On the fundamental dimension of the Cartesian product of compacta with the fundamental dimension 2

by

Stanisław Spież (Warszawa)

Abstract. In this paper we prove that if X is a compactum with $\mathrm{Fd}(X) = \mathrm{Fd}(X\times S^1) = 2$ then $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(S^1\times Y)$ for any compactum Y with $\mathrm{Fd}(Y) > 0$. If a continuum $X\subset E^3$ has a nontrivial shape then $\mathrm{Fd}(X\times Y)\geqslant \mathrm{Fd}(S^1\times Y)$ for any compactum Y. The fundamental dimension of the Cartesian product of movable compacta $X_1, X_2, \ldots, X_k$ with $\mathrm{Fd}(X_i) = 2$ (for $i = 1, 2, \ldots, k$) is not less than k.

1. Introduction. Let $G = \{G_n, p_n^m\}$ be an inverse sequence of groups. We say that G contains elements of infinite order (shortly $\mathrm{In}\,G \neq 0$) if there is an index n_0 such that for each $m\geqslant n_0$ the image $p_{n_0}^m(G_m)$ contains elements of infinite order. We say that G has torsion (shortly $\mathrm{Tor}\,G \neq 0$) if there is an index n_0 such that for any $m\geqslant n_0$ the homomorphism $p_{n_0}^m$ maps some torsion element of G_m onto a nontrivial element. Observe that if $\mathrm{Tor}\,G = 0$ and $\mathrm{In}\,G = 0$ then G is isomorphic with the trivial sequence.

We will prove that if X is a continuum with $\mathrm{In}(\text{pro}-\pi_1(X)) \neq 0$ then $\mathrm{Fd}(X\times Y)\geqslant\mathrm{Fd}(S^1\times Y)$ for any compactum Y (Corollary (2.5)). If X is a non-approximatively 1-connected continuum in E^3, then $\mathrm{In}(\text{pro}-\pi_1(X)) \neq 0$ (if $X\subset E^3$ is an approximatively 1-connected continuum then X has the trivial shape or the shape of a bouquet of 2-spheres). Thus if a continuum $X\subset E^3$ has a nontrivial shape then $\mathrm{Fd}(X\times Y)\geqslant\mathrm{Fd}(S^1\times Y)$ for any compactum Y.

S. Nowak [8] has proved that if Y is a $\mathscr{F}$-compactum ($Y\in\mathscr{F}$), i.e.

$$\mathrm{Fd}(Y) = \max\{n|\,H^n(X, G) \neq 0 \text{ for any abelian group } G\},$$

then $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X)+\mathrm{Fd}(Y)$ for every compactum Y with $\mathrm{Fd}(X) \neq 2$. The present author [10] has constructed an example of a continuum X with $\mathrm{Fd}(X) = 2$ such that $\mathrm{Fd}(X\times Y)<\mathrm{Fd}(X)+\mathrm{Fd}(Y)$ for any compactum Y with $\mathrm{Fd}(Y) \neq 0$.

For a continuum X with $\mathrm{Tor}(\text{pro-}\pi_1(X)) \neq 0$ we have $\mathrm{Fd}(X\times Y)\geqslant 2+\mathrm{Fd}(Y)$ for any compactum $Y\in\mathscr{F}$ (Corollary (3.4)).

If $X_1, X_2, \ldots, X_k$ are continua with $\mathrm{Fd}(X_i) = \mathrm{Fd}(X_i\times S^1) = 2$, then $\mathrm{Fd}(X_1\times X_2\times\ldots\times X_k) = k$ (if $k\geqslant 2$) and $\mathrm{Fd}(X_1\times Y) = \mathrm{Fd}(S^1\times Y)$ for any compactum Y with $\mathrm{Fd}(Y)>0$ (Theorems (4.1) and (4.2)).

We prove that if X is a movable non-approximatively 2-connected continuum with $\mathrm{Fd}(X)=2$, then $\mathrm{Fd}(X\times Y)=\mathrm{Fd}(S^2\times Y)$ for any compactum Y. If $X_1, X_2, \ldots, X_k$ are movable continua with $\mathrm{Fd}(X_i)=2$ then $\mathrm{Fd}(X_1\times X_2\times\ldots\times X_k)\geqslant k$ (Corollary (6.8)). A similar fact does not hold for continua with fundamental dimension $\geqslant 3$. For any integer $n\geqslant 3$, there exists [9] a family $\{X_i\}_{i=1}^{\infty}$ with fundamental dimension n such that $\mathrm{Fd}(X_1\times X_2\times\ldots\times X_k)=n$ for any k. We give an example of a family $\{X_i\}_{i=1}^{\infty}$ of continua with fundamental dimension 2 such that $\mathrm{Fd}(X_1\times X_2\times\ldots\times X_k)=2$ for any k.

We assume that the reader is familiar with some elementary facts from shape theory ([1], [2]).

I wish to thank dr. A. Kadlof and dr. hab. S. Nowak for their valuable remarks.

2. Continua with $\mathrm{In}(\text{pro-}\pi_1(X))\neq 0$. We first prove the following

(2.1) LEMMA. *Let $f\colon X\to Y$ be a map of CW-complexes such that*

$$f^*\colon H^n(Y,\mathfrak{B})\to H^n(X,\mathfrak{B}_f)\quad (n\geqslant 0) \tag{2.2}$$

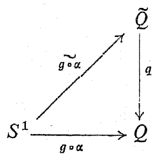
is a nontrivial homomorphism for some local system of abelian groups $\mathfrak{B}$ on Y ($\mathfrak{B}_f$ is the local system of abelian groups on X induced by $\mathfrak{B}$ and f). If $g_i\colon P_i\to Q_i$ is a map of CW-complexes such that $(g_i)_\#(\pi_1(P_i))$ contains an element of infinite order for $(1\leqslant i\leqslant k)$, then the map

$$f\times g_1\times\ldots\times g_k\colon X\times P_1\times\ldots\times P_k\to Y\times Q_1\times\ldots\times Q_k$$

is not deformable to the $(n+k-1)$-skeleton $(Y\times Q_1\times\ldots\times Q_k)^{(n+k-1)}$ of $Y\times Q_1\times\ldots\times Q_k$ (i.e. there is no homotopy $h_t\colon X\times P_1\times\ldots\times P_k\to Y\times Q_1\times\ldots\times Q_k$, $0\leqslant t\leqslant 1$, such that $h_0=f\times g_1\times\ldots\times g_k$ and

$$h_1(X\times P_1\times\ldots\times P_k)\subset (Y\times Q_1\times\ldots\times Q_k)^{(n+k-1)}).$$

Proof. For simplicity of notation, we will prove this lemma in the case of $k=1$ (the proof in the general case is the same) and denote $g=g_1$, $P=P_1$, $Q=Q_1$. Since $g_\#(\pi_1(P))$ contains an element of infinite order, there is a map $\alpha\colon S^1\to P$ such that $(g\circ\alpha)_\#\colon \pi_1(S^1)\to\pi_1(Q)$ is a monomorphism. Let $q\colon\tilde{Q}\to Q$ be a covering such that $\mathrm{im}(g\circ\alpha)_\# = \mathrm{im}\,q_\#$ and let $\widetilde{g\circ\alpha}$ be a lifting of the map $g\circ\alpha\colon S^1\to Q$, i.e. we have the following commutative diagram:



The map $q_\#$ is a monomorphism, and thus the map $(\widetilde{g\circ\alpha})_\#$ is an isomorphism. Let K be a space of homotopy type $(Z,1)$ (i.e. $\pi_1(K)\cong Z$ and $\pi_n(K)=0$ for every $n>1$) such that $K\supset\tilde{Q}$ and the 2-skeleton $K^{(2)}$ of K is equal to the 2-skeleton $Q^{(2)}$ of Q. The map $i\circ(\widetilde{g\circ\alpha})\colon S^1\to K$, where $i\colon\tilde{Q}\to K$ is the inclusion map, induces the isomorphism

$$(i\circ(\widetilde{g\circ\alpha}))_\#\colon \pi_1(S^1)\to\pi_1(K);$$

so (both S^1 and K are spaces of homotopy type $(Z,1)$) the map $i\circ(\widetilde{g\circ\alpha})$ is a homotopy equivalence. Thus

$$(i\circ(\widetilde{g\circ\alpha}))^*\colon H^1(K,Z)\to H^1(S^1,Z)$$

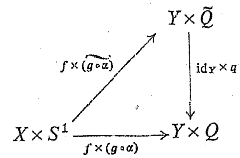
is an isomorphism.

From the Künneth formula (see [13], [9]) and (2.2) it follows that the map $f\times(i\circ(\widetilde{g\circ\alpha}))\colon X\times S^1\to Y\times K$ induces the nontrivial homomorphism

$$(f\times(i\circ(\widetilde{g\circ\alpha})))^*\colon H^{n+1}(Y\times K,\mathfrak{B}\otimes Z)\to H^{n+1}(X\times S^1,\mathfrak{B}_f\otimes Z).$$

Thus the map $f\times(i\circ(\widetilde{g\circ\alpha}))$ is not deformable to the n-skeleton $(Y\times K)^{(n)}$ of $Y\times K$ and so the map $f\times(\widetilde{g\circ\alpha})\colon X\times S^1\to Y\times\tilde{Q}$ is not deformable to the n-skeleton $(Y\times\tilde{Q})^{(n)}$ of $Y\times\tilde{Q}$.

