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Spaces defined by topological games, II *

by

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Abstract. The paper reports some results on the game $G(K, X)$ introduced in [7]. The main results: 1. The space favorable for Player I is the union of countably many K-scattered subsets. 2. Reduction theorems for actions of Player I. 3. Covering characterization of the spaces favorable for Player II. 4. Indeterminacy of the game in ZFC.

The main object of this work is the topological game $G(K, X)$, so the present paper is a continuation of [7]. Some of the results included here were announced earlier in [8] and [9]. The game $G(K, X)$ was used recently for proving general sum theorems for the dimension dim by the author and Y. Yajima [10] and for the dimension Ind by Y. Yajima [12]. Furthermore, a general product theorem for paracompact spaces involving that game was established by Y. Yajima in [13].

Section 1 contains the following: if Player I has a winning strategy in $G(K, X)$, then X is the union of countably many K-scattered subsets. In sections 2 and 3 there are introduced auxiliary games $G^{\#}(K, X)$ and $G^{+}(K, X)$ in order to prove reduction theorems concerning the actions of Player I. Section 4 introduces a convenient equivalent form of the game $G(K, X)$, denoted by $G'(K, X)$. A modification of that game involving G_δ sets and thus denoted by $G^{\delta}(K, X)$ is studied in section 5. The dual game $G^*(K, X)$ to the game $G'(K, X)$ is introduced in section 6; it provides, as a by-product, a covering characterization of spaces favorable for Player II. Finally, in section 7, the indeterminacy of $G(K, X)$ in ZFC is established.

For the topological background and undefined notions we refer to R. Engelking's monograph [1]. Each space considered here is assumed to be completely regular. N denotes the set of positive integers. 2^X denotes the family of closed subsets of the space X. K denotes a class of spaces such that (i) K contains all singletons, and (ii) K is invariant with respect to closed subspaces, i.e., $X \in K$ implies $2^X \subset K$. $\mathbf{1}$, $\mathbf{F}$, $\mathbf{C}$ and $\mathbf{D}$ denote the classes of all singletons, finite spaces, compact spaces, and discrete spaces respectively. $\mathbf{D}K$, $\mathbf{L}K$ and $\mathbf{S}K$ denote the classes of spaces being free unions of spaces from K, locally K, and K-scattered, respectively. Inspite of the notation used in [7], $I(K, X)$ ($II(K, X)$) denotes the following statement: Player I (Player II, resp.) has a winning strategy in $G(K, X)$. For the modifications

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of the game $G(K, X)$, the statements on winning strategies are defined similarly, e.g., $II^*(K, X)$ means that Player II has a winning strategy in $G^*(K, X)$.

1. K-scattered subsets.

1.1. DEFINITION. A family $\mathscr{S}$ of pairwise disjoint closed subsets of a space X is said to be *scattered* if for each nonvoid subset H of $\bigcup \mathscr{S}$ there is $S \in \mathscr{S}$ such that $S \cap H$ is a nonvoid H-open subset of H.

1.2. LEMMA. *Let Y and Z be closed subsets of a space X, where $Z \subset Y$. Then there is a sequence*

$$\langle \mathscr{S}_1(Y, Z), \mathscr{S}_2(Y, Z), \ldots \rangle$$

of scattered families of sets so that

$$Y - Z = \bigcup \{\bigcup \mathscr{S}_n(Y, Z) \colon n \in N\}.$$

Proof. We define a transfinite sequence

$$\langle \mathscr{G}_0, \mathscr{G}_1, \ldots, \mathscr{G}_\xi, \ldots \colon \xi < \gamma \rangle$$

