On the fundamental dimension of the Cartesian product of compacta with the fundamental dimension 2

by

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Abstract. In this paper we prove that if $X$ is a compactum with $\text{Fd}(X) = \text{Fd}(X \times S^1) = 2$ then $\text{Fd}(X \times Y) = \text{Fd}(S^1 \times Y)$ for any compactum $Y$ with $\text{Fd}(Y) > 0$. If a continuum $X \subseteq \mathbb{E}^3$ has a nontrivial shape then $\text{Fd}(X \times Y) \geq \text{Fd}(S^1 \times Y)$ for any compactum $Y$. The fundamental dimension of the Cartesian product of movable compacta $X_1, X_2, \ldots, X_k$ with $\text{Fd}(X_i) = 2$ (for $i = 1, 2, \ldots, k$) is not less than $k$.

1. Introduction. Let $G = \{G_\alpha, p_\alpha^\beta\}$ be an inverse sequence of groups. We say that $G$ contains elements of infinite order (shortly $\text{In}G \neq 0$) if there is an index $n_0$ such that for each $m \geq n_0$ the image $p_\alpha^\beta(G_\alpha)$ contains elements of infinite order. We say that $G$ has torsion (shortly $\text{Tor}G \neq 0$) if there is an index $n_0$ such that for any $m \geq n_0$ the homomorphism $p_\alpha^\beta$ maps some torsion element of $G_\alpha$ onto a nontrivial element. Observe that if $\text{Tor}G = 0$ and $\text{In}G = 0$ then $G$ is isomorphic with the trivial sequence.

We will prove that if $X$ is a continuum with $\text{In}(\text{pro-}\pi_1(X)) \neq 0$ then $\text{Fd}(X \times Y) \geq \text{Fd}(S^1 \times Y)$ for any compactum $Y$ (Corollary (2.5)). If $X$ is a nonapproximatively 1-connected continuum in $E^3$, then $\text{In}(\text{pro-}\pi_1(X)) \neq 0$ (if $X \subseteq \mathbb{E}^3$ is an approximatively 1-connected continuum then $X$ has the trivial shape or the shape of a bouquet of 2-spheres). Thus if a continuum $X \subseteq \mathbb{E}^3$ has a nontrivial shape then $\text{Fd}(X \times Y) \geq \text{Fd}(S^1 \times Y)$ for any compactum $Y$.

S. Nowak [5] has proved that if $X$ is a $\mathcal{F}$-compactum ($X \in \mathcal{F}$), i.e.

$$\text{Fd}(Y) = \max \{n | H^n(X, G) \neq 0 \text{ for any abelian group } G\},$$

then $\text{Fd}(X \times Y) = \text{Fd}(X) + \text{Fd}(Y)$ for every compactum $Y$ with $\text{Fd}(X) \neq 2$. The present author [10] has constructed an example of a continuum $X$ with $\text{Fd}(X) = 2$ such that $\text{Fd}(X \times Y) < \text{Fd}(X) + \text{Fd}(Y)$ for any compactum $Y$ with $\text{Fd}(Y) \neq 0$.

For a continuum $X$ with $\text{Tor}(\text{pro-}\pi_1(X)) \neq 0$ we have $\text{Fd}(X \times Y) \geq 2 + \text{Fd}(Y)$ for any compactum $Y \in \mathcal{F}$ (Corollary (3.4)).

If $X_1, X_2, \ldots, X_k$ are continua with $\text{Fd}(X_i) = \text{Fd}(X_i \times S^1) = 2$, then $\text{Fd}(X_1 \times X_2 \times \ldots \times X_k) = k$ (if $k \geq 3$) and $\text{Fd}(X_i \times Y) = \text{Fd}(S^1 \times Y)$ for any compactum $Y$ with $\text{Fd}(Y) > 0$ (Theorems (4.1) and (4.2)).

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We prove that if $X$ is a movable non-approximatively 2-connected continuum with $\text{Fd}(X) = 2$, then $\text{Fd}(X \times Y) = \text{Fd}(S^1 \times Y)$ for any compactum $Y$. If $X_1, X_2, \ldots, X_k$ are movable continua with $\text{Fd}(X_i) = 2$ then $\text{Fd}(X_1 \times X_2 \times \cdots \times X_k) \geq k$ (Corollary 6.8). A similar fact does not hold for continua with fundamental dimension $\geq 3$. For any integer $n \geq 3$, there exists a family $\{X_i\}_{i=1}^n$ of continua with fundamental dimension $n$ such that $\text{Fd}(X_i \times X_j \times \cdots \times X_k) = n$ for any $i$. We give an example of a family $\{X_i\}_{i=1}^n$ of continua with fundamental dimension 2 such that $\text{Fd}(X_1 \times X_2 \times \cdots \times X_k) = 2$ for any $k$. We assume that the reader is familiar with some elementary facts from shape theory (cf. [1], [2]).

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2. Continue with $\text{In} \circ \pi_1(X) \neq 0$. We prove the following lemma. Let $f: X \to Y$ be a map of CW-complexes such that

\[ f^*: H^n(Y, \mathbb{B}) \to H^n(X, \mathbb{B}) \quad (n \geq 0) \]

is a nontrivial homomorphism for some local system of abelian groups $\mathbb{B}$ on $Y$ (if $f$ is the local system of abelian groups on $X$ induced by $f$). If $g_1, P_1 \to Q_1$ is a map of CW-complexes such that $(g_1)_*\pi_1(P_1)$ contains an element of infinite order for $\{i \leq i \leq k\}$, then the map

\[ f \times g_1 \times \cdots \times g_k: X \times P_1 \times \cdots \times P_k \to Y \times Q_1 \times \cdots \times Q_k \]

is not deformable to the $(n+k-1)$-skelaton $Y \times Q_1 \times \cdots \times Q_k \times 0 \times \cdots \times 0 \times Q_0$ (i.e. there is no homotopy $h_1: X \times P_1 \times \cdots \times P_k \to Y \times Q_1 \times \cdots \times Q_k \times 0 \times \cdots \times 0 \times Q_0$ for $0 \leq i \leq 1$, such that $h_1 = f \times g_1 \times \cdots \times g_k$ and

\[ h_1(x, P_1 \times \cdots \times P_k) \subset (Y \times Q_1 \times \cdots \times Q_k \times 0^{n+k-1}) \]

Proof. For simplicity of notation, we will prove this lemma in the case of $k = 1$ (the proof in the general case is the same) and denote $g = g_1, P = P_1, Q = Q_1$. Since $g \pi_k(P)$ contains an element of infinite order, there is a map $\pi: S^k \to P$ such that $(g \pi)(\pi) \subset \pi_1(Q)$ is a monomorphism. Let $q: \bar{Q} \to Q$ be a covering such that $\text{im}(g \pi) = \pi_1(Q)$, and $g \pi(x) = q(x)$ for all $x \in X$. We have the following commutative diagram:

\[ \begin{array}{ccc}
S^1 & \to & \bar{Q} \\
\downarrow & & \downarrow \\
\bar{T} & \to & Q
\end{array} \]

The map $q_*$ is a monomorphism, and thus the map $(g \pi)_*$ is an isomorphism. Let $K$ be a space of homotopy type $(Z, I)$ (i.e. $\pi_1(K) = Z$ and $\pi_2(K) = 0$ for every $n > 1$) such that $\bar{Q} \to Q$ and the 2-skeleton $K^{(2)}$ of $K$ is equal to the 2-skeleton $Q^{(2)}$ of $Q$. The map $i : (g \pi)_* : S^1 \to \bar{Q}$, where $i : \bar{Q} \to K$ is the inclusion map, induces the isomorphism

\[ (i \circ (g \pi*)_*)_* : \pi_1(K) \to \pi_1(K) \]

so (both $S^1$ and $K$ are spaces of homotopy type $(Z, I)$) the map $i : (g \pi*)$ is a homotopy equivalence. Thus

\[ (i \circ (g \pi*)_*)^* : H^1(K, Z) \to H^1(S^1, Z) \]

is an isomorphism.