The following diagram commutes:



and $\mathrm{id}_Y\times q$ is a covering map. Thus the map $f\times(g\circ\alpha)$ is not deformable to the n-skeleton $(Y\times Q)^{(n)}$ of $Y\times Q$ (if there is a homotopy $h_t\colon X\times S^1\to Y\times Q$, $0\leqslant t\leqslant 1$, such that $h_0=f\times(g\circ\alpha)$ and $h_1(X\times S^1)\subset(Y\times Q)^{(n)}$, then we can lift this homotopy to a homotopy $\tilde{h}_t\colon X\times S^1\to Y\times\tilde{Q}$ for which $\tilde{h}_0=f\times(\widetilde{g\circ\alpha})$ and $\tilde{h}_1(X\times S^1)\subset(Y\times\tilde{Q})^{(n)}$, which is impossible). Finally $f\times g\colon X\times P\to Y\times Q$ is not deformable to the n-skeleton $(Y\times Q)^{(n)}$ of $Y\times Q$.

(2.3) COROLLARY. *Let Y be a continuum with $\mathrm{In}(\text{pro-}\pi_1(Y))\neq 0$ and let X be a compactum with $\mathrm{Fd}(X)\geqslant 3$. Then $\mathrm{Fd}(X\times Y)\geqslant\mathrm{Fd}(X)+1$.*

Proof. Let $\mathrm{Fd}(X)=n<\infty$. Then X has the shape of the inverse limit of an inverse sequence $\{X_n, p_n^m\}$ of polyhedra with dimension $\leqslant n$. Since $\mathrm{Fd}(X)=n$, we can assume that the map $p_1^m\colon X_m\to X_1$ is not deformable to the $(n-1)$-skeleton $X_1^{(n-1)}$ of X_1. Thus from the deformation theorem of obstruction theory (Exercise B-7, Chapter IV of [4] or [3]) it follows that there is a local system of abelian groups $\mathfrak{B}$ on X_1 (we can take $\mathfrak{B}$ such that $\mathfrak{B}|X_1^{(n-1)}=\pi_n(X_1, X_1^{(n-1)})$ such that the homomorphism

$$(p_1^m)^*\colon H^n(X_1,\mathfrak{B})\to H^n(X_m,\mathfrak{B}_m)$$

is nontrivial (here $\mathfrak{B}_m$ is a local system of abelian groups on X_m induced by $\mathfrak{B}$ and p_1^m).

We can assume that Y is the inverse limit of an inverse sequence of polyhedra $\{Y_n, q_n^m\}$ such that $(q_1^m)_{\#}(\pi_1(Y_m))$ contains an element of infinite order for each $m \geqslant 1$. By Lemma (2.1) the map $p_1^m \times q_1^m\colon X_m \times Y_m \to X_1 \times Y_1$ is not deformable to the n-skeleton $(X_1 \times Y_1)^{(n)}$ of $X_1 \times Y_1$ for any $m \geqslant 1$, and so $\mathrm{Fd}(X \times Y) \geqslant n+1$.

By a similar argument to the above we obtain

(2.4) Corollary. *Let Y_i be a continuum with* $\mathrm{In}(\text{pro-}\pi_1(Y_i)) \neq 0$ *for each $i = 1, \ldots, k$. Then* $\mathrm{Fd}(Y_1 \times Y_2 \times \ldots \times Y_k) \geqslant k$.

We also obtain

(2.5) Corollary. *Let Y be a continuum with* $\mathrm{In}(\text{pro-}\pi_1(Y)) \neq 0$. *Then* $\mathrm{Fd}(X \times Y) \geqslant \mathrm{Fd}(X \times S^1)$ *for any compactum X.*

Proof. By Corollaries (2.3) and (2.4) the inequality $\mathrm{Fd}(X \times Y) \geqslant \mathrm{Fd}(X \times S^1)$ holds for any continuum with $\mathrm{Fd}(X) \neq 2$. If $\mathrm{Fd}(X) = \mathrm{Fd}(X \times S^1) = 2$ then evidently $\mathrm{Fd}(X \times Y) \geqslant \mathrm{Fd}(X \times S^1)$. Let $\mathrm{Fd}(X \times S^1) > \mathrm{Fd}(X) = 2$. Then by Corollary (2.3) we have $\mathrm{Fd}(X \times Y \times S^1) \geqslant \mathrm{Fd}(X \times S^1) + 1 = 4$; thus $\mathrm{Fd}(X \times Y) \geqslant 3 = \mathrm{Fd}(X \times S^1)$.

S. Nowak [9] has proved a similar result for continua with fundamental dimension 1. If $\mathrm{Fd}(Y) = 1$ then $\mathrm{In}(\text{pro-}\pi_1(Y)) \neq 0$, and so we have given a new (simpler) proof of the results of S. Nowak.

Let X be a continuum in E^3 with a nontrivial shape. If X is approximatively 1-connected then X has the shape of a bouquet of 2-spheres. If X is not approximatively 1-connected then $\mathrm{In}(\text{pro-}\pi_1(X)) \neq 0$. Thus by Corollary (2.5) we obtain the following

(2.6) Corollary. *If a continuum $X \subset E^3$ has nontrivial shape then* $\mathrm{Fd}(X \times Y) \geqslant \mathrm{Fd}(S^1 \times Y)$ *for any compactum Y.*

3. Continua with $\mathrm{Tor}(\text{pro-}\pi_1(X)) \neq 0$. By a pseudoprojective plane of order $m \geqslant 2$ we mean the matching of disc D and its boundary S^1 by a covering map $\alpha\colon S^1 \to S^1$ of degree m. Now we prove the following

(3.1) Lemma. *Let $g\colon P \to Y$ be a map of a pseudoprojective plane P (of order $m \geqslant 2$) into a CW-complex Y which induces a nontrivial epimorphism $g_{\#}\colon \pi_1(P) \to \pi_1(Y)$. Then $g^*\colon H^2(Y, Z) \to H^2(P, Z)$ is a nontrivial homomorphism (in fact, g^* is a monomorphism).*

Proof. Let $\pi_1(Y) \cong Z_k$ and let K be a CW-complex of homotopy type $(Z_k, 1)$ such that $Y \subset K$ and $Y^{(2)} = K^{(2)}$. The composition $f = i \circ g$ (where $i\colon Y \to K$ is the inclusion) induces the nontrivial epimorphism $f_{\#}\colon \pi_1(P) \to \pi_1(K)$. Let a be a generator of $\pi_1(P) = Z_m$. Then $b = f_{\#}(a)$ is a generator of $\pi_1(K) = Z_k$. Let $m = l \cdot k$. Because K is a space of homotopy type $(Z_k, 1)$, the homotopy class of the map f is determined by the homomorphism $f_{\#}$. So f induces up to the chain

equivalence the following homomorphism of a complex (with free operators) $C(P)$ into a complex (with free operators) $C(K)$ in the sense of [14]:

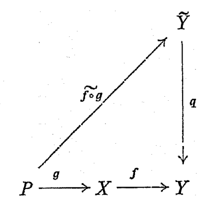
$$\begin{array}{ccccccccc}
0 \leftarrow & Z & \xleftarrow{\varepsilon} & ZZ_m & \xleftarrow{\delta} & ZZ_m & \xleftarrow{\sigma} & ZZ_m & \leftarrow 0 \ldots \\
 & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & \\
0 \leftarrow & Z & \xleftarrow{\varepsilon'} & ZZ_k & \xleftarrow{\delta'} & ZZ_k & \xleftarrow{\sigma'} & ZZ_k & \xleftarrow{\delta'} ZZ_k \ldots
\end{array}$$

where $\varepsilon(n_0 + n_1 a + \ldots + n_{m-1} a^{m-1}) = (n_0 + n_1 + \ldots + n_{m-1})$ and $\varepsilon'(n_0 + n_1 b + \ldots + n_{k-1} b^{k-1}) = (n_0 + n_1 + \ldots + n_{k-1})$; $\delta(x) = (a-1)x$ and $\sigma(x) = (1 + a + \ldots + a^{m-1})x$ for $x \in ZZ_m$; $\delta'(x) = (b-1)x$ and $\sigma'(x) = (1 + b + \ldots + b^{k-1})x$ for $x \in ZZ_k$; $f_0(1) = 1$, $f_1(1) = 1$ and $f_2(1) = 1$.

The homomorphism $\varepsilon'\colon ZZ_k \to Z$ is a 2-cocycle of complex $C(K)$. Suppose that $f^*\colon H^2(K, Z) \to H^2(P, Z)$ is a trivial homomorphism. So there is a homomorphism $\tau\colon ZZ_m \to Z$ such that $\tau(1 + a + \ldots + a^{m-1}) = \varepsilon' f_2(1) = 1$. But $\tau(1 + a + \ldots + a^{m-1}) = m \cdot s$, and so $m \cdot s = 1$, which is impossible. Thus g^* is nontrivial.

