where F_n is a free R-module,*

(2.4) $f_n^m(F_m) \subset F_n$,

(2.5) *the composition $r_n \circ f_n^m \circ i_m\colon E \to E$, where $i_m\colon E \to E_m$ is the inclusion and $r_n\colon E_n \to E$ is the retraction which maps F_n onto 0, is an isomorphism,*

(2.6) $\{E_n, f_n^m\}$ *satisfies the Mittag-Lefler condition,*

(2.7) $\underleftarrow{\lim}\{E_n, f_n^m\}$ *and E are isomorphic,*

then for integer n there is an integer m such that $f_n^m(F_m) = 0$.

Proof. By (2.6), we may assume (we take a subsequence) that

(2.8) $$\operatorname{im} f_n^m = \operatorname{im} f_n^{n+1} \quad \text{for every } m > n.$$

Since the image $f_n^{n+1}(F_{n+1})$ is a free submodule of F_n, F_{n+1} is a direct sum of modules F'_{n+1} and F''_{n+1} such that $f_n^{n+1}|F'_{n+1}$ is a monomorphism and $f_n^{n+1}|F''_{n+1}$ is trivial. Thus $f_n^{n+1} = g_n \circ \tilde{r}_{n+1}$, where

$$\tilde{r}_{n+1}\colon E_{n+1} = E \oplus F'_{n+1} \oplus F''_{n+1} \to \tilde{E}_n = E \oplus F'_{n+1}$$

is the retraction which maps F''_{n+1} onto 0 and $g_n = f_n^{n+1}|\tilde{E}_n$. Observe that, by (2.4), (2.5) and (2.8), it follows that $f_n^{n+1}(F'_{n+1}) \supset F'_n$. So the map $\tilde{f}_n^{n+1} = \tilde{r}_{n+1} \circ g_{n+1}\colon \tilde{E}_{n+1} \to \tilde{E}_n$ is an epimorphism and $\tilde{f}_n^{n+1}(F'_{n+2}) = F'_{n+1}$. Observe that $\tilde{E}_n = E \oplus F'_{n+1}$ can be embedded into the inverse limit $\underleftarrow{\lim}\{\tilde{E}_n, \tilde{f}_n^m\}$ which is isomorphic to E (for every n). Since E is finitely generated, F'_{n+1} is trivial for every n, and so $f_n^{n+1}|F_{n+1}$ is trivial.

We will prove

(2.9) THEOREM. *Let Q be a connected aspherical (i.e. $\pi_2(Q)$ is trivial) 2-polyhedron such that $\Delta_Z(\pi_1(Q))$ is residually nilpotent. Let X be a pointed movable continuum with $\operatorname{Fd} X = 2$. If $\check{\pi}_1(X) \cong \pi_1(Q)$ and $\check{H}_2(X, R) = H_2(Q, R)$ for a principal entire ring R then $\operatorname{Sh}(X) = \operatorname{Sh}(Q)$.*

Proof. By Proposition (1.3) the continuum X has the shape of the inverse limit of an inverse sequence $\{Q_n, p_n^m\}$ such that $Q_n = Q \vee l_n S^2$ and $(p_n^m)_\#$ is the isomorphism of 1-homotopy groups. The composition $r_n \circ p_n^m \circ i_m\colon Q \to Q$, where $i_m\colon Q \to Q \vee l_m S^2$ is the inclusion and $r_n\colon Q_n = Q \vee l_n S^2 \to Q$ is a retraction, induces an isomorphism of 1-homotopy groups. Since Q is a space of $K(\pi, 1)$-type (i.e. $\pi_n(Q)$ is trivial for $n \geqslant 2$), $r_n \circ p_n^m \circ i_m$ is a homotopy equivalence. So the homomorphism

$$(r_n \circ p_n^m \circ i_m)_*\colon H_2(Q, R) \to H_2(Q, R)$$

is an isomorphism. The R-module $H_2(Q_n, R)$ is a direct sum of $H_2(Q, R)$ and $F_n = H_2(l_n S^2, R) = \bigoplus_{l_n} R$. Since $\pi_2(Q)$ is trivial, the homomorphism

$$(p_n^m)_*\colon H_2(Q_m, R) \to H_2(Q_n, R)$$

maps F_m into F_n. The sequence $\{H_2(Q_m, R), (p_n^m)_*\}$ satisfies conditions (2.3)–(2.7). Thus, by Lemma (2.2), we may assume (if necessary we choose a subsequence) that $(p_n^m)_*(F_m) = 0$ for every $m > n$.