From the Künneth formula (see [13], [9]) and (2.2) it follows that the map $f \times (i \circ (g \pi_*)) : X \times S^1 \to Y \times K$ induces the nontrivial homomorphism

\[ (f \times (i \circ (g \pi_*)))^* : H^*(Y \times K, \mathbb{B} \otimes Z) \to H^*(X \times S^1, \mathbb{B} \otimes Z) \]

Thus the map $f \times (i \circ (g \pi_*))$ is not deformable to the $n$-skeleton $(Y \times K)^{(n)}$ of $Y \times K$ and so the map $f \times (g \pi_*) : X \times S^1 \to Y \times Q$ is not deformable to the $n$-skeleton $(Y \times Q)^{(n)}$ of $Y \times Q$.

The following diagram commutes:

\[ \begin{array}{ccc}
X & \times & S^1 \\
\downarrow & \searrow & \downarrow \\
Y & \times & Q
\end{array} \]

and $i \circ q$ is a covering map. Thus the map $f \times (g \pi_*)$ is not deformable to the $n$-skeleton $(Y \times Q)^{(n)}$ of $Y \times Q$ (if there is a homotopy $h_1: X \times S^1 \to Y \times Q, 0 \leq i \leq 1$, such that $h_0 = f \times (g \pi_*)$ and $h_1(x, S^1) = (Y \times Q)^{(0)}$), then we can lift this homotopy to a homotopy $\tilde{h}_1: X \times S^1 \to Y \times Q$, for which $\tilde{h}_0 = f \times (g \pi_*)$ and $\tilde{h}_1(x, S^1) = (Y \times Q)^{(0)}$, which is impossible. Finally $f \times (g \pi_*)$ is not deformable to the $n$-skeleton $(Y \times Q)^{(n)}$ of $Y \times Q$.

(2.3) Corollary. Let $Y$ be a continuum with $\text{In}(\pi_1(X)) \neq 0$ and let $X$ be a continuum with $\text{Fd}(X) \neq 0$. Then $\text{Fd}(X \times Y) = \text{Fd}(X) + 1$.

Proof. Let $\text{Fd}(X) = n < \infty$. Then $X$ has the shape of the inverse limit of an inverse sequence $\{X_i, f_i^k\}$ of polyhedra with dimension $< n$. Since $\text{Fd}(X) = n$, we can assume that the map $f_i^k: X_i \to X_j$ is not deformable to the $(n-1)$-skeleton $X_i^{(n-1)}$ of $X_i$. Thus from the deformation theorem of obstruction theory (Exercise B-7, Chapter IV of [4] or [3]) it follows that there is a local system of abelian groups $\mathbb{B}$ on $X_i$ (we can take $\mathbb{B}$ such that $\mathbb{B}(X_i^{(n-1)}) = \pi_1(X_i, X_i^{(n-1)})$) such that the homomorphism

\[ (p_i^k)^* : H^i(X_i, \mathbb{B}) \to H^i(X_j, \mathbb{B}_j) \]

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is nontrivial (here \( B_a \) is a local system of abelian groups on \( X_m \) induced by \( \mathcal{B} \) and \( p^f \)).

We can assume that \( Y \) is the inverse limit of an inverse sequence of polyhedra \( \{ Y_m, q_m \} \) such that \( (q_m)_* (p_{m+1}) \) contains an element of infinite order for each \( m \geq 1 \). By Lemma (2.1) the map \( p_{m} \times \pi_1 (Y_{m+1}) \times X_m \times Y_m \to Y_m \times Y_{m+1} \) is not deforming to the \( n \)-skeleton \( (X \times Y)[m] \) for any \( m \geq 1 \), and so \( \text{Fd}(X \times Y) \geq n+1 \).

By a similar argument to the one we obtain

\[ Y \times Y \]

(2.4) **Corollary.** Let \( Y_1 \) be a continuum with \( \text{In}(p_{n}(Y)_1) \neq 0 \) for each \( i = 1, \ldots, k \). Then \( \text{Fd}(Y_1 \times Y_2 \times \ldots \times Y_k) \geq k \).

We also obtain

\[ Y \times Y \]

(2.5) **Corollary.** Let \( Y \) be a continuum with \( \text{In}(p_{n}(Y)) \neq 0 \). Then \( \text{Fd}(X \times Y) \geq \text{Fd}(X \times S^1) \) for any compactum \( X \).

**Proof.** By Corollaries (2.3) and (2.4) the inequality \( \text{Fd}(X \times Y) \geq \text{Fd}(X \times S^1) \) holds for any continuum with \( \text{Fd}(X) \neq 2 \). If \( \text{Fd}(X) = \text{Fd}(X \times S^1) = 2 \), then evidently \( \text{Fd}(X \times Y) \geq \text{Fd}(X \times S^1) \). Let \( \text{Fd}(X \times S^1) = \text{Fd}(X) = 2 \). Then by Corollary (2.3) we have \( \text{Fd}(X \times Y \times S^1) \geq \text{Fd}(X \times S^1) + 1 = 4 \); thus \( \text{Fd}(X \times Y) \geq \text{Fd}(X \times S^1) = 4 \).

S. Nowak [9] has proved a similar result for continua with fundamental dimension 1. If \( \text{Fd}(Y) = 1 \) then \( \text{In}(p_{n}(Y)) \neq 0 \), and so we have given a new (simpler) proof of the results of S. Nowak.

Let \( X \) be a continuum in \( E^3 \) with a nontrivial \( \mathcal{B} \). If \( X \) is approximatively 1-connected then \( X \) has the shape of a bouquet of 2-spheres. If \( X \) is not approximatively 1-connected then \( \text{In}(p_{n}(X)) \neq 0 \). Thus by Corollary (2.5) we obtain the following

\[ Y \times Y \]

(2.6) **Corollary.** If a continuum \( X \subset E^3 \) has nontrivial shape then \( \text{Fd}(X \times Y) \geq \text{Fd}(X \times S^1) \) for any compactum \( Y \).

3. **Continua with \( \text{Tor}(p_{n}(Y)) \neq 0 \).** By a pseudo-projective class of order \( m \geq 2 \) we mean the matching of disc \( D \) and its boundary \( S^1 \) by a covering map \( \mathcal{B} : S^1 \to S^1 \) of degree \( m \). Now we prove the following

\[ Y \times Y \]

(3.1) **Lemma.** Let \( g : P \to X \) be a map of a pseudo-projective plane \( P \) (of order \( m \geq 2 \)) into a CW-complex \( Y \) which induces a nontrivial homomorphism \( g_{\mathcal{B}} : \pi_i (P) \to \pi_i (Y) \). Then \( g^*: H^i (Y, Z) \to H^i (P, Z) \) is a nontrivial homomorphism (in fact, \( g^* \) is a monomorphism).