(3.2) Lemma. *Let $f\colon X \to Y$ be a map of CW-complexes such that $f_{\#}(a)$ is nonzero for some torsion element $a \in \pi_1(X)$. Then the map $f \times \mathrm{id}_{S^n}\colon X \times S^n \to Y \times S^n$ is not deformable to the $(n+1)$-skeleton $(Y \times S^n)^{(n+1)}$ of $Y \times S^n$.*

Proof. Let $g\colon P \to X$ be a map of a pseudoprojective plane (of order m) such that $g_{\#}(\pi_1(P))$ is a (finite) subgroup of $\pi_1(X)$ generated by the element $a \in \pi_1(X)$. Since $f_{\#}(a)$ is nonzero, the homomorphism $(f \circ g)_{\#}\colon \pi_1(P) \to \pi_1(Y)$ is nontrivial. Let $q\colon \tilde{Y} \to Y$ be the covering such that $(f \circ g)_{\#}(\pi_1(P)) = q_{\#}(\pi_1(\tilde{Y})) \cong Z_k$ $(k \geqslant 2)$ and let $\widetilde{f \circ g}$ be a lifting of the map $f \circ g$, i.e. the following diagram commutes:



Since $q_{\#}$ is a monomorphism, $(\widetilde{f \circ g})_{\#}$ is a nontrivial epimorphism.

By Lemma (3.1). the map $\widetilde{f \circ g}$ induces the nontrivial homomorphism $(\widetilde{f \circ g})^*\colon H^2(\tilde{Y}, Z) \to H^2(P, Z)$. By the Künneth formula the map $(\widetilde{f \circ g}) \times \mathrm{id}_{S^n}\colon P \times S^n \to \tilde{Y} \times S^n$ induces the nontrivial homomorphism

$$((\widetilde{f \circ g}) \times \mathrm{id}_{S^n})^*\colon H^{n+2}(\tilde{Y} \times S^n, Z) \to H^{n+2}(P \times S^n, Z).$$

Thus the map $(\widetilde{f\circ g})\times \mathrm{id}_{S^n}$ is not deformable to the $(n+1)$-skeleton of $\tilde{Y}\times S^n$. It follows that the map $(f\circ g)\times \mathrm{id}_{S^n}$ (and so also the map $f\times \mathrm{id}_{S^n}$) is not deformable to the $(n+1)$-skeleton of $Y\times S^n$.

(3.3) COROLLARY. *Let X be a continuum with* $\mathrm{Tor}(\text{pro-}\pi_1(X)) \neq 0$. *Then* $\mathrm{Fd}(X\times S^n)\geqslant n+2$.

By a result of S. Nowak [8] we obtain the following

(3.4) COROLLARY. *Let X be a continuum with* $\mathrm{Fd}(X)\leqslant 2$ *and* $\mathrm{Tor}(\text{pro-}\pi_1(X))\neq 0$. *Then* $\mathrm{Fd}(X) = c[X] = 2$ (*where $c[X]$ is the maximal number n such that there is a Čech local system of abelian groups $\mathfrak{B}$ on X such that $H^n(X,\mathfrak{B}) \neq 0$*) *or equivalently* $\mathrm{Fd}(Y\times X) = \mathrm{Fd}\,Y+2$ *for every continuum* $Y\in\mathscr{F}$.

4. Continua with $\mathrm{Fd}(X\times S^1) = 2$. The class of these continua contains all continua with $\mathrm{Fd}(X) = 1$ and all continua with $c[X]<\mathrm{Fd}(X) = 2$ (see [8]).

If X is an approximatively 1-connected continuum with $\mathrm{Fd}(X) = 2$, then X has the shape of the inverse limit of an inverse sequence of bouquets of 2-spheres and it is easy to see [8] that then $\mathrm{Fd}(X\times S^1) = 3$.

Let Y be a continuum with $\mathrm{Fd}(Y) = \mathrm{Fd}(Y\times S^1) = 2$. Then $\text{pro-}\pi_1(Y)$ is not isomorphic to the trivial sequence and $\mathrm{Tor}(\text{pro-}\pi_1(Y)) = 0$ by Corollary (3.3). Thus $\mathrm{In}(\text{pro-}\pi_1(Y)) \neq 0$.

Let Y_i be a continuum with $\mathrm{Fd}(Y_i) = \mathrm{Fd}(Y_i\times S^1) = 2$ for $i = 1,2,\ldots,k$, $k\geqslant 2$. Then $\mathrm{Fd}((Y_1\times S^1)\times(Y_2\times S^1)\times\ldots\times(Y_k\times S^1))\leqslant 2k$. Since $S^1\times S^1\times\ldots\times S^1$ is an $\mathscr{F}$-compactum, it follows by the theorem of Nowak [8] (see the introduction) that $\mathrm{Fd}(Y_1\times Y_2\times\ldots\times Y_k)\leqslant k$. Thus, by Corollary (2.4), we obtain the following

(4.1) THEOREM. *Let Y_i be a continuum with* $\mathrm{Fd}(Y_i) = \mathrm{Fd}(Y_i\times S^1) = 2$ *for each* $i = 1,2,\ldots,k$, $k\geqslant 2$. *Then* $\mathrm{Fd}(Y_1\times Y_2\times\ldots\times Y_k) = k$.

Let Y be a continuum with $\mathrm{Fd}(Y) = \mathrm{Fd}(Y\times S^1) = 2$. If X is a continuum with $\mathrm{Fd}(X) = \mathrm{Fd}(X\times S^1)<\infty$ (it can hold only if $\mathrm{Fd}(X) = 2$) then by Theorem (4.1) we have $\mathrm{Fd}(X\times Y) = 2 = \mathrm{Fd}(X\times S^1)$. If X is a continuum with $0<\mathrm{Fd}(X)<\mathrm{Fd}(X\times S^1)$, we have $\mathrm{Fd}(X\times Y)\leqslant\mathrm{Fd}(X)+1$ (see [9], [10]), and so $\mathrm{Fd}(X\times Y)\leqslant\mathrm{Fd}(X\times S^1)$. Thus by Corollary (2.5) we have

(4.2) THEOREM. *Let Y be a continuum with* $\mathrm{Fd}(Y) = \mathrm{Fd}(Y\times S^1) = 2$. *Then* $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X\times S^1)$ *for any compactum X with* $\mathrm{Fd}(X)>0$.

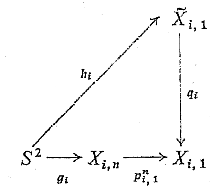
Let Y be an $\mathscr{F}$-compactum. If $\mathrm{Fd}(X) \neq 2$ or $2 = \mathrm{Fd}(X)<\mathrm{Fd}(X\times S^1)$, then by the theorem of S. Nowak we have $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X)+\mathrm{Fd}(Y)$. If $2 = \mathrm{Fd}(X) = \mathrm{Fd}(X\times S^1)$, then by Theorem (4.2) $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(S^1\times Y) = \mathrm{Fd}(Y)+1$. Thus we obtain the following (see [5]):

(4.3) COROLLARY. *If Y is an $\mathscr{F}$-compactum and X is a compactum with* $\mathrm{Fd}(X)>0$, *then* $\mathrm{Fd}(Y\times X)>\mathrm{Fd}(Y)$.

5. Non-approximatively 2-connected continua. We will prove the following

(5.1) THEOREM. *Let X_i be a non-approximatively 2-connected continuum with* $\mathrm{Fd}(X_i) = 2$ *for each* $i = 1,2,\ldots,k$. *Then* $\mathrm{Fd}(X_1\times X_2\times\ldots\times X_k) = 2k$.

Proof. Each X_i has the shape of the inverse limit of an inverse sequence $\underline{X_i} = \{X_{i,n}, p_{i,n}^m\}$ of 2-dimensional polyhedra. We can assume that the map $p_{i,1}^n\colon X_{i,n}\to X_{i,1}$ induces the nontrivial homomorphism $f_{\#,2}\colon \pi_2(X_{i,n})\to\pi_2(X_{i,1})$. Let $g_i\colon S^2\to X_{i,n}$ be a map such that $p_{i,1}^n\circ g_i$ is homotopically nontrivial. Let $q_i\colon \tilde{X}_{i,1}\to X_{i,1}$ be the universal covering and let $h_i\colon S^2\to\tilde{X}_{i,1}$ be a lifting of $p_{i,1}^n\circ g_i$. So we have the following commutative diagram:



The map $h_i\colon S^2\to\tilde{X}_{i,1}$ is homotopically nontrivial and $\tilde{X}_{i,1}$ is a 1-connected 2-dimensional polyhedron; so $h_i^*\colon H^2(\tilde{X}_{i,1},Z)\to H^2(S^2,Z)$ is a nontrivial homomorphism. From the Künneth formula it follows that the map

$$h_1\times\ldots\times h_k\colon S^2\times\ldots\times S^2\to\tilde{X}_{1,1}\times\ldots\times\tilde{X}_{k,1}$$

induces a nontrivial homomorphism

$$(h_1\times\ldots\times h_k)^*\colon H^{2k}(\tilde{X}_{1,1}\times\ldots\times\tilde{X}_{k,1},Z)\to H^{2k}(S^2\times\ldots\times S^2,Z).$$

So the map $h_1\times\ldots\times h_k$ is not deformable to the $(2k-1)$-skeleton

$$(\tilde{X}_{1,1}\times\ldots\times\tilde{X}_{k,1})^{(2k-1)}$$

of $\tilde{X}_{1,1}\times\ldots\times\tilde{X}_{k,1}$ and thus the map $p_{1,1}^n\times\ldots\times p_{k,1}^n$ is not deformable to

$$(X_{1,1}\times\ldots\times X_{k,1})^{(2k-1)}.$$

Thus $\mathrm{Fd}(X_1\times\ldots\times X_k) = 2k$.