We consider $\pi_2(Q_n)$ as the $Z(\pi_1(Q))$-module $\bigoplus_{l_n} Z(\pi_1(Q))$. Since $(p_n^m)_*(F_m) = 0$, the image of the $Z(\pi_1(Q))$-homomorphism

$$(p_n^m)_{\#,2}\colon \pi_2(Q_m) \to \pi_2(Q_n)$$

is included in $\bigoplus_{l_n} \Delta_Z(\pi_1(Q))$. Since X is movable, we may assume (we choose a subsequence if necessary) that $\operatorname{im}(p_n^m)_{\#,2} = \operatorname{im}(p_n^{n+1})_{\#,2}$ for every $m>n$, and so in particular $\operatorname{im}((p_n^{n+1})_{\#,2}\circ(p_{n+1}^{n+2})_{\#,2}) = \operatorname{im}(p_n^{n+1})_{\#,2}$. Thus, by Lemma (2.1), the homomorphism $(p_n^{n+1})_{\#,2}$ is trivial. Thus the map p_n^{n+1} is homotopically equivalent to a map which maps every 2-sphere S^2 of $Q_{n+1} = Q\vee l_{n+1}S^2$ onto the base point of $Q_n = Q\vee l_n S^2$. So the inverse sequence $\{Q_n, p_n^m\}$ is homotopically equivalent to an inverse sequence $\{Q'_n, p'^m_n\}$ such that $Q'_n = Q$ and p'^m_n is a map which induces an isomorphism of 1-homotopy groups, and so p'^m_n is a homotopy equivalence. So X and Q have the same shape.

Let α be a group property. We say that a group G is residually α if, for every nontrivial element $x\in G$, there exists a normal subgroup N_x of G such that $x\notin N_x$ and the group G/N_x has property α.

Let H be a normal subgroup of a group G of index p^k, where p is a prime integer. If H is a residually "finite p-group" then G is also a residually "finite p-group".

The fundamental group of an orientable surface is a residually "finite 2-group" (see [9]). For every non-orientable surface M we have a two-fold covering $q\colon \tilde{M}\to M$, where $\tilde{M}$ is an orientable surface. The image $q_\#(\pi_1(\tilde{M}))$ is a normal subgroup of $\pi_1(M)$ of index 2. Since $q_\#$ is a monomorphism, $q_\#(\pi_1(\tilde{M}))$ is a residually "finite 2-group", and so also $\pi_1(M)$ is a residually "finite 2-group". Thus we have the following

(2.10) Proposition. *The fundamental group of a surface is a residually "finite 2-group".*

The problem of characterizing the groups G with $\Delta_Z(G)$ residually nilpotent has been solved by Lichtman (see [15] or [19], Theorem 2.30, p. 92). In particular ([19], Theorem 2.11, p. 84), if G is a residually "nilpotent p-group of bounded exponent", then $\Delta_Z(G)$ is residually nilpotent. Since every finite p-group is nilpotent, by Theorem (2.9) and Proposition (2.10) we have the following

(2.11) Corollary. *Let M be a surface with $\pi_2(M) = 0$. Let X be a pointed movable continuum. If* $\operatorname{Fd} X = 2$, $\check{\pi}_1(X) = \pi_1(M)$ *and* $\check{H}_2(X, Z_2) = Z_2$ *then* $\operatorname{Sh}(X) = \operatorname{Sh}(M)$.

Since the free product of residually "finite p-groups" is also residually "finite p-group" (see [10] or [16]), we have the following

(2.12) Corollary. *Let Q be a finite bouquet of aspherical surfaces. If X is a pointed movable continuum with* $\operatorname{Fd} X = 2$, $\check{\pi}_1(X) = \pi_1(Q)$ *and* $\check{H}_2(X, R) = H_2(Q, R)$ *for a principal entire ring R, then* $\operatorname{Sh}(X) = \operatorname{Sh}(Q)$.

3. A shape characterization of a bouquet of the projective plane and 2-spheres. The projective plane we denote by P and let p be the base point of the bouquet $P\vee kS^2$. By $\tilde{Z}$ we denote the group of integers Z with involution; so $\tilde{Z}$ is a ZZ_2-module on which the group Z_2 acts nontrivially.

Let ε_0 be the element of $\pi_2(P\vee kS^2, p)$ induced by the composition of a covering map $(S^2, s)\to(P, p)$ and the inclusion $(P, p)\to(P\vee kS^2, p)$. Let ε_i be an element of $\pi_2(P\vee kS^2, p)$ induced by the inclusion $(S^2, s)\to(P\vee kS^2, p)$ which maps S^2 onto the ith 2-sphere of $P\vee kS^2$. The ZZ_2-submodule M_0 of the ZZ_2-module $\pi_2(P\vee kS^2, p)$ generated by ε_0 is isomorphic to $\tilde{Z}$. The ZZ_2-submodule M_i of $\pi_2(P\vee kS^2, p)$ generated by ε_i is isomorphic to ZZ_2. The ZZ_2-module $\pi_2(P\vee kS^2, p)$ is isomorphic to $\bigoplus_{i=0}^{k} M_i$. By a we denote the generator of the group Z_2. Now we will prove the following:

(3.1) Lemma. *Let $f_i\colon (P, p)\to(P\vee kS^2, p)$ be a map which induces an isomorphism of the fundamental groups, $i = 1, 2$. If $(f_1)_{\#,2} = (f_2)_{\#,2}\colon \pi_2(P, p)\to\pi_2(P\vee kS^2, p)$ then f_1 and f_2 are homotopic* rel. p.