**Proof.** Let \( \pi_i (Y) \cong Z_m \) and let \( K \) be a CW-complex of homotopy type \( (Z_m, 1) \) such that \( Y = K \) and \( Y^{(m)} = K^{(m)} \). The composition \( f = i \circ g \) (where \( i : Y \to K \) is the inclusion) induces the nontrivial homomorphism \( f_* : \pi_i (P) \to \pi_i (K) \). Let \( a \) be a generator of \( \pi_i (P) \cong Z_m \). Then \( b = f_* (a) \) is a generator of \( \pi_i (Y) \cong Z_m \). Let \( m = l \cdot k \). Because \( K \) is a space of homotopy type \( (Z_m, 1) \), the homotopy class of the map \( f \) is determined by the homomorphism \( f_* : \pi_i (P) \to \pi_i (K) \).

Since \( f_* \) is a monomorphism, \( \widetilde{f} \) is a nontrivial homomorphism.

By Lemma (3.1), the map \( f_* \) induces the nontrivial homomorphism \( \widetilde{f}^* : H^*(Y, Z) \cong H^*(P, Z) \). By the Künneth formula the map \( (f_* \otimes 1) : P \times S^1 \to Y \times S^1 \) induces the nontrivial homomorphism

\[ (f_* \otimes 1)^*: H^*(P \times S^1, Z) \to H^*(Y \times S^1, Z) \]
Thus the map $\langle f \circ g \rangle \times id_{p^a}$ is not deformable to the $(n+1)$-skeleton of $\mathcal{Y} \times S^a$. It follows that the map $f \circ g \times id_{p^a}$ (and so also the map $\times id_{p^a}$) is not deformable to the $(n+1)$-skeleton of $Y \times S^a$.

(3.3) **COROLLARY.** Let $X$ be a continuum with $\text{Tor}(\text{pro-}\pi_1(X)) \neq 0$. Then $\text{Fd}(X \times S^a) \geq n+2$.

By a result of S. Nowak [8] we obtain the following

(4.4) **COROLLARY.** Let $X$ be a continuum with $\text{Fd}(X) \leq 2$ and $\text{Tor}(\text{pro-}\pi_1(X)) \neq 0$. Then $\text{Fd}(X) = c[X] = 2$ (where $c[X]$ is the maximal $n$ such that there is a Cech local system of abelian groups $\mathbb{B}$ on $X$ such that $H^i(Y, \mathbb{B}) \neq 0$) or equivalently $\text{Fd}(Y \times X) = Fd Y + 2$ for every continuum $Y \in \mathcal{S}$.

4. **Continua with $\text{Fd}(X \times S^a) = 2$.** The class of these continua contains all continua with $\text{Fd}(X) = 1$ and all continua with $c[X] < \text{Fd}(X) = 2$ (see [8]).

If $X$ is an approximatively 1-connected continuum with $\text{Fd}(X) = 2$, then $X$ has the shape of the inverse limit of an infinite sequence of bouquets of 2-spheres and it is easy to see [8] that then $\text{Fd}(X \times S^a) = 2$.

Let $Y$ be a continuum with $\text{Fd}(Y) = \text{Fd}(Y \times S^a) = 2$. Then $\text{pro-}\pi_1(Y)$ is not isomorphic to the trivial sequence and Tor(pro-\pi_1(Y)) = 0 by Corollary (3.3). Thus $\text{In}(\text{pro-}\pi_1(Y)) = 0$.

Let $Y_1$ be a continuum with $\text{Fd}(Y_1) = \text{Fd}(Y_1 \times S^a) = 2$ for $i = 1, 2, \ldots, k$, $k \geq 2$. Then $\text{Fd}(Y_i \times S^a) = (\text{Fd}(Y_1 \times S^a)_i \times \text{Fd}(Y_1 \times S^a)_{i+1}) \leq 2k$. Since $S^a \times S^a = S^a$ is an $\mathcal{S}$-compactum, it follows by the theorem of Nowak [8] (see the introduction) that $\text{Fd}(Y_1 \times Y_2 \times \ldots \times Y_k) \leq k$. Thus, by Corollary (2.4), we obtain the following

(4.1) **THEOREM.** Let $Y_i$ be a continuum with $\text{Fd}(Y_i) = \text{Fd}(Y_i \times S^a) = 2$ for each $i = 1, 2, \ldots, k$, $k \geq 2$. Then $\text{Fd}(Y_1 \times Y_2 \times \ldots \times Y_k) = k$.

Let $Y$ be a continuum with $\text{Fd}(Y) = \text{Fd}(Y \times S^a) = 2$. If $X$ is a continuum with $\text{Fd}(X) = \text{Fd}(X \times S^a) < \infty$ (it can only hold if $\text{Fd}(X) = 2$) then by Theorem (4.1) we have $\text{Fd}(X \times S^a) = 2 = \text{Fd}(X \times S^a)$. If $X$ is a continuum with $0 < \text{Fd}(X)$, then we have $\text{Fd}(X \times Y) < \text{Fd}(X \times Y) + 1$ (see [9], [10]), and so $\text{Fd}(X \times Y) < \text{Fd}(X \times S^a)$, Thus by Corollary (2.5) we have

(4.2) **THEOREM.** Let $Y$ be a continuum with $\text{Fd}(Y) = \text{Fd}(Y \times S^a) = 2$. Then $\text{Fd}(X \times Y) = \text{Fd}(X \times S^a)$ for any compactum $X$ with $\text{Fd}(X) > 0$.

Let $Y$ be an $\mathcal{S}$-compactum. If $\text{Fd}(X) = 2$ or $2 = \text{Fd}(X) = \text{Fd}(X \times S^a)$, then by the theorem of S. Nowak we have $\text{Fd}(X \times Y) = \text{Fd}(X \times Y)$. If $2 = \text{Fd}(X \times S^a)$, then by Theorem (4.2) $\text{Fd}(X \times Y) = \text{Fd}(X \times S^a) = \text{Fd}(X) + 1$. Thus we obtain the following (see [3]):

(4.3) **COROLLARY.** If $X$ is an $\mathcal{S}$-compactum and $X$ is a compactum with $\text{Fd}(X) > 0$, then $\text{Fd}(X \times Y) = \text{Fd}(Y)$.

5. **Non-approximatively 2-connected continua.** We will prove the following

(5.1) **THEOREM.** Let $X_i$ be a non-approximatively 2-connected continuum with $\text{Fd}(X_i) = 2$ for each $i = 1, 2, \ldots, k$. Then $\text{Fd}(X_1 \times X_2 \times \ldots \times X_k) = 2k$.