By a similar argument to the above one can prove the following

(5.2) PROPOSITION. *Let a compactum X have the shape of the inverse limit of an inverse sequence $\{X_i, p_1^i\}$ of polyhedra such that the image of the homomorphism* $(p_1^i)^*\colon H^k(X_1,\mathfrak{B})\to H^k(X_i,\mathfrak{B}_i)$ *contains elements of infinite order for some local system of coefficients $\mathfrak{B}$ on X_1 for every i (here $\mathfrak{B}_i$ denotes the local system of coefficients induced by $\mathfrak{B}$ and p_1^i). Then* $\mathrm{Fd}(X\times Y)\geqslant k+2$ *for any non-approximatively 2-connected compactum Y with* $\mathrm{Fd}(X) = 2$.

The above proposition generalizes a result of S. Nowak [8]; case $X = S^k$.

(5.3) PROPOSITION. $\mathrm{Fd}(X\times Y)\leqslant\mathrm{Fd}(S^n\times Y)$ *for any compactum X with* $\mathrm{Fd}(X) = n$.

Proof. If $\mathrm{Fd}(Y)<\mathrm{Fd}(Y\times S^1)$ (this always holds if $2\neq\mathrm{Fd}(Y)<\infty$), then by the theorem of Nowak [8] $\mathrm{Fd}(S^n\times Y) = n+\mathrm{Fd}(Y)$; thus $\mathrm{Fd}(X\times Y)\leqslant\mathrm{Fd}(X)+$

$+\mathrm{Fd}(Y) = \mathrm{Fd}(S^n\times Y)$. If $\mathrm{Fd}(X)>0$ and $\mathrm{Fd}(Y) = \mathrm{Fd}(Y\times S^1)$, then by Theorem (4.2)

$$\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X\times S^1) \leqslant \mathrm{Fd}(X)+1 = \mathrm{Fd}(S^n\times Y).$$

Now we will prove the following

(5.4) THEOREM. *Let X be a movable non-approximatively 2-connected continuum with $\mathrm{Fd}(X) = 2$. Then $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(S^2\times Y)$ for any compactum Y.*

Proof. By Proposition (5.3) we have to prove that $\mathrm{Fd}(X\times Y) \geqslant \mathrm{Fd}(S^2\times Y)$. The continuum X has the shape of the inverse limit of an inverse sequence $\underline{X} = \{X_i, p_i^j\}$ of 2-dimensional connected polyhedra such that

(5.5) $(p_i^j)_{\#,2}\colon \pi_2(X_j, x_j) \to \pi_2(X_i, x_i)$ is a nontrivial homomorphism for any $i<j$;

(5.6) for any $2\leqslant i<j$ there exists a map $r_i^j\colon X_i \to X_j$ such that $p_{i-1}^j \circ r_i^j \simeq p_{i-1}^i$.

For every i we choose $x_i \in X_i$ such that $p_i^{i+1}(x_{i+1}) = x_i$. We can assume that $r_i^j(x_i) = x_j$ $(i<j)$. By (5.6), for any $2\leqslant i<j$ there exists an automorphism $h_{i,j}\colon \pi_2(X_i, x_i) \to \pi_2(X_i, x_i)$ (induced by a path) such that

(5.7) $$(p_{i-1}^j \circ r_i^j)_{\#,2} = h_{i-1,j} \circ (p_{i-1}^i)_{\#,2}.$$

Let $\varphi_i\colon \tilde{X}_i \to X_i$ be the universal covering and let us choose $\tilde{x}_i \in \tilde{X}_i$ such that $\varphi_i(\tilde{x}_i) = x_i$ (for every i). Let $\tilde{p}_i^j\colon \tilde{X}_j \to \tilde{X}_i$ and $\tilde{r}_i^j\colon \tilde{X}_i \to \tilde{X}_j$ be the liftings of the maps $p_i^j \circ \varphi_j\colon \tilde{X}_j \to X_i$ and $r_i^j \circ \varphi_i\colon \tilde{X}_i \to X_j$, respectively, such that $\tilde{p}_i^j(\tilde{x}_j) = \tilde{x}_i$ and $\tilde{r}_i^j(\tilde{x}_i) = \tilde{x}_j$.

For any element a of an abelian group G denote by $k(a)$ the greatest integer such that $a = k(a)\cdot b$ for some element $b \in G$. ($k(e) = \infty$ for the trivial element $e \in G$.) If $f\colon G \to H$ is a homomorphism, then $k(a) \leqslant k(f(a))$ for any $a \in G$. If $f\colon G \to H$ is an isomorphism, then $k(a) = k(f(a))$ for any $a \in G$.

Let $2\leqslant i<j$. Let $b = (p_{i-1}^i)_{\#,2}(a)$ be an element of $\mathrm{im}(p_{i-1}^i)_{\#,2} \subset \pi_2(X_{i-1}, x_{i-1})$ with the smallest integer $k(b)$. By (5.7) we have

$$k\big((p_{i-1}^i)_{\#,2}(p_i^j \circ r_i^j)_{\#,2}(a)\big) = k\big(h_{i-1,j}(p_{i-1}^i)_{\#,2}(a)\big) = k\big((p_{i-1}^i)_{\#,2}(a)\big) = k(b).$$

It follows that $k\big((p_i^j \circ r_i^j)_{\#,2}(a)\big) = 1$. Let $(p_i^j \circ r_i^j)_{\#,2}(a) = (\varphi_i)_{\#,2}(c)$ and $(r_i^j)_{\#,2}(a) = (\varphi_j)_{\#,2}(d)$. Since $(\varphi_i)_{\#,2}$ is an isomorphism, c is a primitive element (i.e. $k(c) = 1$) of the free abelian group $\pi_2(\tilde{X}_i, \tilde{x}_i)$. Since $(\tilde{X}_i, \tilde{x}_i)$ has the homotopy type of a bouquet of 2-spheres, there exists a map $f\colon (\tilde{X}_i, \tilde{x}_i) \to (S^2, x)$ such that the composition $f\circ g_c$ is homotopic to the identity map $1_{(S^2,x)}$, where $g_c\colon (S^2, x) \to (\tilde{X}_i, \tilde{x}_i)$ is a map which represents the element $c \in \pi_2(\tilde{X}_i, \tilde{x}_i)$. Since $c = (\tilde{p}_i^j)_{\#,2}(d)$,

(5.8) the composition $f\circ \tilde{p}_i^j \circ g_d$ is homotopic to $1_{(S^2,x)}$ where g_d represents the element $d \in \pi_2(\tilde{X}_j, \tilde{x}_j)$.

Let $Y = \underleftarrow{\lim}(Y_i, q_i^j)$, where Y_i are polyhedra. The continuum $X\times Y$ has the shape of the inverse limit of the inverse sequence $\{X_i\times Y_i, p_i^j\times q_i^j\}$. Suppose that $\mathrm{Fd}(X\times Y) = n$. Then for any i there exists a j such that the map $p_i^j\times q_i^j\colon X_j\times Y_j \to X_i\times Y_i$ is deformable to the n-skeleton of $X_i\times Y_i$. Since the map $(p_i^j\circ\varphi_j)\times q_i^j\colon \tilde{X}_j\times Y_j \to X_i\times Y_i$ is also deformable to the n-skeleton of $X_i\times Y_i$, by the covering homotopy property the map $\tilde{p}_i^j\times q_i^j\colon \tilde{X}_j\times Y_j \to \tilde{X}_i\times Y_i$ is deformable to the n-skeleton of $\tilde{X}_i\times Y_i$. It follows by (5.8) that the map $1_{S^2}\times q_i^j\colon S^2\times Y_j \to S^2\times Y_i$ is deformable to the n-skeleton of $S^2\times Y_i$; thus $\mathrm{Fd}(S^2\times Y) \leqslant n = \mathrm{Fd}(X\times Y)$.

6. Movable continua with $\mathrm{Tor}(\text{pro-}\pi_1(X)) \neq 0$ and $\mathrm{Fd}(X) = 2$. Let the pseudoprojective plane P of order n', where n' is a positive integer, be the space formed from the unit disc $D = \{x\in R^2 \mid \|x\|\leqslant 1\}$ by the identification on $S^1 = \{x\in R^2 \mid \|x\| = 1\}$ in polar coordinates $(1,\theta) \equiv \{1, \theta+2\pi/n'\}$.