of families of subsets of $Y - Z$ and a transfinite sequence

$$\langle E_0, E_1, \ldots, E_\xi, \ldots \colon \xi < \gamma \rangle$$

of closed subsets of Y as follows. We set $\mathscr{G}_0 = 0$ and $E_0 = Y$. Let α be an ordinal such that $\mathscr{G}_\xi$ and E_ξ are already defined for each $\xi \leqslant \alpha$. We choose $\mathscr{G}_{\alpha+1}$ to be a family of pairwise disjoint subsets of $E_\alpha - Z$ so that each $G \in \mathscr{G}_{\alpha+1}$ is an E_α-open F_σ-set in E_α and $\bigcup \mathscr{G}_{\alpha+1}$ is dense in $E_\alpha - Z$. Furthermore, we set $E_{\alpha+1} = E_\alpha - \bigcup \mathscr{G}_{\alpha+1}$. Let λ be a limit ordinal such that $\mathscr{G}_\xi$ and E_ξ are already defined for each $\xi < \lambda$. Then we set $\mathscr{G}_\lambda = 0$ and $E_\lambda = \bigcap \{E_\xi \colon \xi < \lambda\}$. It is easy to show that each E_ξ is closed in X and each $G \in \mathscr{G}_\xi$ is a F_σ-set in X. Clearly, there is an ordinal, say β, such that $E_\beta = Z$; let γ be the least ordinal with that property. Let us put $\mathscr{G} = \bigcup \{\mathscr{G}_\xi \colon \xi < \gamma\}$. For each $G \in \mathscr{G}$ we choose a sequence $\langle F_1(G), F_2(G), \ldots \rangle$ of closed subsets of X so that $G = \bigcup \{F_n(G) \colon n \in N\}$. Finally, we set $\mathscr{S}_n(Y, Z) = \{F_n(G) \colon G \in \mathscr{G}\}$ for each $n \in N$. Now, we have

$$\bigcup \{\bigcup \mathscr{S}_n(Y, Z) \colon n \in N\} = Y - Z.$$

Let $n \in N$ and let H be a nonvoid subset of $\bigcup \mathscr{S}_n(Y, Z)$. Then there is the least ordinal $\xi < \gamma$ such that $H \cap F_n(G) \neq 0$ for some $G \in \mathscr{G}_\xi$. Since $G \cap E_{\xi+1} = 0$ and $F_n(G) \subset G$, the set $F_n(G)$ is relatively open in $E_\xi \cap \bigcup \mathscr{S}_n(Y, Z)$. Hence $F_n(G) \cap H$ is H-open nonvoid subset of H. The proof is complete.

1.3. THEOREM. *If Player I has a winning strategy in $G(K, X)$, then X is the union of a countable family of its K-scattered subsets.*

Proof. Let s be a winning strategy of Player I in $G(K, X)$. Without loss of generality we may assume that $s(E_0, E_1, \ldots, E_{2k}) \neq 0$ whenever $\langle E_0, E_1, \ldots, E_{2k} \rangle$ is a partial play of $G(K, X)$ with $E_{2k} \neq 0$. For each finite sequence φ of natural numbers we define a family $\mathscr{T}(\varphi)$ of subsets of X, and subsets $X(\varphi)$ and $Y(\varphi)$ of X as follows. We set $\mathscr{T}(\emptyset) = \{\langle X \rangle\}$, $X(\emptyset) = X$ and $Y(\emptyset) = s(X)$. For $k_1 \in N$ we set

$$\mathscr{T}(k_1) = \{\langle E_0, E_1, E_2 \rangle \colon E_0 = X,\ E_1 = s(E_0) \text{ and } E_2 \in \mathscr{S}_{k_1}(E_0, E_1)\},$$

$$X(k_1) = \bigcup \{E_2 \colon \langle E_0, E_1, E_2 \rangle \in \mathscr{T}(k_1)\}, \quad \text{and}$$

$$Y(k_1) = \bigcup \{s(E_0, E_1, E_2) \colon \langle E_0, E_1, E_2 \rangle \in \mathscr{T}(k_1)\},$$

where $\mathscr{S}_{k_1}$ is defined in Lemma 1.1. Proceeding by induction, we set