**Proof.** Each $X_i$ has the shape of the inverse limit of an infinite sequence $X_i = \{X_i, \alpha_i, p_i^a\}$ of 2-dimensional polyhedra. We can assume that the map $p_i^a_+ \cdot X_i \to X_i \to X_i \cdot p_i^a$ induces the nontrivial homomorphism $\pi_2(X_i) \to \pi_1(X_i)$. Let $g_i: S^a \to X_i$ be a map such that $p_i^a \circ g_i$ is homotopically nontrivial. Let $h_i: S^a \to X_i$ be the universal covering and let $h_i^*: S^a \to X_i$ be a lifting of $p_i^a \circ g_i$. So we have the following commutative diagram:

![Diagram](image)

The map $h: S^a \to X_{k+1}$ is homotopically nontrivial and $X_{k+1}$ is a 1-connected 2-dimensional polyhedron; so $h^*: H^2(X_{k+1}, Z) \to H^2(S^a, Z)$ is a nontrivial homomorphism. From the K"unneth formula it follows that the map

$$h_1 \times \ldots \times h_k: S^a \times \ldots \times S^a \to X_{k+1} \times \ldots \times X_{k+1}$$

induces a nontrivial homomorphism

$$(h_1 \times \ldots \times h_k)^*: H^2(X_{k+1} \times \ldots \times X_{k+1}, Z) \to H^2(S^a \times \ldots \times S^a, Z).$$

So the map $h_1 \times \ldots \times h_k$ is not deformable to the $(2k-1)$-skeleton of $X_{k+1} \times \ldots \times X_{k+1}$ and thus the map $p_i^a_\times \ldots \times p_i^a$ is not deformable to

$$(X_1 \times \ldots \times X_k)_{(2k-1)}.$$  

Thus $\text{Fd}(X_1 \times \ldots \times X_k) = 2k$.

By a similar argument to the above one can prove the following

(5.2) **PROPOSITION.** Let a compactum $X$ have the shape of the inverse limit of an infinite sequence $\{X_i, p_i^a\}$ of polyhedra such that the image of the homomorphism $(p_i^a)^*: H_1(X_i, \mathbb{B}) \to H_1(X, \mathbb{B})$ contains elements of infinite order for some local system of coefficients $\mathbb{B}$ on $X_i$ for any $i$, then $\text{Fd}(X) = 2$.

The above proposition generalizes a result of S. Nowak [8]; case $X = S^a$.

(5.3) **PROPOSITION.** $\text{Fd}(X \times Y) = \text{Fd}(S^a \times Y)$ for any compactum $X$ with $\text{Fd}(X) = n$.

**Proof.** If $\text{Fd}(X) < \text{Fd}(Y)$ (this always holds if $2 \leq \text{Fd}(Y) < \infty$), then by the theorem of Nowak [8] $\text{Fd}(S^a \times Y) = n + \text{Fd}(Y)$; thus $\text{Fd}(X \times Y) < \text{Fd}(X)$.
Now we will prove the following

(5.4) THEOREM. Let $X$ be a movable non-approximately 2-connected continuum with $\text{Fd}(X) = 2$. Then $\text{Fd}(X \times Y) = \text{Fd}(X) + \text{Fd}(Y) \geq 6$ for any continuum $Y$.

Proof. By Proposition (5.3) we have to prove that $\text{Fd}(X \times Y) \geq \text{Fd}(X) + \text{Fd}(Y)$.

The continuum $Y$ has the shape of the inverse limit of a two-dimensional connected polyhedra such that

(5.5) $(p_{a,1}^i, \pi_i(Y, x_i)) \to \pi_i(X, x_i)$ is a nontrivial homomorphism for any $i < j$.

(5.6) for any $2 < i < j$ there exists a map $r_i: X_i \to X_j$ such that $p_{a,1}^i \circ r_i = p_{a,1}^j$.

Fix $i,j \leq n$. For every $i$ we have $x_i \in X_i$ such that $p_{a,1}^i(x_i) = x_i$. We can assume that $r_i(x_i) = x_i$ (i.e., $f_i$ is an idempotent). By (5.6), for any $2 < i < j$ there exists a $\alpha_i(x_i) \to \pi_1(X_i, x_i)$ (induced by a path) such that

(5.7) $(p_{a,1}^i \circ r_i)(x_i) = \alpha_i(x_i)$.

Let $\psi: Y \times X_i \to X_i$ be the universal covering and let us choose $x_i \in X_i$ such that $\psi(x_i) = x_i$ (for every $i$). Let $p_i: X_i \to X_i$ and $p_i: X_i \to X_i$ be the liftings of the maps $p_{a,1}^i \circ r_i: X_i \to X_i$ and $p_{a,1}^i \circ r_i: X_i \to X_i$, respectively, such that $p_i(x_i) = x_i$ and $p_i(x_i) = x_i$ for any $i, j < n$.

For any element $a$ of an abelian group $G$ denote by $k(a)$ the greatest integer such that $a \leq k(a)$. For some element $b \in G$, $k(a) = \infty$ if $a$ is the trivial element $e \in G$. If $f: G \to H$ is a homomorphism, then $k(f(a)) = k(a)$ for any $a \in G$.

Let $2 < i < j$. Let $b = (p_{a,1}^i \circ r_i, a) = (\alpha_i(x_i), a)$ be an element of the $n$-simplex $a$ with the smallest integer $k(b)$. By (5.7) we have

$k((p_{a,1}^i \circ r_i)(x_i), a) = k((p_{a,1}^i \circ r_i)(x_i), a) = k(b)$.

It follows that

$k((p_{a,1}^i \circ r_i)(x_i), a) = 1$. Let $(p_{a,1}^i \circ r_i, a)(x_i) = (\alpha_i(x_i), a)$. Since $(\alpha_i(x_i), a)$ is an isomorphism, $c$ is a primitive element (i.e., $k(c) = 1$) of the free abelian group $\pi_1(X_i, x_i)$.

(5.8) The composition $f \circ g_0$ is homotopic to $1_{\pi_0(X_i, x_i)}$.

Let $Y = \lim(Y_i, q_i)$, where $Y_i$ are polyhedra. The continuum $X \times Y$ has the shape of the inverse limit of the inverse sequence $(X_i \times Y_i, p_{a,1}^i \times q_i)$. Suppose that $\text{Fd}(X \times Y) = n$. Then for any $i$ there exists a $a$ such that the map $(p_{a,1}^i \times q_i): X_i \times Y_i \to X_i \times Y_i$ is deformable to the $n$-skelaton of $X_i \times Y_i$, by the covering homotopy property the map $p_{a,1}^i \times q_i: X_i \times Y_i \to X_i \times Y_i$ is deformable to the $n$-skelaton of $X_i \times Y_i$. It follows by (5.8) that the map $1_{\pi_0(X_i, x_i)}: X_i \times Y_i \to X_i \times Y_i$ is deformable to the $n$-skelaton of $X_i \times Y_i$; thus $\text{Fd}(X_i \times Y_i) \leq n$. 2

6. Movable continua with $\text{Tor}(\pi_0(X)) = 0$ and $\text{Fd}(X) = 2$. Let $p_{a,1}$ be a two-dimensional planar $n$-simplex, where $n$ is a positive integer, be the space formed from the unit disc $D = \{ [x \in \mathbb{R}^2] \mid 0 < \|x\| < 1\}$ by the identification on $S^1 = \{ [x \in \mathbb{R}^2] \mid \|x\| = 1\}$ in polar coordinates $(1, \theta) = \{ (1, \theta + 2\pi n) \}$.

We will prove the following

(6.1) LEMMA. Let $\mathcal{K}$ be a local system of coefficients on $P$ such that $\mathcal{K}(x)$ is isomorphic with the integral group ring $\mathbb{Z} \mathcal{K}$ for each $x \in P$ and the group $\pi_1(P, x)$ is $\mathcal{K}$-acts on $\mathcal{K}$ under an epimorphism $\phi_1: \pi_1(P, x) \to \pi_1$. If $q: P \to P$ is the universal covering, then the kernel of the homomorphism $\phi^*: \pi_1(P, \mathcal{K}) \to \pi_1(P, \mathcal{K})$ is equal to

\[ (1 + a + \ldots + a^{n-1}) \mathcal{K} \mathcal{K} \mathcal{K} + (1 + a + \ldots + a^{n-1}) \mathcal{K} \mathcal{K} \mathcal{K} = \mathcal{K} \mathcal{K} \mathcal{K} + (1 + a + \ldots + a^{n-1}) \mathcal{K} \mathcal{K} \mathcal{K} \]

where $a$ is a generator of the group $\pi_1$ and $n = n'$ (here $\mathcal{K}$ is the local system of coefficients on $P$ induced by $\mathcal{K}$ and $q$).