We will prove the following

(6.1) LEMMA. *Let $\mathscr{K}$ be a local system of coefficients on P such that $\mathscr{K}(x)$ is isomorphic with the integral group ring ZZ_n (for each $x\in P$) and the group $\pi_1(P) = Z_n$, acts on $\mathscr{K}$ under an epimorphism $\varphi\colon \pi_1(P) \to Z_n$. If $q\colon \tilde{P} \to P$ is the universal covering, then the kernel of the homomorphism $q^*\colon H^2(P, \mathscr{K}) \to H^2(P, \mathscr{K}_q)$ is equal to*

$$(1+a+\ldots+a^{n-1})ZZ_n/s(1+a+\ldots+a^{n-1})ZZ_n \subset H^2(P, \mathscr{K})$$
$$= ZZ_n/s(1+a+\ldots+a^{n-1})ZZ_n,$$

where a is a generator of the group Z_n and $s\cdot n = n'$ (here $\mathscr{K}_q$ is the local system of coefficients on P induced by $\mathscr{K}$ and q).

Proof. Let $\varphi_i\colon D \to D$ be the rotation of the angle $2i\pi/n'$. Let the natural projection $r\colon D \to P$ be simplicial with respect to triangulations K' and K (of D and P respectively) and let $\varphi_1\colon D \to D$ be simplicial with respect to K'. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be all 2-simplexes of K' oriented coherently; we will denote by the same symbols 2-simplexes of K with the orientation induced by r. We can assume that $\sigma_1 \in K'$ has exactly one 1-face which is contained in S^1. Let $x_{\sigma_1} \in |\sigma_1| \cap S^1$.

The universal covering space $\tilde{P}$ is formed from the n'-copies of the unit disc $\bigcup_{i=0}^{n'-1} D\times\{i\}$ by the identification on $\bigcup_{i=0}^{n'-1} S^1\times\{i\}$

$$(x, 0) \equiv (x, 1) \equiv \ldots \equiv (x, n'-1).$$

We consider $D\times\{i\}$ as a subset of $\tilde{P}$, and let $\tilde{K}$ be the triangulation on $\tilde{P}$ induced by triangulations $K'\times\{i\}$ on $D\times\{i\}$. We can assume that $q(x, i) = r(\varphi_i(x))$ for $x\in D$, $i = 0, 1, \ldots, n'-1$.

Let $C^2(K, \mathscr{K})$ be a group of 2-cochains in the sense of [12]. Any 2-cochain $c \in C^2(K, \mathscr{K})$ is cohomology equivalent to a 2-chain $c' \in C^2(K, \mathscr{K})$ which is concentrated on σ_1, i.e. $c'(\sigma_1) = z \in \mathscr{K}(x_{\sigma_1})$ and $c'(\sigma_i) = 0 \in \mathscr{K}(x_{\sigma_i})$ for each $i \neq 1$ $(x_{\sigma_i} \in |\sigma_i|)$. Let $\tau_0 = \sigma_1\times\{0\}, \tau_1, \ldots, \tau_{n'-1}$ $(|\tau_i| \subset D\times\{i\})$ be the simplexes of $\tilde{K}$ which are mapped by q onto σ_1. Then $c'q$ is the 2-cochain of $C^2(\tilde{K}, \mathscr{K}_q)$ which has nontrivial values only on simplexes $\tau_0, \tau_1, \ldots, \tau_{n'-1}$. The 2-cochain $c'q$ is cohomology equivalent to the cochain $\bar{c} \in C^2(\tilde{K}, \mathscr{K}_q)$ which has nontrivial values only on simplexes $\sigma_1 \times \{i\}$ and $\bar{c}(\sigma_1 \times \{i\}) = a^i z \in \mathscr{K}_q(x)$, for $i = 0, 1, \ldots, n'-1$, where x is a point of $\tilde{P}$ such that $q(x) = x_{\sigma_1}$ and $x \in |\sigma_1 \times \{i\}|$. The 2-cochain $\bar{c}$ is cohomologically trivial iff $a^i \cdot z = z$ for $i = 0, 1, \ldots, n'-1$. This last holds iff $z \in (1+a+\ldots+a^{n-1})ZZ_n$ (here we identify $\mathscr{K}_q(x)$ with ZZ_n); it proves the lemma.

Now we prove the following

(6.2) LEMMA. *Let $f\colon P \to W$ be a map of the pseudoprojective plane P of order n' into a 2-dimensional* CW*-complex which induces a nontrivial homomorphism $f_{\#}\colon \pi_1(P) \to \pi_1(W)$. Let $q\colon \tilde{P} \to P$ be the universal covering. If n^2 does not divide n', where n is the order of* $\operatorname{im} f_{\#}$, *then the map fq is homotopically nontrivial.*

Proof. Let $p\colon \tilde{W} \to W$ be a covering such that $p_{\#}(\pi_1(\tilde{W})) = f_{\#}(\pi_1(P))$. Then a lifting $\tilde{f}\colon P \to \tilde{W}$ of the map f induces an epimorphism $\tilde{f}_{\#}\colon \pi_1(P) \to \pi_1(\tilde{W})$. Let $K(P)$ and $K(\tilde{W})$ be spaces of type $(Z_{n'}, 1)$ and type $(Z_n, 1)$, respectively, such that $(K(\tilde{W}))^{(2)} = \tilde{W}$ and $(K(P))^{(2)} = P$. The map $g\colon K(P) \to K(\tilde{W})$, such that $g(x) = \tilde{f}(x)$ for each $x \in P$, induces an epimorphism of 1-homotopy groups. We have the following commutative diagram:

$$\text{(6.3)}\qquad \begin{array}{ccc} P & \xrightarrow{\ \tilde{f}\ } & \tilde{W} \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ K(P) & \xrightarrow{\ g\ } & K(\tilde{W}) \end{array}$$

where j_1 and j_2 are the inclusions. One can easily see that $j_1^*\colon H^2(K(P), Z) \to H^2(P, Z)$ is an isomorphism, and also that $g^*\colon H^2(K(\tilde{W}), Z) \to H^2(K(P), Z)$ is a monomorphism; thus $(j_2 \circ \tilde{f})^* = (\tilde{f})^* \circ j_2^*\colon H^2(K(\tilde{W}), Z) \to H^2(P, Z)$ is a monomorphism. If $[c] = j_2^*([d]) \in H^2(\tilde{W}, Z)$, where $[d]$ is a generator of the group $H^2(K(\tilde{W}), Z) = Z_n$, then $(\tilde{f})^*[c] = [c\tilde{f}]$ is an element of the group $H^2(P, Z)$ of order n.

We can assume that the map $f\colon P \to W$ is simplicial with respect to some triangulations K and L (of P and W respectively) and that the map $r\colon D \to P$ is simplicial with respect to the triangulations K' and K. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be all 2-simplexes of K' oriented coherently; we will denote by the same symbols 2-simplexes of K with the orientation induced by r. Let $l_i = (c\tilde{f})(\sigma_i)$ for $i = 1, 2, \ldots, k$; since $[c\tilde{f}]$ is an element of the group $H^2(P, Z) = Z_n$, of order n,

(6.4) the greatest common divisor of integers n' and $(l_1+l_2+\ldots+l_k)$ is equal to s, where $n' = s \cdot n$.

Let $\mathfrak{B}$ be a local system of coefficients on $\tilde{W}$ such that $\mathfrak{B}(x)$ is isomorphic with the integral group ring ZZ_n (for each $x \in \tilde{W}$) and the group $\pi_1(\tilde{W}) = Z_n$ acts freely on $\mathfrak{B}$. The map $\tilde{f}\colon P \to \tilde{W}$ is simplicial with respect to the triangulations K, and $\tilde{L}$ where $\tilde{L}$ is the triangulation on $\tilde{W}$ induced by the map p and the triangulation L. Let $\tau_1, \tau_2, \ldots$ be all 2-simplexes of $\tilde{L}$ with a once chosen orientation. Let us choose a point $x_i \in |\tau_i|$ for each τ_i and a point $y_j \in |\sigma_j|$ for each σ_j such that $\tilde{f}(y_j) = x_i$ for a certain x_i.