Proof. For a map $f\colon (P, p)\to(P\vee kS^2, p)$, we can define a map $f'\simeq f$ rel. p (using Borsuk's homotopy extension theorem) which maps the sum $\bigcup_{j=1}^{m} D_j$ of discs $D_1, D_2, \ldots, D_m$ ($D_j\subset P\setminus\{p\}$) into kS^2 and maps $(P\setminus\bigcup_{j=1}^{m} D_j)$ into P. We can find a family of mutually disjoint arcs L_j, $j = 1, 2, \ldots, m-1$, such that

$L_j\cap D_j$ and $L_j\cap D_{j+1}$ are the ends of L_j,

L_j is disjoint with $D_{j'}$, if $j'\neq j, j+1$,

$f'|L_j$ is homotopically trivial rel. the ends of L_j.

Thus by Borsuk's homotopy extension theorem we can get a map $f''\simeq f'$ rel. p which maps a disc D into kS^2 ($p\in D$) and maps $(P\setminus D)$ into P.

Let $g\colon P\to P/\dot{D} = P\vee S^2$ be the natural projection and let $\tilde{f}\colon (P\vee S^2, p)\to(P\vee kS^2, p)$ be the map such that $\tilde{f}\circ g = f''$. Thus $f\simeq\tilde{f}\circ g$ rel. p. By the definition of $\tilde{f}$ we have

$$\tilde{f}(P)\subset P \quad\text{and}\quad \tilde{f}(S^2)\subset kS^2. \tag{3.2}$$

Observe that

$$g_{\#,2}(\varepsilon_0) = \varepsilon_0+(a-1)\varepsilon_1\,. \tag{3.3}$$

Since $f_{\#,2}$ is a ZZ_2-homomorphism, we have

$$f_{\#,2}(\varepsilon_0) = l_0+l_1(a-1)\varepsilon_1+\ldots+l_k(a-1)\varepsilon_k \tag{3.4}$$

where $l_0, l_1, \ldots, l_k$ are integers. By (3.2)–(3.4) we obtain

$$\tilde{f}_{\#,2}(\varepsilon_0) = l_0\varepsilon_0\,,$$
$$\tilde{f}_{\#,2}(\varepsilon_1) = l_1\varepsilon_1+l_2\varepsilon_2+\ldots+l_k\varepsilon_k\,.$$

Since $(f_1)_{\#,2} = (f_2)_{\#,2}$, we have $(\tilde{f}_1)_{\#,2} = (\tilde{f}_2)_{\#,2}$.

Olum [18] has proved that maps of (P, p) onto itself which induce isomorphisms of the fundamental groups and which induce the same homomorphisms of 2-homotopy groups are homotopic rel. p. It follows that the maps $\tilde{f}_1$ and $\tilde{f}_2$ are homotopic rel. p. Thus f_1 and f_2 are also homotopic rel. p.

(3.5) Remark. Olum [18] has proved that maps of pseudoprojective plane (P_k, p) onto itself which induce isomorphisms of the fundamental groups and which induce the same homomorphism of 2-homotopy groups are homotopic rel. p. Lemma (3.1) will still be valid if we replace the projective plane P by a pseudoprojective plane P_k. The proof of this generalization of Lemma (3.1) is similar to the above but a little longer.

Let Q be a polyhedron with $H_2(Q) = 0$. By s_i we denote a generator of the group $H_2(Q \vee lS^2)$ which corresponds to the ith 2-sphere of the bouquet $Q \vee lS^2$. Let q be the base point of the bouquet $Q \vee lS^2$. We will prove the following

(3.6) Lemma. *Let* $(\underline{Q}, \underline{q}) = \{(Q_n, q), f_n^m\}$ *be an inverse sequence such that* $Q_n = Q \vee l_n S^2$. *If* $\underline{Q} = \{Q_n, f_n^m\}$ *is a movable sequence then there is an inverse sequence* $(\underline{Q}', \underline{q}') = \{(Q_n', q), g_n^m\}$, $Q_n' = Q \vee k_n S^2$, *homotopically equivalent to* $(\underline{Q}, \underline{q})$ *and a sequence of integers* $\{k_n'\}$, $k_n' \leqslant k_n$, *such that*

(3.7) *for every* n *there is an* $m(n)$ *such that* $Q_n' = Q_{m(n)}$

$$(3.8)\qquad (g_n^{n+1})_*(s_i) = 0 \quad \text{if} \quad k_{n+1}' < i \leqslant k_{n+1},$$

$$(3.9)\qquad (g_{n+1}^{n+2})_*(s_i) = s_i \quad \text{if} \quad 1 \leqslant i \leqslant k_{n+1}',$$

where $(g_n^{n+1})_*$ *is the homomorphism of the second homology group over the coefficient integer group* Z.