Proof. Let $q: P \to P$ be the rotation of the angle $2\pi n'$. Let the natural projection $r: D \to P$ be simplicial with respect to triangulations $K$ and $\mathcal{K}$ of $D$ and $P$, respectively, and let $q: D \to P$ be simplicial with respect to $K$. Let $s_1, s_2, \ldots, s_n$ be all $n$-simplices of $K'$ oriented coherently; we will denote by the same symbols $n$-simplices of $K$ with the orientation induced by $r$. We can assume that $s_i \in K'$ has exactly one $1$-face which is contained in $S^1$.

The universal covering space $\bar{P}$ is formed from $n'$-copies of the unit disc $D$ by the identification on $\bigcup_{i=0}^{n'} S^1 \times \{i\}$

We consider $D \times \{i\}$ as a subset of $\bar{P}$, and let $\mathcal{K}$ be the triangulation on $P$ induced by triangulations $K'$ on $D \times \{i\}$. We can assume that $q(i, i) = q(x, y)$ for $x \in D$, $i = 0, 1, \ldots, n'-1$.

Let $C^2(\mathcal{K}, \mathcal{K})$ be a group of 2-cochains in the sense of [12]. Any 2-cochain $c \in C^2(\mathcal{K}, \mathcal{K})$ is coboundary equivalent to a 2-chain $c' \in C^2(\mathcal{K}, \mathcal{K})$ which is concentrated on $s_1$, i.e., $c'(s_1) = c(q(s_1)) = c(s_1) = 0$, and $c'(s_i) = 0$ for $i$ such that $s_i \in K'$.

Let $\gamma = \gamma \mathcal{K} \mathcal{K} = \{ \gamma \mathcal{K} \mathcal{K} \} \subset D \times \{i\}$ be the simplex of $K$ which are mapped by $q$ onto $s_1$. Then $c' \gamma$ is the 2-cochain of $C^2(\mathcal{K}, \mathcal{K})$ which has nontrivial values only on simplices $s_i, s_i, \ldots, s_{n'-1}$. The 2-cochain $c' \gamma$ is
cohomology equivalent to the cochain $e \in C^1(\mathbb{R}, \mathbb{R})$ which has nontrivial values only on simplexes $e_1 \times [t]$ and $e(\xi, x) = d_2 e \in C^2(\mathbb{R}, \mathbb{R})$, for $i = 0, 1, \ldots, n-1$, where $x$ is a point of $P$ such that $q(x) = x_0$, and $e \in [x_1 \times [t]]$. The 2-cochain $e$ is cohomologically trivial iff $d_2 e = 0$ for $i = 0, 1, \ldots, n-1$. This last holds iff $e \in \{1 + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_{i} \}$ of $\mathbb{Z}^n$, where we identify $\mathbb{R}^n$ with $\mathbb{Z}$; it proves the lemma.

Now we prove the following

**Lemma (6.2).** Let $f : P \to W$ be a map of the pseudoprojective plane $P$ of order $n$ into a 2-dimensional CW-complex which induces a nontrivial homomorphism $f_* : \pi_1(P) \to \pi_1(W)$. Let $q : P \to P$ be the universal covering. If $n$ does not divide $n'$, where $n'$ is the order of $f_*$, then the map $f : P \to W$ is homotopically nontrivial.

**Proof.** Let $f : P \to W$ be a covering such that $\pi_1(f_*(P)) = \pi_1(W)$. Then a lifting $\tilde{f} : P \to W$ of the map $f$ induces an epimorphism $f_* : \pi_1(P) \to \pi_1(W)$. Let $K(P)$ and $K(W)$ be spaces of type $(Z_n, 1)$ and type $(Z_n, 1)$, respectively, such that $(K(P), 2) \to W$ and $(K(W), 2) \to P$. The map $g : K(P) \to K(W)$ such that $g(x) = \tilde{f}(x)$ for each $x \in P$, induces an epimorphism of 1-homotopy groups. We have the following commutative diagram:

$$
\begin{array}{ccc}
P & \xrightarrow{f} & W \\
\downarrow{j_1} & & \downarrow{j_2} \\
K(P) & \xrightarrow{g} & K(W)
\end{array}
$$

where $j_1$ and $j_2$ are the inclusions. One can easily see that $j_2^* H^1(K(P), \mathbb{Z}) \to H^1(W, \mathbb{Z})$ is an isomorphism, and also that $g^* H^1(K(P), \mathbb{Z}) \to H^1(K(W), \mathbb{Z})$ is a monomorphism; thus $(j_2^* + j_1^*)^* = (j_2^* + j_1^*)^* H^1(K(P), \mathbb{Z}) \to H^1(W, \mathbb{Z})$ is a monomorphism. If $[d] = [f] + [d] \in H^1(W, \mathbb{Z})$, where $[d]$ is a generator of the group $H^1(K(P), \mathbb{Z}) \to Z_n$, then $[f]^*[d] = [f]^*[d]$ is an element of the group $H^1(W, \mathbb{Z})$ of order $n$.

We can assume that the map $f : P \to W$ is simplicial with respect to some triangulations $K$ and $L$ of $P$ and $W$ respectively and that the map $f : D \to P$ is simplicial with respect to $K$ and $L$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n'}$ be all $2$-simplexes of $K$ oriented coherently; we will denote by the same symbols $2$-simplexes of $K$ with the orientation induced by $\sigma$. Let $l_1 = \{\sigma \}$ for $i = 1, 2, \ldots, k$; since $[\sigma]$ is an element of the group $H^1(P, \mathbb{Z}) = Z_n$, of order $n$.

The greatest common divisor of integers $n'$ and $(l_1 + l_2 + \cdots + l_k)$ is equal to $s$, where $s = s' n$.

Let $\mathcal{B}$ be a local system of coefficients on $W$ such that $\mathcal{B}(x)$ is isomorphic with the integral group ring $\mathbb{Z}n'$ for each $x \in W$ and the group $\pi_1(W) = Z_n$ acts freely on $\mathcal{B}$. The map $f : P \to W$ is simplicial with respect to the triangulations $K$ and $L$ where $L$ is the triangulation on $W$ induced by the map $p$ and the triangulation $L$. Let $v_1, v_2, \ldots$ be all $2$-simplexes of $L$ with a chosen orientation. Let us choose a point $x_i \in v_i$ for each $v_i$ and a point $y_j \in v_j$ for each $y_j$ such that $f(y_j) = x_i$ for a certain $x_i$.