We assume that $Z \subset ZZ_n$ (i.e. we identify an integer $k \in Z$ with $(k+0\cdot a+\ldots+0\cdot a^{n-1}) \in ZZ_n$, where a is a generator of Z_n). Let $\varphi_j\colon ZZ_n \to \mathfrak{B}(x_j)$ be an isomorphism. For the cocycle $c \in \operatorname{Hom}(C_2(\tilde{W}), Z)$ we define a cocycle $c' \in C^2(\tilde{W}, \mathfrak{B})$ such that $c'(\tau_j) = \varphi_j(c(\tau_j)) \in \mathfrak{B}(x_j)$. Let $\mathfrak{B}_{\tilde{f}}$ be a local system of coefficients on P induced by $\mathfrak{B}$ and $\tilde{f}$. Then the cocycle $c'\tilde{f} \in C^2(P, \mathfrak{B}_{\tilde{f}})$ is cohomological to the cocycle $\bar{c} \in C^2(P, \mathfrak{B}_{\tilde{f}})$ such that $\bar{c}(\sigma_i) = 0$ for $i = 2, 3, \ldots, k$ and $\bar{c}(\sigma_1) = m_0+m_1 a+\ldots+m_{n-1}a^{n-1}$ where $m_0+m_1+\ldots+m_{n-1} = l_1+l_2+\ldots+l_k$ (here we identify $\mathfrak{B}_{\tilde{f}}(y_1)$ with ZZ_n). Suppose that the cocycle $\bar{c}q \in C^2(\tilde{P}, \mathfrak{B}_{\tilde{f}q})$ is cohomologically trivial. Thus by Lemma (6.1) we have $m_0 = m_1 = \ldots = m_{n-1}$. By condition (6.4), the greatest common divisor of the integers $n \cdot m_0$ and $n' = n \cdot s$ is equal to s. Thus n divides s and so n^2 divides n', which is impossible by the assumption of the lemma.

From Lemma (6.2) we obtain the following

(6.5) COROLLARY. *Let $f\colon P \to W$ be a map as in Lemma* (6.2). *Then $f_{\#,2}\colon \pi_2(P) \to \pi_2(W)$ is a nontrivial homomorphism.*

Let us formulate the following

(6.6) THEOREM. *A movable continuum X with* $\operatorname{Tor}(\text{pro-}\pi_1(X)) \neq 0$ *and* $\operatorname{Fd}(X) = 2$ *is non-approximatively 2-connected.*

Proof. We can assume that (X, x) is the inverse limit of an inverse sequence $\{(X_n, x_n), p_n^m\}$ of 2-dimensional connected polyhedra such that the homomorphism $(p_1^n)_{\#}\colon \pi_1(X_n, x_n) \to \pi_1(X_1, x_1)$ maps a torsion element of $\pi_1(X_n, x_n)$ onto a nontrivial element for each n. Since X is a movable continuum and thus is uniformly movable (see [7] and [11]), there is an $m > 1$ and a sequence of maps $r_n^m\colon (X_m, x_m) \to (X_n, x_n)$ (for each n) such that

$$\text{(6.7)}\qquad p_n^{n+1} \circ r_{n+1}^m \simeq r_n^m \quad\text{and}\quad p_1^n \circ r_n^m = p_1^m \quad\text{for } n \geqslant m.$$

Let a be a torsion element of the group $\pi_1(X_m, x_m)$ such that $b = (p_1^m)(a)$ is a nontrivial element of the group $\pi_1(X_1, x_1)$. Let $a_i = (r_i^m)_{\#}(a) \in \pi_1(X_i, x_i)$ for each i. Denote by l_i the order of the element a_i in $\pi_1(X_i, x_i)$ and by l the order of the element a in $\pi_1(X_m, x_m)$. Since the maps $p_i^{i+1} r_{i+1}^m$ an r_i^m are homotopic, there is an automorphism (induced by a closed path) $h\colon \pi_1(X_i, x_i) \to \pi_1(X_i, x_i)$ such that $h(p_i^{i+1} r_{i+1}^m)_{\#} = (r_i^m)_{\#}$. Thus the elements $(p_i^{i+1})_{\#}(a_{i+1}) = (p_i^{i+1} r_{i+1}^m)_{\#}(a)$ and a_i have the same order in the group $\pi_1(X_i, x_i)$. It follows that $l_i \leqslant l_{i+1}$. Also $l_i \leqslant l$ for any i. Thus there is an integer j such that $l_i = l_j$ for $i > j$. Let $f_i\colon (P, p) \to (X_i, x_i)$ be a map of the pseudoprojective plane of order l_j such that $\operatorname{im}(f_i)_{\#}$ is a subgroup of $\pi_1(X_i, x_i)$ generated by the element a_i for each $i > j$. Then the map $p_j^i \circ f_i\colon (P, p) \to (X_j, x_j)$ induces a monomorphism $(p_j^i \circ f_i)_{\#}\colon \pi_1(P, p) \to \pi_1(X_j, x_j)$ for each $i > j$ and thus by Corollary (6.5) the homomorphism $(p_j^i \circ f_i)_{\#,2}\colon \pi_2(P, p) \to \pi_2(X_j, x_j)$ is not trivial, and so $(p_j^i)_{\#,2}\colon \pi_2(X_i, x_i) \to \pi_2(X_j, x_j)$ is not trivial for each $i > j$. Thus (X, x) is non-approximatively 2-connected.

From Theorem (6.6) and the previous results follows

(6.8) COROLLARY. *Let X_i be a movable continuum with $1 \leqslant \operatorname{Fd}(X_i) \leqslant 2$ for $i = 1, 2, \ldots, k$. Then* $\operatorname{Fd}(X_1 \times X_2 \times \ldots \times X_k) \geqslant k$.

Proof. If X_i is an approximatively 1-connected continuum, then X_i is non-approximatively 2-connected. Thus by Theorem (6.6) if X_i is an approximatively 2-connected continuum, then $\operatorname{In}(\text{pro-}\pi_1(X_i)) \neq 0$. Thus by Corollary (2.5) and Theorem (5.4) it follows that $\operatorname{Fd}(X_1 \times X_2 \times \ldots \times X_k) \geqslant \operatorname{Fd}(Y_1 \times Y_2 \times \ldots \times Y_k)$, where Y_i is equal to S^1 or S^2.

Remark. S. Nowak [9] has given an example of a family $\{X_i\}_{i=1}^{\infty}$ of polyhedra with $\operatorname{Fd}(X_i) = n \geqslant 3$ such that $\operatorname{Fd}(X_1 \times \ldots \times X_k) = n$ for any k.

7. Examples. We will give an example of an approximatively 2-connected continuum Y with $\operatorname{Tor}(\text{pro-}\pi_1(Y)) \neq 0$ and $\operatorname{Fd}(X) = 2$; thus the assumption of movability in Theorem (6.6) is essential.

(7.1) EXAMPLE. Let P_k be a pseudoprojective plane of order k. We consider P_k as a CW-complex with one 0-cell $\{p_k\}$, one 1-cell and one 2-cell. Then $C(P_k)$ is the following complex with free operators (in the sense of [14]):

$$0 \longleftarrow Z \xleftarrow{\ \varepsilon\ } ZZ_k \xleftarrow{\ a-1\ } ZZ_k \xleftarrow{\ 1+a+\ldots+a^{k-1}\ } ZZ_k \longleftarrow 0$$

where a is a generator of Z_k. Let $f\colon (P_{k^2}, p_{k^2}) \to (P_k, p_k)$ be a map of pseudoprojective planes (of orders k^2 and k respectively) which is a homeomorphism on 1-skeletons of P_{k^2} and P_k and which induces the following homomorphism of complexes $C(P_{k^2})$ and $C(P_k)$:

$$\begin{array}{ccccccccc}
0 \longleftarrow & Z & \xleftarrow{\ \varepsilon'\ } & ZZ_{k^2} & \xleftarrow{\ a-1\ } & ZZ_{k^2} & \xleftarrow{\ 1+a+\ldots+a^{k^2-1}\ } & ZZ_{k^2} & \longleftarrow 0\\
 & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & \\
0 \longleftarrow & Z & \xleftarrow{\ \varepsilon\ } & ZZ_k & \xleftarrow{\ b-1\ } & ZZ_k & \xleftarrow{\ 1+b+\ldots+b^{k-1}\ } & ZZ_k & \longleftarrow 0
\end{array}$$

where $f_0(1) = 1$, $f_1(1) = 1$ and $f_2(1) = 1+b+\ldots+b^{k-1}$. This map f induces the epimorphism $f_{\#}\colon \pi_1(P_{k^2}, p_{k^2}) \to \pi_1(P_k, p_k)$ and the trivial homomorphism $f_{\#,2}\colon \pi_2(P_{k^2}, p_{k^2}) \to \pi_2(P_k, p_k)$.

Let k be a fixed integer $\geqslant 2$. Let Y_n be the pseudoprojective plane of order $k^{(2^n)}$ and let $q_n^{n+1}\colon (Y_{n+1}, y_{n+1}) \to (Y_n, y_n)$ be the map described above. Then $(Y, y) = \underleftarrow{\lim}\{(Y_n, y_n), q_n^{n+1}\}$ is an approximatively 2-connected continuum with $\operatorname{Tor}(\text{pro-}\pi_1(Y, y)) \neq 0$ and $\operatorname{Fd}(Y) = 2$.

Let n be a positive integer and let π be an inverse sequence of groups (abelian groups if $n > 1$). We say that a continuum (X, x) is a space of shape type (π, n) if $\text{pro-}\pi_n(X, x)$ is isomorphic to π and $\text{pro-}\pi_i(X, x)$ is isomorphic to the trivial sequence for every $i \neq n$. One can prove (using the Whitehead theorem in shape theory [7]) the following

(7.2) PROPOSITION [1]. *Let* (X, x) *and* (Y, y) *be continua of shape type* (π, n) *with finite fundamental dimension. Then* $\operatorname{Sh}(X, x) = \operatorname{Sh}(Y, y)$.