Proof. We may assume that for every n there is a map $r_{n+1}\colon Q_{n+1} \to Q_{n+2}$ such that $f_n^{n+2} \circ r_{n+1} \simeq f_n^{n+1}$ (if necessary we take a subsequence). Since $H_2(Q_n) = H_2(l_n S^2)$ is a free abelian group, for every n there is a free abelian group $\tilde{F}_{n+1}$ such that $H_2(Q_{n+1}) = \tilde{F}_{n+1} \oplus G_{n+1}$ where $G_{n+1} = \ker(f_n^{n+1})_*$. Observe that $(f_n^{n+1})_* | \tilde{F}_{n+1}$ is a monomorphism. Let

$$F_{n+1} = (f_{n+1}^{n+2} \circ r_{n+1})_*(\tilde{F}_{n+1})$$

for every n. We know that

$$(f_n^{n+1})_*(F_{n+1}) = (f_n^{n+2} \circ r_{n+1})_*(F_{n+1}) = (f_n^{n+1})_*(F_{n+1}) = \operatorname{im}(f_n^{n+1})_* .$$

If $a \in H_2(Q_{n+1})$, then $(f_n^{n+1})_*(a) = (f_n^{n+1})_*(b)$ and so $a - b \in \ker(f_n^{n+1})_* = G_{n+1}$ for some element $b \in F_{n+1}$. Thus $H_2(Q_{n+1}) = F_{n+1} + G_{n+1}$. Since $(f_n^{n+1})_* | \tilde{F}_{n+1} = (f_n^{n+2} \circ r_{n+1})_* | \tilde{F}_{n+1}$ is a monomorphism, $(f_n^{n+1})_* | F_{n+1}$ is a monomorphism; it follows that $F_{n+1} \cap G_{n+1} = F_{n+1} \cap \ker(f_n^{n+1})_* = 0$. Thus $H_2(Q_{n+1}) = F_{n+1} \oplus G_{n+1}$. Observe that $F_{n+1} \subset \operatorname{im}(f_{n+1}^{n+2})_* = (f_{n+1}^{n+2})_*(F_{n+2})$. It follows that F_{n+2} is a direct sum of free groups F_{n+1}' and F_{n+2}'' such that $(f_{n+1}^{n+2})_* | F_{n+2}'$ is an isomorphism onto F_{n+1} and $(f_{n+1}^{n+2})_*(F_{n+2}'') \subset G_{n+1}$.

By induction we can define for every $n \geqslant 2$ a minimal set $a_1^n, a_2^n, \ldots, a_k^n$ of generators of $H_2(Q_n)$ such that $a_1^n, a_2^n, \ldots, a_{k_n'}^n$ are generators of F_n, $a_{k_n'+1}^n, \ldots, a_{k_n}^n$ are generators of G_n and

$$(f_n^{n+1})_*(a_i^{n+1}) = a_i^n \quad \text{if} \quad 1 \leqslant i \leqslant k_n' .$$

There is a homotopy equivalence $f_n\colon (Q_n, q) \to (Q_n, q)$ such that $f_n | Q$ is the inclusion and $(f_n)_*(s_i) = a_i^n$. Let $g_n\colon (Q_n, q) \to (Q_n, q)$ be a map such that the composition maps $f_n \circ g_n$ and $g_n \circ f_n$ are homotopic to the identity map on (Q_n, q). Observe that the inverse sequence $\{(Q_n, q), f_n^m\}$, where $g_n^m = g_n \circ f_n^m \circ f_m$, satisfies the required conditions.

(3.10) Remark. If we additionally assume in Lemma (3.6) that $H_2(\underline{\lim}\, \underline{Q}, G)$ is finitely generated for a nontrivial group G, then we can require that $k_n' = k$ for every n, where k is the rank of the group $H_2(\underline{\lim}\, \underline{Q})$.

Let X be a pointed movable continuum with $\operatorname{Fd} X = 2$, $\check{\pi}_1(X) = Z_2$ and $\check{H}_2(X, G)$ finitely generated for a nontrivial group G. Then by Proposition (1.3), Lemma (3.6) and Remark (3.10) the pointed continuum (X, x) has the pointed shape of the inverse limit of an inverse sequence $\{(X_n, x_n), f_n^m\}$ such that

$$(3.11)\qquad (X_n, x_n) = (P \vee k_n S^2, p) ,$$

(3.12) $\{\pi_1(X_n, x_n), (f_n^m)_\#\}$ is an iso-sequence,

$$(3.13)\qquad (f_n^{n+1})_*(s_i) = \begin{cases} s_i & \text{if } 1 \leqslant i \leqslant k , \\ 0 & \text{if } k < i < k_{n+1} , \end{cases}$$

where k is the rank of the group $\check{H}_2(X)$.

Since (X, x) is movable, we may assume that

(3.14) for every n there is a map $r_{n+1}\colon (X_{n+1}, x_{n+1}) \to (X_{n+2}, x_{n+2})$ such that $f_n^{n+2} \circ r_{n+1} \simeq f_n^{n+1}$ rel. x_{n+1}.