We assume that $Z \subseteq Z_n'$, i.e. we identify an integer $\xi \in Z$ with $(k \cdot 0 + a + \cdots + 0 \cdot a^{-1}) \in Z_n'$, where $a$ is a generator of $Z_n$. Let $\xi : Z_n' \to \mathcal{B}(x)$ be an isomorphism. For the cocycle $c \in H^1(P, \mathcal{B})$, we define a cocycle $c' \in H^1(P, \mathcal{B})$ such that $c(x_i) = \xi(s_j + \xi(s_i)) \in \mathcal{B}(x_j)$. Let $\mathcal{B}$ be a local system of coefficients on $P$ induced by $\mathcal{B}$ and $f$. Then the cocycle $c' \in H^1(P, \mathcal{B})$ is homological to the cocycle $c \in H^1(P, \mathcal{B})$ such that $c(x_j) = 0$ for $i = 1, 2, \ldots, k$ and $c(x_i) = 0$ for $i = m_1 + \alpha_{n-1} + \cdots + \alpha_{i}$ where $m_1 + \alpha_{n-1} + \cdots + \alpha_{i} = l_1 + l_2 + \cdots + l_k$ (here we identify $\mathcal{B}(x_1)$ with $\mathbb{Z}n'$). Suppose that the cocycle $c' \in H^1(P, \mathcal{B})$ is homological trivial. Thus by Lemma (6.1) we have $m_1 = \cdots = m_{n-1}$. By condition (6.6), the greatest common divisor of the integers $m_i$ and $n' = s$ is equal to $s$. Thus $n'$ divides $s$ and so $n'$ divides $s'$, which is impossible by the assumption of the lemma.

From Lemma (6.2) we obtain the following

**Corollary (6.5).** Let $f : P \to W$ be a map as in Lemma (6.2). Then $f_* : \pi_1(P) \to \pi_1(W)$ is a nontrivial homomorphism.

Let us formulate the following

**Theorem (6.6).** A movable continuum $X$ with Tor(pro-$\pi_1$, $X)$ $\neq 0$ and $\text{Fd}(X)$ $= 2$ is non-approximatively 2-connected.

**Proof.** We can assume that $X$ is the inverse limit of an inverse sequence $\{X_n, x_n, p_{n+1}^n \}$ of 2-dimensional connected polyhedra such that the homomorphism $(p_{n+1}^n)_* : \pi_1(X_{n+1}, x_{n+1}) \to \pi_1(X_n, x_n)$ maps a torsion element of $\pi_1(X_{n+1}, x_{n+1})$ onto a nontrivial element for each $n$. Since $X$ is a movable continuum and thus is uniformly movable (see [7] and [11]), there is an $m > 1$ and a sequence of maps $\pi_1^n : (X_n, x_n) \to (X_m, x_m)$ (for each $n$) such that

$$p_{n+1}^n \circ \pi_1^n \approx \pi_1^n \quad \text{and} \quad p_{n+1}^n \circ \pi_1^n \approx \pi_1^n \quad \text{for} \quad n \geq n.$$
Proof. If \( X_t \) is an approximatively 1-connected continuum, then \( X_t \) is non-approximatively 2-connected. Thus by Theorem (6.6) if \( X_t \) is an approximatively 2-connected continuum, then \( \text{th}(\text{pro-\pi}_{n}(X_t)) \neq 0 \). Thus by Corollary (2.5) and Theorem (5.4) it follows that \( \text{Fo}(X_t \times X_2 \times \ldots \times X_p) \geq \text{Fo}(Y_1 \times Y_2 \times \ldots \times Y_n) \), where \( Y_i \) is equal to \( S^1 \) or \( S^2 \).

Remark. S. Nowak [9] has given an example of a family \( \{X_j\}_{j=1}^n \) of polyhedra with \( \text{Fo}(X_t) = n \geq 3 \) such that \( \text{Fo}(X_t \times X_2 \times \ldots \times X_p) = n \) for any \( k \).

7. Examples. We will give an example of an approximatively 2-connected continuum \( Y \) with \( \text{Tor}(\text{pro-\pi}_{n}(Y)) = 0 \) and \( \text{Fo}(X_t) = 2 \); thus the assumption of movability in Theorem (6.6) is essential.

(7.1) Example. Let \( P_k \) be a pseudoprojective plane of order \( k \). We consider \( P_k \) as a CW-complex with one 0-cell \( p_0 \), one 1-cell and one 2-cell. Then \( C(P_k) \) is the following complex with free operators (in the sense of [14]):

\[
\begin{array}{ccc}
0 & \rightarrow & Z \rightarrow Z(k) = \cdots \rightarrow Z(1) \\
& \uparrow & \downarrow s & \downarrow s & \downarrow s & \downarrow s \\
0 & \rightarrow & Z \rightarrow Z(k) = \cdots \rightarrow Z(1) \\
& \downarrow s & \downarrow s & \downarrow s & \downarrow s
\end{array}
\]

where \( s \) is a generator of \( Z(k) \). Let \( f: (P_1, P_0) \rightarrow (P_k, P_0) \) be a map of pseudoprojective planes (of orders \( k \) and \( k \) respectively) which is a homeomorphism on 1-skeletons of \( P_1 \) and \( P_k \) and which induces the following homomorphism of complexes \( C(P_1) \) and \( C(P_k) \):

\[
\begin{array}{ccc}
0 & \rightarrow & Z \rightarrow Z = \cdots \rightarrow Z \\
& \uparrow b & \downarrow s & \downarrow s & \downarrow s & \downarrow s & \downarrow s \\
0 & \rightarrow & Z \rightarrow Z = \cdots \rightarrow Z
\end{array}
\]

where \( f_0(1) = 1 \), \( f_1(1) = 1 \), \( f_2(1) = 1 \), \( f_3(1) = 1 \). This map \( f \) induces the epimorphism \( f_* \): \( \pi_1(P_1, P_0) \rightarrow \pi_1(P_k, P_0) \) and the trivial homomorphism \( f_*: \pi_2(P_1, P_0) \rightarrow \pi_2(P_k, P_0) \).

Let \( k \) be a fixed integer \( \geq 2 \). Let \( Y_k \) be the pseudoprojective plane of order \( k \) and let \( \mathfrak{g}^{n+1} = (Y_{n+1}, Y_n) \rightarrow (Y_n, x_n) \) be the map described above. Then \( (Y, y) = \lim \{ (Y_{n+1}, Y_n) \} \) is an approximatively 2-connected continuum with \( \text{Tor}(\text{pro-\pi}_{n}(Y, y)) \neq 0 \) and \( \text{Fo}(Y, y) = 2 \).

The next example shows that the assumption of movability in Corollary (6.8) is essential.

(7.3) Example. Let \( Y(k) \) be the continuum defined in Example (7.1). If \( k \) and \( l \) are integers (distinct prime \( k \cdot l \geq 1 \), \( l > 1 \)), then one can check that \( \text{pro-\pi}_{n}(Y(k) \times Y(l)) \) and \( \text{pro-\pi}_{n}(Y(l) \times Y(k)) \) are isomorphic. Since \( Y(k) \times Y(l) \) and \( Y(l) \times Y(k) \) are continua of shape \((\pi_1, 1)\) with finite dimension, by Proposition (7.2) we have \( \text{Sh}(Y(k) \times Y(l)) = \text{Sh}(Y(l) \times Y(k)) \).

Now we will give an example of continua with \( \text{Fo}(X_t) = 2 \) such that \( X_t \) is non-approximatively 2-connected and \( \text{Fo}(X_t \times X_2 \times \ldots \times X_p) = 2 \) (compare Theorem (5.4)).