[1] The proof of this proposition is given in the appendix (Section 8).

The next example shows that the assumption of movability in Corollary (6.8) is essential.

(7.3) EXAMPLE. Let $Y(k)$ be the continuum defined in Example (7.1). If k and l are integers relatively prime ($k > 1$, $l > 1$), then one can check that $\text{pro-}\pi_1(Y(k) \times Y(l))$ and $\text{pro-}\pi_1(Y(k \cdot l))$ are isomorphic. Since $Y(k) \times Y(l)$ and $Y(k \cdot l)$ are continua of shape type $(\underline{\pi}, 1)$ with finite dimension, by Proposition (7.2) we have $\operatorname{Sh}(Y(k) \times Y(l)) = \operatorname{Sh}(Y(k \cdot l))$. Thus, if $k_1, k_2, \ldots, k_n$ are different primes, $Y(k_1) \times Y(k_2) \times \ldots \times Y(k_n)$ has the same shape as $Y(k_1 \cdot k_2 \cdot \ldots \cdot k_n)$, and so $\operatorname{Fd}(Y(k_1) \times Y(k_2) \times \ldots \times Y(k_n)) = 2$.

Now we will give an example of continua with $\operatorname{Fd}(X) = \operatorname{Fd}(Y) = 2$ such that X is non-approximatively 2-connected and $\operatorname{Fd}(X \times Y) = 2$ (compare Theorem (5.4)).

(7.4) EXAMPLE. Let $(Y, y) = \underleftarrow{\lim}\{(Y_n, y_n), q_n^{n+1}\}$ be the continuum from Example (7.1). We will consider the suspension of a k-adic selenoid as the inverse limit of an inverse sequence $\{(X_n, x_n), p_n^{n+1}\}$ of 2-spheres where $p_n^{n+1}\colon S^2 \to S^2$ is a map of degree $k^{(2^n)}$.

Let $(W_n, w_n) = (X_n \times \{y_n\} \cup \{x_n\} \times Y_n, (x_n, y_n)) \subset (X_n \times Y_n, (x_n, y_n))$. The element of $\pi_2(W_n, w_n)$ represented by the imbedding $(X_n, x_n) \to (W_n, w_n)$ we denote by ε. Let $r_n^{n+1}\colon (W_{n+1}, w_{n+1}) \to (W_n, w_n)$ be a map satisfying the following conditions:

$$(7.5)\qquad (r_n^{n+1} | Y_{n+1}) = q_n^{n+1}\colon (Y_{n+1}, y_{n+1}) \to (Y_n, y_n) \subset (W_n, w_n),$$

the map

$$(7.6)\qquad (r_n^{n+1} | X_{n+1})\colon (X_{n+1}, x_{n+1}) \to (W_n, w_n),$$

induces the homomorphism of 2-homotopy groups such that $(r_n^{n+1} | X_{n+1})_{\#,2}(\varepsilon') = (1+a+\ldots+a^{k^{(2^n)}-1})\varepsilon$, where ε' is a generator of the group $\pi_2(X_{n+1}, x_{n+1})$ and a is a generator of the group $\pi_1(W_n, w_n)$ (we consider $\pi_2(W_n, w_n)$ as a $Z\pi_1(W_n, w_n)$-module).

We will prove that $(W, w) = \underleftarrow{\lim}\{(W_n, w_n), r_n^{n+1}\}$ and $(X \times Y, (x, y))$ have the same shape. Let $i_n\colon (W_n, w_n) \to (X_n \times Y_n, (x_n, y_n))$ be the inclusion map. One can easily see that $i_n \circ r_n^{n+1}$ is homotopic to $(p_n^{n+1} \times q_n^{n+1}) \circ i_{n+1}$. Thus $\underline{i} = \{i_n\}\colon (W, w) \to (X \times Y, (x, y))$ is a shape morphism. It is easy to see that $\text{pro-}\pi_1(\underline{i})$ is an isomorphism. Let $(\tilde{Y}_n, \tilde{y}_n)$, $(\widetilde{X_n \times Y_n}, \widetilde{(x_n, y_n)}) = (X_n \times \tilde{Y}_n, (x_n, \tilde{y}_n))$ and $(\tilde{W}_n, \tilde{w}_n)$ be the universal coverings of (Y_n, y_n), $(X_n \times Y_n, (x_n, y_n))$ and (W_n, w_n), respectively and let $\tilde{q}_n^{n+1}\colon (\tilde{Y}_{n+1}, \tilde{y}_{n+1}) \to (\tilde{Y}_n, \tilde{y}_n)$ and $\tilde{i}_n\colon (\tilde{W}_n, \tilde{w}_n) \to (X_n \times \tilde{Y}_n, (x_n, \tilde{y}_n))$ be the lifting of q_n^{n+1} and i_n, respectively (we can assume that $\tilde{i}_n$ is an inclusion map).

Let $g_n\colon (X_{n+1}, x_{n+1}) \to (W_n, w_n)$ be a map such that

$$(g_n)_{\#,2}(\varepsilon') = (1+a+\ldots+a^{k^{(2^n)}-1})(\varepsilon)$$

and let $f_{n+1} \cong \tilde{g}_n \circ s_{n+1}$ where $\tilde{g}_n\colon (X_{n+1}, x_{n+1}) \to (\tilde{W}_n, \tilde{w}_n)$ is a lifting of g_n and $s_{n+1}\colon (X_{n+1} \times Y_{n+1}, (x_{n+1}, y_{n+1})) \to (X_{n+1}, x_{n+1})$ is the natural projection. One can check that

$$\tilde{r}_n^{n+2} \cong f_n \circ (p_n^{n+1} \times \tilde{q}_n^{n+1}) \circ \tilde{i}_{n+2}, \tag{7.7}$$

$$\tilde{i}_n \circ \tilde{r}_n^{n+1} \circ \tilde{f}_{n+1} \cong (p_n^{n+2} \times \tilde{q}_n^{n+2}). \tag{7.8}$$

Thus the map of sequences

$$\tilde{\boldsymbol{i}} = \{\tilde{i}_n\}: \{(\tilde{W}_n, \tilde{w}_n), \tilde{r}_n^{n+1}\} \to \{(X_n \times \tilde{Y}_n, (x_n, \tilde{y}_n)), p_n^{n+1} \times \tilde{q}_n^{n+1}\}$$

is a homotopy equivalence (in the sense of [6]). Since the covering maps induce isomorphisms of the homotopy groups for $n \geqslant 2$, pro-$\pi_n(\boldsymbol{i})$ is an isomorphism for $n \geqslant 2$. By the Whitehead theore in shape theory (see [7] or [1] and [2]) $\boldsymbol{i}$ is a shape equivalence. So (W, w) and $(X \times Y, (x, y))$ have the same shape and thus

$$\mathrm{Fd}(X \times Y, (x, y)) = \mathrm{Fd}(W, w) = 2.$$

8. Appendix — the proof of Proposition (7.2). Let X and Y be continua of shape type $(\underline{\pi}, k)$ with finite fundamental dimension. We will show (Lemma (8.3)) that there is a shape morphism $f: X \to Y$ which induces isomorphisms of pro-homotopy groups in all dimensions. By the Whitehead theorem in shape theory (see [7] or [1] and [2]) it follows that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. We will first prove the following

(8.1) LEMMA. *Let $g: X \to Y$ be a map of topological spaces such that for any map $\varphi: S^n \to X$ the composition $g \circ \varphi$ is homotopically trivial. Let f_0 and f_1 be maps of an n-dimensional CW-complex K into X. If $f_0|K^{(n-1)} = f_1|K^{(n-1)}$ then $g \circ f_0$ is homotopic $g \circ f_1$ rel. $K^{(n-1)}$.*

Proof. Let $\alpha: (B^n, S^{n-1}) \to (K, K^{(n-1)})$ be a characteristic map of some n-cell in K. We define the map

$$\varphi: (B^n \times \{0, 1\}) \cup (S^{n-1} \times [0, 1]) \to X$$

by

$$\varphi(x, i) = f_i \alpha(x) \quad \text{if} \quad x \in B^n,\ i = 0 \text{ or } 1,$$
$$\varphi(x, t) = f_0 \alpha(x) \quad \text{if} \quad x \in S^{n-1},\ t \in [0, 1]$$

(of course $f_0 \alpha(x) = f_1 \alpha(x)$ if $x \in S^{n-1}$).

Since $(B^n \times \{0, 1\}) \cup (S^{n-1} \times [0, 1])$ is an n-sphere, the map $g \circ \varphi$ is homotopically trivial, and so there is an extension $\psi: B^n \times [0, 1] \to Y$ of the map $g \circ \varphi$. So the maps $g f_0 \alpha$ and $g f_1 \alpha$ are homotopic rel. S^{n-1}. It follows that the maps $g f_0$ and $g f_1$ are homotopic rel. $K^{(n-1)}$.