Observe that

$$(3.15)\qquad (f_{n+1}^{n+2} \circ r_{n+1})_*(s_i) = \begin{cases} s_i & \text{if } 1 \leqslant i \leqslant k , \\ 0 & \text{if } k < i \leqslant k_n . \end{cases}$$

Let $f\colon (P \vee mS^2, p) \to (P \vee nS^2, p)$ be a map which induces the homomorphism

$$f_*\colon H_2(P \vee mS^2) \to H_2(P \vee nS^2)$$

such that

$$(3.16)\qquad f_*(s_i) = \begin{cases} s_i & \text{if} \quad 1 \leqslant i \leqslant k \\ 0 & \text{if} \quad k < i \leqslant m . \end{cases}$$

Then f induces the ZZ_2-homomorphism

$$\tilde{f} = f_{\#,2}\colon \bigoplus_{i=0}^{m} M_i \to \bigoplus_{i=0}^{n} M_i$$

given by

(3.17) $\tilde{f}(\varepsilon_i) = r_i^0\varepsilon_0 + \sum_{j=1}^{n}(1-a)r_i^j\varepsilon_j + \delta_i\varepsilon_i$ for $i = 0, 1, \ldots, m$ where r_i^j are integers, $\delta_i = 1$ if $1 \leqslant i \leqslant k$, $\delta_0 = 0$ and $\delta_i = 0$ if $k < i \leqslant m$.

If f induces the isomorphism of the fundamental groups then r_0^0 is odd (see [18]).

With the ZZ_2-homomorphism $\tilde{f}\colon \bigoplus_{i=0}^{m} M_i \to \bigoplus_{i=0}^{n} M_i$ given by (3.17) we associate a matrix

$$M(\tilde{f}) = (a_i^j|\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n)$$

(which has m columns and n rows) such that

(3.18) $a_0^j = r_0^j$, $a_i^j = 2r_i^j + \delta_i^j$ if $1 \leqslant i \leqslant k$ (here $\delta_i^j = 0$ if $i \neq j$ and $\delta_i^i = 1$) and $a_i^j = r_i^j$ if $k < i \leqslant m$.

Let $M'(\tilde{f}) = (a_i^j|\ i, j = 0, 1, \ldots, k)$. If $a_0^0 = r_0^0$ is odd then a_i^i is odd for $i = 0, 1, \ldots, k$ and a_i^j is even if $0 \leqslant j < i \leqslant k$; thus $\det M'(\tilde{f}) \neq 0$. So $\operatorname{rank} M(\tilde{f}) \geqslant k+1$. Let us prove the following

(3.19) LEMMA. *Let $\tilde{f}\colon \bigoplus_{i=0}^{m} M_i \to \bigoplus_{i=0}^{n} M_i$ be a ZZ_2-homomorphism given by (3.17) where r_0^0 is odd. If there is a ZZ_2-homomorphism $\tilde{g}\colon \bigoplus_{i=0}^{m} M_i \to \bigoplus_{i=0}^{m} M_i$ given by*

$$\tilde{g}(\varepsilon_i) = s_i^0\varepsilon_0 + \sum_{j=1}^{n}(1-a)s_i^j\varepsilon_j + \delta_i\varepsilon_i \quad \textit{for} \quad i = 0, 1, \ldots, m,$$

where s_i^j are integers and δ_i is as in (3.17), such that $\tilde{f}\circ\tilde{g} = \tilde{f}$, then $\operatorname{rank} M(\tilde{f}) = k+1$.

Proof. We have to prove that $\operatorname{rank}(M(\tilde{f})) \leqslant k+1$. From $\tilde{f}\circ\tilde{g} = \tilde{f}$ it follows that $M(\tilde{f})\cdot N = 0$, where $N = (b_i^j|\ i, j = 0, 1, \ldots, m)$ is the matrix such that $b_0^0 = s_0^0 - 1$, $b_i^j = s_i^j$ if $0 \leqslant j \leqslant k$ and $(i, j) \neq (0, 0)$, $b_i^j = 2s_i^j - \delta_i^j$ if $k < j \leqslant m$. Let $N' = (b_i^j|\ i, j = k+1, \ldots, m)$. Observe $\det N' \neq 0$, thus $\operatorname{rank} N \geqslant m-k$. It follows that $\operatorname{rank} M(\tilde{f}) \leqslant k+1$.