(7.4) Example. Let \((Y, x) = \lim \{ (Y_{n+1}, Y_n) \} \) be the continuum from Example (7.1). We will consider the suspension of a \( k \)-adic skeleton as the inverse limit of an inverse sequence \( \{ (Y_{n+1}, Y_n) \} \) of \( 2 \)-spaces where \( f^{n+1} = S^2 = S^2 \) is a map of degree \( k^{n+1} \).

Let \((W_n, w_0) = (Y_n \times x_n) \cup (Y_n \times Y_{n+1}, x_{n+1}) \) be the map described above. Then \( \text{pr}_{n+1}: (W_n, w_0) \rightarrow (W_{n+1}, w_0) \) be a map satisfying the following conditions:

\[
\begin{align*}
(\ast_{n+1}^{n+1}) &= (Y_{n+1}, Y_n) \\
&= (Y_n, x_n) = (W_n, w_0)
\end{align*}
\]

the map

\[
(\ast_{n+1}^{n+1}|_{X_{n+1}}): (X_{n+1}, x_{n+1}) \rightarrow (W_n, w_0)
\]

induces the homomorphism of 2-homotopy groups such that \( (\ast_{n+1}^{n+1}|_{X_{n+1}}) = (1 + a + \ldots + a^{k^{n+1}-1})w_0 \) where \( a \) is a generator of the group \( \pi_1(Y_{n+1}, x_{n+1}) \).

We will prove that \( (W_n, w_0) = \lim \{ (W_{n+1}, w_0) \} \) and \( (Y_x \times Y_l, x) \) have the same shape. Let \( i: (W_n, w_0) \rightarrow (X_n, x_n, x_{n+1}) \) be the inclusion map. One can easily see that \( i \) is \( \ast_{n+1}^{n+1} \) is homotopic to \( (\ast_{n+1}^{n+1}|_{X_{n+1}}) = \ast_{n+1}^{n+1} \). Thus \( i: (W_n, w_0) \rightarrow (X_n, x_n, x_{n+1}) \) is a shape morphism. It is easy to see that \( \text{pro-\pi}_{n}(i) \) is an isomorphism. Let \( \bar{Y}_{n+1}, \bar{X}_n, \bar{X}_{n+1} \) be the universal coverings of \( (Y_{n+1}, X_n, x_{n+1}) \) and \( (W_n, w_0) \) respectively and let \( \mathfrak{g}^{n+1} = (\bar{Y}_{n+1}, \bar{X}_n, \bar{X}_{n+1}) = (Y_{n+1}, Y_n) \). Let \( i: (W_n, w_0) \rightarrow (X_n, x_n, x_{n+1}) \) be the lifting of \( \mathfrak{g}^{n+1} \) and \( \bar{X}_{n+1} \) respectively (we can assume that \( i \) is an inclusion map).

Let \( g_{E}: (X_{n+1, x_{n+1}+1}) \rightarrow (W_n, w_0) \) be a map such that

\[
(\bar{g}_{E} = \ast_{n+1}^{n+1}) = (1 + a + \ldots + a^{k^{n+1}-1})\ast
\]

and let \( g_{f_{n+1}} \in g_{E} \times g_{n+1} \) where \( g_{f_{n+1}}: (X_{n+1, x_{n+1}+1}) \rightarrow (W_n, w_0) \) is lifting of \( g_{E} \) and \( x_{n+1} : (X_{n+1, x_{n+1}+1}, x_{n+1, x_{n+1}+1}) \rightarrow (X_{n+1, x_{n+1}+1}) \) is the natural projection. One
can check that

\[(7.7) \quad \pi_{g}^{k+2} \cong f_{g}^{\ast}(p_{n}^{k+1} \times g_{n}^{k+1}) \cdot \pi_{n+2}.\]

\[(7.8) \quad \pi_{g}^{k+1} \cdot f_{g}^{\ast} \cong f_{n}^{\ast}(p_{n}^{k+2} \times g_{n}^{k+2}).\]

Thus the map of sequences

\[I = \{I_{i} : \{\{W_{i}, \nu_{i}, \pi_{i}(\nu_{i})\}, \pi_{1}(\nu_{i})^{\ast}\} \to \{X \times Y, \nu_{i}, \pi_{i}(\nu_{i})\}, \pi_{1}(\nu_{i})^{\ast}\}\]

is a homotopy equivalence (in the sense of [6]). Since the covering maps induce isomorphisms of the homotopy groups for \(n \geq 2\), \(\text{pro}_{g}(\mathbb{I})\) is an isomorphism for \(n \geq 2\). By the Whitehead theorem in shape theory (see [7] or [1] and [2]) \(I\) is a shape equivalence. So \((\mathcal{W}, \nu)\) and \((X \times Y, \nu, \pi)\) have the same shape and thus

\[\text{Fd}(X \times Y, (x, y)) = \text{Fd}(W, \nu) = 2.\]

8. Appendix — the proof of Proposition (7.2). Let \(X\) and \(Y\) be continua of shape type \((\pi_{g}, k)\) with finite fundamental dimension. We will show (Lemma (8.3)) that there is a shape morphism \(f : X \to Y\) which induces isomorphisms of pro-homotopy groups in all dimensions. By the Whitehead theorem in shape theory (see [7] or [1] and [2]) it follows that \(\text{Sh}(X) = \text{Sh}(Y)\). We will first prove the following

\[\text{(8.1) Lemma. Let } g : X \to Y \text{ be a map of topological spaces such that for any map } f : S^{g} \to X \text{ the composition } g \circ f \text{ is homotopically trivial. Let } f_{0} \text{ and } f_{1} \text{ be maps of an } n \text{-dimensional CW-complex } K \text{ into } X. \text{ If } f_{0}(K^{n-1}) = f_{1}(K^{n-1}) \text{ then } g \circ f_{0} \text{ is homotopic to } g \circ f_{1} \text{ rel. } K^{n-1}.\]

Proof. Let \(z : (B^{n}, S^{n-1}) \to (K, K^{n-1})\) be a characteristic map of some \(n\)-cell in \(K\). We define the map

\[\varphi : (B^{n}, 0) \cup (S^{n-1} \times [0, 1]) \to X\]

by

\[\varphi(x, t) = f_{0}(z(x)) \text{ if } x \in B^{n}, t = 0 \text{ or } 1,\]

\[\varphi(x, t) = f_{1}(z(x)) \text{ if } x \in S^{n-1}, t \in [0, 1]\]

(of course \(f_{0}(z(x)) = f_{1}(z(x)) \text{ if } x \in S^{n-1}\)).

Since \((B^{n}, 0) \cup (S^{n-1} \times [0, 1])\) is an \(n\)-sphere, the map \(g \circ \varphi\) is homotopically trivial, and so there is an extension \(\varphi : B^{n} \times [0, 1] \to Y\) of the map \(g \circ \varphi\). So the maps \(g_{0} z_{0} \text{ and } g_{1} z_{1} \text{ are homotopic rel. } S^{n-1}\). It follows that the maps \(g_{0} z_{0} \text{ and } g_{1} z_{1} \text{ are homotopic rel. } K^{n-1}\).

If \(L\) is a subcomplex of \(CW\)-complex \(K\), then the pair \((K, L)\) has the homotopy extension property for any topological space. Thus we can obtain the following

\[\text{(8.2) Corollary. Let } g_{j} : X_{j} \to X_{j+1} \text{ be a map of topological spaces such that for any map } \varphi : S^{g_{j}} \to X_{j} \text{ the composition } g_{j} \circ \varphi \text{ is homotopically trivial } (j = 1, 2, ..., m). \text{ Let } f_{0} \text{ and } f_{1} \text{ be maps of an } (n + m) \text{-dimensional } CW\text{-complex } K\]

into \(X_{1}\). If \(f_{0}(K^{n}) = f_{1}(K^{n})\) then the maps \(g_{0} \circ \varphi_{n-1} \circ ... \circ g_{1} \circ f_{1}, \quad i = 0, 1, \text{ are homotopic rel. } K^{n}\).