$$\mathscr{T}(k_1, \ldots, k_n, k_{n+1}) = \{\langle E_0, E_1, \ldots, E_{2n}, E_{2n+1}, E_{2n+2} \rangle \colon \langle E_0, E_1, \ldots, E_{2n} \rangle \in \mathscr{T}(k_1, \ldots, k_n),$$
$$E_{2n+1} = s(E_0, E_1, \ldots, E_{2n}) \text{ and } E_{2n+2} \in \mathscr{S}_{k_{n+1}}(E_{2n}, E_{2n+1})\},$$

$$X(k_1, \ldots, k_n, k_{n+1}) = \bigcup \{E_{2n+2} \colon \langle E_0, E_1, \ldots, E_{2n}, E_{2n+1}, E_{2n+2} \rangle \in \mathscr{T}(k_1, \ldots, k_n, k_{n+1})\}$$

and

$$Y(k_1, \ldots, k_n, k_{n+1}) = \bigcup \{s(E_0, E_1, \ldots, E_{2n}, E_{2n+1}, E_{2n+2}) \colon \langle E_0, E_1, \ldots, E_{2n}, E_{2n+1}, E_{2n+2} \rangle \in \mathscr{T}(k_1, \ldots, k_n, k_{n+1})\}.$$

Let us notice that

$$X(k_1, \ldots, k_n) = Y(k_1, \ldots, k_n) \cup \bigcup \{X(k_1, \ldots, k_n, k) \colon k \in N\}.$$

Now, we claim that the sets $Y(\varphi)$ constitute a cover of X. For, suppose there is a point x_0 in X which is not covered. Then $x_0 \notin Y(\emptyset) = s(X)$, and therefore there is a $k_1 \in N$ and $\langle E_0, E_1, E_2 \rangle \in \mathscr{T}(k_1)$ so that $x_0 \in E_2$. Since $E_2 \subset X(k_1)$ and $x_0 \notin Y(k_1)$, there is a $k_2 \in N$ and sets E_3 and E_4 with $\langle E_0, E_1, E_2, E_3, E_4 \rangle \in \mathscr{T}(k_1, k_2)$, so that $x_0 \in E_4$. Since $E_4 \subset X(k_1, k_2)$ and $x_0 \notin Y(k_1, k_2)$, we can find $k_3 \in N$ and $\langle E_0, E_1, \ldots, E_5, E_6 \rangle \in \mathscr{T}(k_1, k_2, k_3)$ with $x_0 \in E_6$, and so on. Continuing in that manner we get a play $\langle E_0, E_1, \ldots \rangle$ of $G(K, X)$ so that $E_1 = s(E_0)$ and $E_{2n+1} = s(E_0, E_1, \ldots, E_{2n})$ for each $n \in N$. Since s is a winning strategy, we have $\bigcap \{E_{2n} \colon n \in N\} = 0$. On the other hand, we have $x_0 \in \bigcap \{E_{2n} \colon n \in N\}$, that yields a contradiction. Thus our claim is true. Finally, we shall show that $Y(\varphi)$ is K-scattered for each φ. Clearly, $Y(\emptyset) \in K$. Let $k_1 \in N$ and let H be a nonvoid relatively closed subset of $Y(k_1)$. Since $Y(k_1) \subset X(k_1)$ and $X(k_1)$ is the union of the scattered family $\mathscr{S}_{k_1}(E_0, E_1)$, where $E_0 = X$ and $E_1 = s(E_0)$, there is $E_2 \in \mathscr{S}_{k_1}(E_0, E_1)$ so that $E_2 \cap H$ is nonvoid and relatively closed and open in H. Since $Y(k_1) \cap E_2 = s(E_0, E_1, E_2) \neq 0$, it follows that

$$H \cap s(E_0, E_1, E_2) \neq 0, \qquad H \cap s(E_0, E_1, E_2) \in K,$$

and $H \cap s(E_0, E_1, E_2)$ is relatively closed and open in H. Thus $Y(k_1)$ is K-scattered.