Now we prove the following

(3.20) LEMMA. *Let $M = (a_i^j|\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n)$ be a matrix with integer coefficients such that a_i^i is odd if $0 \leqslant i \leqslant k$ and a_i^j is even if $0 \leqslant j < i \leqslant k$. If $\operatorname{rank} M = k+1$, then there are vectors $\beta_i = (b_i^j|\ j = 0, 1, \ldots, n)$ with integer coefficients $(i = 0, 1, \ldots, k)$ such that every column $\alpha_{i'} = (a_{i'}^j|\ j = 0, 1, \ldots, n)$ of the matrix M $(i' = 0, 1, \ldots, m)$ is a linear combination with integer coefficients of the vector $\beta_0, \beta_1, \ldots, \beta_k$ and $b_i^j = 0$ if $0 \leqslant j < i \leqslant k$ (it follows that b_i^i is odd for $i = 0, 1, \ldots, k$). If a_i^j is even for $1 \leqslant i \leqslant k$ and $k < j \leqslant n$, then we may require that b_i^j should be even if $1 \leqslant i < j \leqslant n$.*

Proof. Since $\operatorname{rank} M = k+1$, there is a submodule E of dimension $k+1$ of the Z-module $\bigoplus_{i=0}^{n} Z$ such that every column $\alpha_{i'}$ of M is an element of E. Let $\beta_i = (b_i^j|\ j = 0, 1, \ldots, n)$, $i = 0, 1, \ldots, k$, be a base of the free module E. Since $\det M' \neq 0$ where $M' = (a_i^j|\ i, j = 0, 1, \ldots, k)$, we can assume that $b_j^i = 0$ if $0 \leqslant j < i \leqslant k$ (compare the proof of Theorem 1, § 2, XV, [14]). It is easy to see that b_i^i is odd for $i = 0, 1, \ldots, k$.

The column $\alpha_{i'}$, $1 \leqslant i' \leqslant k$, is a linear combination with integer coefficients of the vectors $\beta_0, \beta_1, \ldots, \beta_k$ such that the coefficient at $\beta_{i'}$ is odd and the coefficient at β_i is even if $p \leqslant i \leqslant i'$. It follows (by induction from the last column) that if the matrix $(a_i^j|\ 1 \leqslant i \leqslant k,\ k < j \leqslant n)$ has all coefficients even then the matrix $(b_i^j|\ 1 \leqslant i \leqslant k,\ k < j \leqslant n)$ also has all coefficients even. It is easy to construct a base $\beta_0, \beta_1, \ldots, \beta_k$ with the required property.

Now we will prove the following

(3.21) LEMMA. *Let a map $f\colon (P \vee mS^2, p) \to (P \vee nS^2, p)$ induce the isomorphism of the fundamental groups and a ZZ_2-homomorphism $\tilde{f} = f_{\#,2}$ of the 2-homotopy modules given by (3.17). If $\operatorname{rank} M(\tilde{f}) = k+1$, then there are the maps*

$$f_1\colon (P \vee mS^2, p) \to (P \vee kS^2, p) \quad \textit{and} \quad f_2\colon (P \vee kS^2, p) \to (P \vee nS^2, p)$$

such that $f_1 \circ f_2 \cong f$ rel. p.

Proof. Since the map f induces the isomorphism of the fundamental groups, r_0^0 is odd. The matrix $M = M(\tilde{f})$ satisfies the assumptions of Lemma (3.20). Let $\beta_0, \beta_1, \ldots, \beta_k$ be vectors as in Lemma (3.20). Let $b_i^i = 2c_i^i + 1$ for $i = 1, 2, \ldots, k$ and $b_i^j = 2c_i^j$ if $1 \leqslant i < j \leqslant n$. Let

$$\alpha_i = \sum_{j=0}^{k} t_i^j\beta_j \quad \text{for} \quad i = 0, 1, \ldots, m$$

where t_i^j are integers (α_i is the ith column of the matrix $M = M(\tilde{f})$). Since $(a_i^j - \delta_i^j)$ is even if $1 \leqslant i \leqslant k$, $(t_i^j - \delta_i^j)$ is even if $1 \leqslant i \leqslant k$. Let $t_i^j = 2s_i^j + \delta_i^j$ for $i = 1, 2, \ldots, k$. We define ZZ_2-homomorphisms

$$\tilde{f}_1\colon \bigoplus_{i=0}^{m} M_i \to \bigoplus_{i=0}^{k} M_i,$$

$$\tilde{f}_2\colon \bigoplus_{i=0}^{k} M_i \to \bigoplus_{i=0}^{n} M_i$$

as follows:

$$\tilde{f}_1(\varepsilon_i) = t_i^0\varepsilon_0 + \sum_{j=1}^{k}(1-a)t_i^j\varepsilon_j \quad \text{if} \quad i = 0 \text{ or } k < i \leqslant n,$$

$$\tilde{f}_1(\varepsilon_i) = s_i^0\varepsilon_0 + \sum_{j=1}^{k}(1-a)s_i^j\varepsilon_j + \varepsilon_i \quad \text{if} \quad 1 \leqslant i \leqslant k,$$

$$\tilde{f}_2(\varepsilon_0) = b_0^0\varepsilon_0 + \sum_{j=1}^{n}(1-a)b_0^j\varepsilon_j,$$

$$\tilde{f}_2(\varepsilon_i) = \sum_{j=i}^{n}(1-a)c_i^j\varepsilon_j + \varepsilon_i \quad \text{if} \quad 1 \leqslant i \leqslant k.$$

One can check that $\tilde{f}_2 \circ \tilde{f}_1 = \tilde{f}$. Let us observe that t_0^0 and b_0^0 are odd. There are maps

$$f_1\colon (P \vee mS^2, p) \to (P \vee kS^2, p),$$

$$f_2\colon (P \vee kS^2, p) \to (P \vee nS^2, p)$$

inducing the isomorphisms of the fundamental groups and such that $(f_1)_{\#,2} = \tilde{f}_1$ and $(f_2)_{\#,2} = \tilde{f}_2$. By Lemma (3.1) we infer that the map $f_2 \circ f_1$ is homotopic to f rel. p.