Now we will prove the following

\[\text{(8.3) Lemma. Let } k \text{ and } l \text{ be positive integers, } l \geq k. \text{ Let } X = \{X_{n}, p_{n}^{k}\} \text{ be an inverse sequence of connected CW-complexes such that the pro-group } \pi_{1}(X) \text{ is isomorphic to the trivial group for } k = 1, 2, ..., k-1 \text{ and dim } X_{n} \leq 1 \text{ for every } n. \text{ Let } Y = \{Y_{n}, q_{n}^{k}\} \text{ be an inverse sequence of connected CW-complexes such that the pro-group } \pi_{1}(X) \text{ is isomorphic to the trivial group for } l \neq k \text{. Then for any morphism of pro-groups}\]

\[\varphi : \pi_{1}(X) \to \pi_{1}(Y)\]

there exists a morphism\]

\[f : X \to Y\]

such that \(\pi_{1}(f) = \varphi\).

Proof. By the trick of Mareš (see Lemma 8.1.1 and the proof of Theorem 8.3.2 in [2]) we may assume that \(\pi_{1}(X_{n})\) and \(\pi_{1}(X_{n})\) are the trivial groups for \(i = 1, 2, ..., k-1\) and every integer \(n\). Thus we can assume that \(X_{n}\) and \(Y_{n}\) has exactly one 0-cell and has no cells in dimensions 1, 2, ..., \(k-1\) for every \(n\). We may also assume that the map \(q_{n}^{k+1}\) induces the trivial homomorphism\]

\[\pi_{1}(q_{n}^{k+1}) : \pi_{1}(Y_{n+1}) \to \pi_{1}(Y_{n}),\]

for \(i = k+1, 2, ..., l-1\) and every integer \(n\).

The morphism \(\varphi\) is a pair \((\varphi_{n}, \varphi)\) which consists of an increasing map \(\varphi\) of the set of all positive integers \(J\) and of a sequence of homomorphisms\]

\[\varphi_{n} : \pi_{1}(X_{n}) \to \pi_{1}(Y_{n}), \quad n \in J\]

such that\]

\[\varphi_{n} \circ \varphi_{n+1} = \varphi_{n}(q_{n}^{k+1}) \circ \varphi_{n+1} \text{ for } n \in J.\]

By \(x\) we denote the integer \(2l-2k-1\) and by \(r\) the integer \(l-k-1\). Let \(\psi_{n}\) denote the isomorphism of \(k\)-th homotopy groups induced by the inclusion \(X_{(n)}^{(k)} \to X_{(n)}^{(k)}\). There is a map\]

\[g_{n} : X_{(n)}^{(k+1)} \to Y_{n}\]

such that\]

\[\pi_{1}(g_{n}) = \varphi_{n} \circ \psi_{n}.\]

Since the map \(q_{m}^{k+1}\) induces the trivial homomorphism of \((k+m)\)-th homotopy groups for \(m = 1, 2, ..., r\) and \(\text{dim } X_{n} \leq 1\), by induction we can define the map\]

\[g_{n} : X_{(n)}^{(k+1)} \to Y_{n} \circ \psi_{n},\]

which is an extension of the map \(q_{m}^{k+1} \circ q_{n}^{k+1}\). Of course \(\pi_{1}(g_{n}) = \pi_{1}(q_{n}^{k+1}) \circ \varphi_{n}\). By\]

\[\text{(8.4)}\]

we obtain\]

\[\pi_{1}(q_{n}^{k+1} \circ q_{n}^{k+1}) = \pi_{1}(g_{n} + p_{n}^{k+1} \circ q_{n}^{k+1}).\]
Thus the map $g_k = g_{k+1}^{X_k}$ is homotopic (relatively to the base point) to a map which coincides the map $g_k = g_{k+1}^{X_k}$ on the $k$-skeleton $X_k^{X_k}$ of $X_k$. By Corollary (8.2) the maps $g_k = g_{k+1}^{X_k}$ and $g_k = g_{k+1}^{X_k}$ are homotopic (relatively to the base point). Thus the maps $g_k = g_{k+1}^{X_k}$ and $g_k = g_{k+1}^{X_k}$ are homotopic (relatively to the base point), where $g_k = g_{k+1}^{X_k}$.

Thus the maps $g_k$, where $n = k + 1, k \in \mathcal{O}$, define a morphism (in procategory homotopy) of $X$ in a subsequence of $Y$. Since

$$\pi_n(g_k) = \pi_n(g_{k+1}) \circ \varphi_n,$$

we can define required $f$.

References


On a Problem of Silver

by

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Abstract. We show that it is consistent, relative to an $\omega$ sequence of measurable cardinals, for $\mathfrak{b}_\omega$ to be a Rowbottom cardinal and for $\text{DC}_{\omega_2}$ to hold, where $\omega$ is an arbitrary natural number.

Of all the large cardinal axioms which are currently known, the axioms which assert the existence of Rowbottom and Jonsson cardinals are amongst the most interesting hypotheses. Most large cardinal axioms assert, at least when the Axiom of Choice is true, that the cardinal in question is strongly inaccessible. This, however, is not true about Rowbottom and Jonsson cardinals. Indeed, Devlin has shown [5] that it is relatively consistent for $2^\omega$ be a Jonsson cardinal, and Prikry has shown [6] that, assuming the consistency of a measurable cardinal, it is consistent for a Rowbottom cardinal of cofinality $\omega$ to exist.

The above results inspire the following question: How large is the least Rowbottom cardinal? Silver in his thesis [7] hypothesizes that it is relatively consistent that the answer is $\kappa_\omega$, assuming the Axiom of Choice.

The answer to Silver’s question is still not known, and is the only remaining unsolved problem from Silver’s thesis. We have obtained a partial answer to Silver’s question by showing that it is consistent, relative to the existence of an $\omega$ sequence of measurable cardinals, for $\mathfrak{b}_\omega$ to be a Rowbottom cardinal and for a large portion, though not all, of the Axiom of Choice to be true. Specifically, we have proven the following:

**Theorem 1.** Assume that the theory \( \text{ZFC + There is an } \omega \text{ sequence of measurable cardinals} \) is consistent. Let $\mathfrak{n}_0 \in \omega$ be a fixed (though arbitrary) natural number. Then the theory \( \text{ZFC + there exists a Rowbottom filter} \) is consistent.

Note that some strong hypothesis is needed to obtain a model which witnesses Theorem 1 since an unpublished result of Silver shows that if $\mathfrak{n}_0$ is a Rowbottom cardinal, then it must be measurable in some inner model. Note also that other partial results on Silver’s problem have been obtained. In particular, Bull in his thesis [2] showed that, assuming the consistency of a measurable cardinal, the theory \( \text{ZF+V=H}(2^\omega = \kappa_{\omega+1}) \) is consistent.

Before beginning the proof of Theorem 1, we briefly mention some background information. Basically, our notation and terminology are fairly standard.

\[ \vdash \text{Fundamenta Mathematicae CXVII} \]