If L is a subcomplex of CW-complex K, then the pair (K, L) has the homotopy extension property for any topological space. Thus by induction we can obtain the following

(8.2) COROLLARY. *Let $g_j: X_j \to X_{j+1}$ be a map of topological spaces such that for any map $\varphi: S^{n+1} \to X_j$ the composition $g_j \circ \varphi$ is homotopically trivial $(j = 1, 2, \ldots, m)$. Let f_0 and f_1 be maps of an $(n+m)$-dimensional CW-complex K into X_1. If $f_0|K^{(n)} = f_1|K^{(n)}$ then the maps $g_m \circ g_{m-1} \circ \ldots \circ g_1 \circ f_i$, $i = 0, 1$, are homotopic rel. $K^{(n)}$.*

Now we will prove the following

(8.3) LEMMA. *Let k and l be positive integers, $l \geqslant k$. Let $X = \{X_n, p_n^m\}$ be an inverse sequence of connected CW-complexes such that the pro-group $\pi_i(X)$ is isomorphic to the trivial group for $i = 1, 2, \ldots, k-1$ and $\dim X_n \leqslant l$ for every n. Let $Y = \{Y_n, q_n^m\}$ be an inverse sequence of connected CW-complexes such that the pro-group $\pi_i(Y)$ is isomorphic to the trivial group for every $i \neq k$. Then for any morphism of pro-groups*

$$\varphi: \pi_k(X) \to \pi_k(Y)$$

there exists a morphism

$$f: X \to Y$$

such that $\pi_k(f)$ is equivalent to φ.

Proof. By the trick of Mardešić (see Lemma 8.1.1 and the proof of Theorem 8.3.2 in [2]) we may assume that $\pi_i(X_n)$ and $\pi_i(Y_n)$ are the trivial groups for $i = 1, 2, \ldots, k-1$ and every integer n. Thus we can assume that X_n (and Y_n) has exactly one 0-cell and has no cells in dimensions $1, 2, \ldots, k-1$ (for every n). We may also assume that the map q_n^{n+1} induces the trivial homomorphism

$$\pi_i(q_n^{n+1}): \pi_i(Y_{n+1}) \to \pi_i(Y_n)$$

for $i = k+1, 2, \ldots, l$ and every integer n.

The morphism φ is a pair (φ_n, φ) which consists of an increasing map φ of the set of all positive integers J and of a sequence of homomorphisms

$$\varphi_n: \pi_k(X_{\varphi(n)}) \to \pi_k(Y_n), \quad n \in J$$

such that

$$\varphi_n \circ \pi_k(p_{\varphi(n)}^{\varphi(n+1)}) = \pi_k(q_n^{n+1}) \circ \varphi_{n+1} \quad \text{for} \quad n \in J. \tag{8.4}$$

By s we denote the integer $2l-2k-1$ and by r the integer $l-k-1$. Let ψ_n denote the isomorphism of kth homotopy groups induced by the inclusion $X_{\varphi(n)}^{(k-1)} \to X_{\varphi(n)}$. There is a map

$$g': X_{\varphi(n)}^{(k+1)} \to Y_n$$

such that

$$\pi_k(g_n') = \varphi_n \circ \psi_n.$$

Since the map q_{n-m}^{n-m+1} induces the trivial homomorphism of $(k+m)$-th homotopy groups for $m = 1, 2, \ldots, r$ and $\dim X_{\varphi(n)} \leqslant 1$, by induction we can define the map

$$g_n: X_{\varphi(n)} \to Y_{n-r},$$

which is an extension of the map $q_{n-r}^n \circ g_n'$. Of course $\pi_k(g_n) = \pi_k(q_{n-r}^n) \circ \varphi_n$. By (8.4) we obtain

$$\pi_k(q_{n-r}^{n+s-r} \circ g_{n+s}) = \pi_k(g_n \circ p_{\varphi(n)}^{\varphi(n+s)}).$$

Thus the map $g_n \circ p_{\varphi(n)}^{\varphi(n+s)}$ is homotopic (relatively to the base point) to a map which coincides the map $q_{n-r}^{n+s-r} \circ g_{n+s}$ on the k-skeleton $X_{\varphi(n+s)}^{(k)}$ of $X_{\varphi(n+s)}$. By Corollary (8.2) the maps $q_{n-s}^{n-r} \circ g_n \circ p_{\varphi(n)}^{\varphi(n+s)}$ and $q_{n-s}^{n+s-r} \circ g_{n+s}$ are homotopic (relatively to the base point). Thus the maps $\tilde{g}_n \circ p_{\varphi(n)}^{\varphi(n+s)}$ and $q_{n-s}^{n} \circ \tilde{g}_{n+s}$ are homotopic (relatively to the base point), where $\tilde{g}_m = q_{m-s}^{m-r} \circ g_m$.

Thus the maps $\tilde{g}_n$, where $n = k \cdot s+1$, $k \in J$, define a morphism (in procategory homotopy) of X in a subsequence of Y. Since.

$$\pi_k(\tilde{g}_n) = \pi_k(q_{n-s}^{n}) \circ \varphi_n ,$$

we can define required f.

References

[1] K. Borsuk, *Theory of Shape*, Warszawa 1975.

[2] J. Dydak and J. Segal, *Shape theory, an introduction*, Lecture Notes in Mathematics vol. 688, Springer-Verlag.

[3] S. T. Hu, *Cohomology and deformation retracts*, Proc. London Math. Soc. 53 (2) (1951), pp. 191–219.

[4] — *Homotopy Theory*, Academic Press, Inc., New York 1959.

[5] A. Kadlof and S. Spież, *Remark on the fundamental dimension of cartesian product of metric compacta*, to appear.

[6] S. Mardešić and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.

[7] M. Moszyńska, *The Whitehead theorem in the theory of shapes*, Fund. Math. 80 (1973), pp. 221–263.

[8] S. Nowak, *On the fundamental dimension of cartesian product of two compacta*, Bull. Acad. Polon. Sci. 24 (1976), pp. 1021–1028

[9] — *Algebraic theory of fundamental dimension*, Dissertationes Math. 187 (1981).

[10] S. Spież, *An example of a continuum X with* $\mathrm{Fd}(X \times S^1) = \mathrm{Fd}(X) = 2$, to appear.

[11] — *On uniform movability*, Bull. Acad. Polon. Sci. 22 (1974), pp. 43–45.

[12] N. E. Steenrod, *Homology with local coefficients*, Ann. of Math. 44 (1943), pp. 610–627.

[13] — *The topology of fibre bundles*, Princeton University Press, Princeton 1951.

[14] J. H. C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc. 55 (1949), pp. 453–496.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW

Accepté par la Rédaction le 28. 4. 1980

On a Problem of Silver

by

Arthur W. Apter (Newark, N.J.)

Abstract. We show that it is consistent, relative to an ω sequence of measurable cardinals, for $\aleph_\omega$ to be a Rowbottom cardinal and for $\mathrm{DC}_{\aleph_n}$ to hold, where n is an arbitrary natural number.

Of all of the large cardinal axioms which are currently known, the axioms which assert the existence of Rowbottom and Jonsson cardinals are amongst the more interesting hypotheses. Most large cardinal axioms assert, at least when the Axiom of Choice is true, that the cardinal in question is strongly inaccessible. This, however, is not true about Rowbottom and Jonsson cardinals. Indeed, Devlin has shown [3] that it is relatively consistent for $2^{\aleph_0}$ be a Jonsson cardinal, and Prikry has shown [6] that, assuming the consistency of a measurable cardinal, it is consistent for a Rowbottom cardinal of cofinality ω to exist.

The above results inspire the following question: How large is the least Rowbottom cardinal? Silver in his thesis [7] hypotheses that it is relatively consistent that the answer is $\aleph_\omega$, assuming the Axiom of Choice.

The answer to Silver's question is still not known, and is the only remaining unsolved problem from Silver's thesis. We have obtained a partial answer to Silver's question by showing that it is consistent, relative to the existence of an ω sequence of measurable cardinals, for $\aleph_\omega$ to be a Rowbottom cardinal and for a large portion, though not all, of the Axiom of Choice to be true. Specifically, we have proven the following:

THEOREM 1. *Assume that the theory* "ZFC + *There is an ω sequence of measurable cardinals*" *is consistent. Let $n_0 \in \omega$ be a fixed* (*though arbitrary*) *natural number. Then the theory* "$\mathrm{ZF} + \mathrm{DC}_{\aleph_{n_0}} + \aleph_\omega$ *carries a Rowbottom filter*" *is consistent.*

Note that some strong hypothesis is needed to obtain a model which witnesses Theorem 1 since an unpublished result of Silver shows that if $\aleph_\omega$ is a Rowbottom cardinal, then it must be measurable in some inner model. Note also that other partial results on Silver's problem have been obtained. In particular, Bull in his thesis [2] showed that, assuming the consistency of a measurable cardinal, the theory "$\mathrm{ZF} + \forall n \in \omega [2^{\aleph_n} = \aleph_{n+1}] + \aleph_\omega$ is a Rowbottom cardinal $+ \neg \mathrm{AC}_\omega$" is consistent.

Before beginning the proof of Theorem 1, we briefly mention some background information. Basically, our notation and terminology are fairly standard.