We also need the following

(3.22) LEMMA. *Let $\tilde{f}, \tilde{g}\colon \bigoplus_{i=0}^{k} M_i \to \bigoplus_{i=0}^{k} M_i$ be ZZ_2-homomorphism such that*

$$\tilde{f}(\varepsilon_i) = r_i^0 \varepsilon_0 + \sum_{j=1}^{k} (1-a) r_i^j \varepsilon_j + \delta_i \varepsilon_i \quad \text{for} \quad i = 0, 1, \ldots, k,$$

$$\tilde{g}(\varepsilon_i) = s_i^0 \varepsilon_0 + \sum_{j=1}^{k} (1-a) s_i^j \varepsilon_j + \delta_i \varepsilon_i \quad \text{for} \quad i = 0, 1, \ldots, k$$

where r_i^j, s_i^j are integers, r_0^0 is odd, $\delta_0 = 0$ and $\delta_i = 1$ for $i = 1, 2, \ldots, k$. If $\tilde{f} \circ \tilde{g} = \tilde{f}$ then $\tilde{g}$ is the identity ZZ_2-homomorphism.

Proof. Let $M = M(\tilde{f})$, i.e. $M = (a_i^j|\ i, j = 0, 1, \ldots, k)$ where $a_0^j = r_0^j$ and $a_i^j = 2r_i^j + \delta_i^j$ if $1 \leqslant i \leqslant k$. Let $N = (b_i^j|\ i, j = 0, 1, \ldots, k)$ be the matrix as in the proof of Lemma (3.19), i.e. $b_0^0 = s_0^0 - 1$ and $b_i^j = s_i^j$ for the other pairs (i, j). If $\tilde{f} \circ \tilde{g} = \tilde{f}$ then $M \cdot N = 0$. Since $\det M \neq 0$, we have $N = 0$. So $s_0^0 = 1$ and $s_i^j = 0$ for the other pairs (i, j). It follows that $\tilde{g}$ is the identity ZZ_2-homomorphism.

Now we can to prove the following

(3.23) THEOREM. *Let X be a pointed movable continuum with* $\operatorname{Fd} X = 2$ *and $\check{\pi}_1(X) = Z_2$. If $\check{H}_2(X, G)$ is finitely generated for a nontrivial group G, then* $\operatorname{Sh}(X, x) = \operatorname{Sh}(P \vee kS^2, p)$ *for some integer k.*

Proof. We know that (X, x) has the pointed shape of an inverse sequence $(\underline{X}, \underline{x}) = \{(X_n, x_n), f_n^m\}$ satisfying conditions (3.11)–(3.14) (k is the rank of the group $\check{H}_2(X)$). By Lemma (3.19) (we put $\tilde{f} = (f_n^{n+1})_{\#,2}$ and $\tilde{g} = (f_{n+1}^{n+2} \circ r_{n+1})_{\#,2}$) it follows that $\operatorname{rank} M((f_n^{n+1})_{\#,2}) = k+1$ for every n. Thus, by Lemma (3.21), for every n there are maps

$$f_n'\colon (X_{n+1}, x_{n+1}) \to (P \vee kS^2, p)$$

and

$$f_n''\colon (P \vee kS^2, p) \to (X_n, x_n)$$

such that $f_n^{n+1} = f_n'' \circ f_n'$ rel. x_{n+1}. Let $g_n^{n+1} = f_n' \circ f_{n+1}''$. The inverse sequence $(Y, y) = \{(Y_n, y_n), g_n^m\}$, where $(Y_n, y_n) = (P \vee kS^2, p)$ is homotopically equivalent to (X, x); thus $\operatorname{Sh}(\underline{\lim}(Y, y)) = \operatorname{Sh}(X, x)$. Since the rank of the group $H_2(\underline{\lim}\, Y)$ is k and (Y, y) is a movable sequence, by Lemma (3.6) and Remark (3.10) there is an inverse sequence $(Y', y') = \{(Y_n', y_n'), h_n^m\}$ homotopically equivalent to (Y, y)

and such that $(Y_n', y_n') = (P \vee kS^2, p)$ and $(h_n^{n+1})_*(s_i) = s_i$ for $i = 1, 2, \ldots, k$. Since $\{\pi_1(X_n, x_n), (f_n^m)_\#\}$ is an iso-sequence, we may assume that h_n^{n+1} induces the isomorphism of the fundamental groups for every n. Since (Y', y') is a movable sequence, we may assume that for every n there is a map

$$r_{n+1}\colon (Y_{n+1}', y_{n+1}') \to (Y_{n+2}', y_{n+2}')$$

such that $h_n^{n+2} \circ r_{n+1} \cong h_n^{n+1}$ rel. y_{n+1}'. Observe that $(h_{n+1}^{n+2} \circ r_{n+1})_*(s_i) = s_i$ for $i = 1, 2, \ldots, k$. By Lemma (3.22) (we take $\tilde{f} = (h_n^{n+1})_{\#,2}$ and $\tilde{g} = (h_{n+1}^{n+2} \circ r_{n+1})_{\#,2}$) it follows that $(h_{n+1}^{n+2} \circ r_{n+1})_{\#,2}$ is the identity ZZ_2-homomorphism. Observe that

$$(h_{n+1}^{n+2})_{\#,2} = (h_{n+1}^{n+2} \circ r_{n+1} \circ h_{n+1}^{n+2})_{\#,2}.$$

Thus by Lemma (3.22) $(r_{n+1} \circ h_{n+1}^{n+2})_{\#,2}$ is the identity isomorphism. Since the maps $h_{n+1}^{n+2} \circ r_{n+1}$ and $r_{n+1} \circ h_{n+1}^{n+2}$ both induce the isomorphisms of the fundamental groups, by Lemma (3.1) both these maps are homotopic to the identity maps relatively to x_{n+1} and x_{n+2}, respectively; so h_{n+1}^{n+2} is a homotopy equivalence. It follows that $\operatorname{Sh}(\underline{\lim}(Y', y')) = \operatorname{Sh}(P \vee kS^2, p)$ and so $\operatorname{Sh}(X, x) = \operatorname{Sh}(P \vee kS^2, p)$.

4. Some remarks. Let X be a pointed compact FANR. By [8], X has the shape of a pointed CW-complex and there is a "Wall obstruction" (an element of the projective class group $\tilde{K}^0(\pi_1(X))$) which vanishes if and only if X has the pointed shape of a pointed finite simplicial complex (the finite complex may be chosen so as to have dimension $\max(3, \operatorname{Fd} X)$). All the possible Wall obstructions occur among two-dimensional compacta. Since $\tilde{K}^0(Z_{23})$ is not trivial, there is a pointed connected compact FANR with $\operatorname{Fd} X = 2$ and $\check{\pi}_1(X) = Z_{23}$ which does not have the shape of a finite simplicial complex.

M. N. Dyer has proved ([6], p. 242) the following theorem:

(4.1) *Let L be a connected* CW*-complex with the fundamental group $\pi_1(L) \cong Z_p$ such that L is (homotopy) dominated by a finite 2-complex. Then L has the homotopy type of a finite 2-complex if and only if* $\operatorname{Wa}_2[L] = 0$.

Here $\operatorname{Wa}_2[L]$ is the Wall invariant, $\operatorname{Wa}_2[L]$ is the class of the $Z\pi_1(L)$-module $C_2(\tilde{L})/B_2(\tilde{L})$ in the projective class group $\tilde{K}^0(\pi_1(L))$ where $\tilde{L}$ is the universal cover of L, $C(\tilde{L})$ is the cellular chain complex of $\tilde{L}$, and $B_2(\tilde{L}) = \operatorname{im}(\delta_3\colon C_3(\tilde{L}) \to C_2(\tilde{L}))$.

Let X be a compact connected FANR with $\operatorname{Fd} X = 2$. Thus X is shape dominated by a finite CW-complex K with $\dim K = 2$. By [5], X is a pointed FANR, and so X has the shape of a CW-complex L. Since L is shape dominated by K, and K and L are CW-complexes, L is (homotopy) dominated by K. If $\check{\pi}_1(X) \cong Z_p$ then $\pi_1(L) \cong Z_p$. If $\tilde{K}(Z_p)$ is trivial then $\operatorname{Wa}_2[L]$ is trivial (it is known that $\tilde{K}^0(Z_p)$ is trivial for $p = 2, 3, 5, 7, 11, 13, 17, 19$; see [11] or [23]). Thus by (4.1) we obtain the following:

(4.2) PROPOSITION. *Let X be a connected compact* FANR *with* $\operatorname{Fd} X = 2$. *If* $\check{\pi}_1(X) \cong Z_p$ *and* $\tilde{K}^0(Z_p)$ *is trivial, then X has the shape of a finite polyhedron. By* [7], *it follows that X has the shape of a bouquet of the pseudoprojective plane of order p and 2-spheres.*

(4.3) QUESTION. Let X be a pointed movable continuum with $\operatorname{Fd} X = 2$, $\check{\pi}_1(X) \cong Z_p$ and $\check{H}_2(X)$ finitely generated. Is it true that if $\tilde{K}^0(Z_p)$ is trivial then X has the shape of a finite polyhedron? By Theorem 5.1 in [8], it suffices to prove that $\check{\pi}_2(X)$ is finitely generated.

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