Proceeding by induction, assume that for some $n \in N$, the sets $Y(k_1, \ldots, k_m)$ are $\boldsymbol{K}$-scattered whenever $\langle k_1, \ldots, k_m \rangle \in N^m$ and $m \leqslant n$. Let $\langle k_1, \ldots, k_n, k_{n+1} \rangle \in N^{n+1}$ and let H be a nonvoid subset of $Y(k_1, \ldots, k_{n+1})$ so that H is relatively closed in $Y(k_1, \ldots, k_{n+1})$. Since

$$Y(k_1, \ldots, k_{n+1}) \subset X(k_1, \ldots, k_{n+1}) \subset X(k_1, \ldots, k_n) \subset \ldots \subset X(k_1, k_2) \subset X(k_1),$$

there is $E_2 \in \mathscr{S}_{k_1}(E_0, E_1)$ so that $E_2 \cap H \neq 0$ and $E_2 \cap H$ is relatively closed and open in $X(k_1)$; there is $E_4 \in \mathscr{S}_{k_2}(E_2, E_3)$, where $E_3 = s(E_0, E_1, E_2)$, so that $E_4 \cap H \neq 0$, $E_4 \subset E_2$, and E_4 is relatively closed and open in $X(k_1, k_2)$; ...; there is $E_{2n} \in \mathscr{S}_{k_n}(E_{2n-2}, E_{2n-1})$, where $E_{2n-1} = s(E_0, E_1, \ldots, E_{2n-2})$, so that $E_{2n} \cap H \neq 0$, $E_{2n} \subset E_{2n-2}$, and E_{2n} is relatively closed and open in $X(k_1, \ldots, k_n)$; and, finally, there is $E_{2n+2} \in \mathscr{S}_{k_{n+1}}(E_{2n}, E_{2n+1})$, where $E_{2n+1} = s(E_0, E_1, \ldots, E_{2n})$, so that $E_{2n+2} \cap H \neq 0$, $E_{2n+2} \subset E_{2n}$, and E_{2n+2} is relatively closed and open in $X(k_1, \ldots, k_{n+1})$. Since $E_{2n+2} \cap Y(k_1, \ldots, k_{n+1}) = E_{2n+3}$, where $E_{2n+3} = s(E_0, \ldots, \ldots, E_{2n+2})$, and $H \cap E_{2n+2} \neq 0$, it follows that $H \cap E_{2n+3} \neq 0$, $H \cap E_{2n+3} \in \boldsymbol{K}$, and $H \cap E_{2n+3}$ is relatively closed and open in $Y(k_1, \ldots, k_{n+1})$. Therefore all $Y(\varphi)$ are $\boldsymbol{K}$-scattered, and so the proof is complete.

1.4. Remark. In terms introduced by H. H. Wicke and J. M. Worrell, Jr in [11], the above theorem reads: If Player I has a winning strategy in $G(\boldsymbol{K}, X)$, then X is σ-$\boldsymbol{K}$-collectionwise scattered.

1.5. Remark. Assuming that each open subset of X is the union of a σ-locally finite family of closed sets (in particular, that X is totally normal or hereditarily paracompact), the proof of Theorem 1.3 can be modified to get the following conclusion: X is the union of a countable family of its K-scattered closed subsets (cf. [7], Theorem 11.1). That would be the desired result, however, it is not clear how to release the additional assumption.

From Theorem 1.3 above and Corollary 10.2 of [7] we get a partial solution to Problem 1 in [6].

1.6. Corollary. *If X has a closure-preserving cover consisting of compact sets, then X is the union of a countable family of its $\boldsymbol{C}$-scattered subsets.*

Similarly, by 1.3 above and 10.5 of [7] we get

1.7. Corollary. *If X has a closure-preserving cover by finite sets, then it is the union of countably many scattered subsets.*

1.8. Remark. Theorem 6 of [6] provides a stronger conclusion than Corollary 1.7: the scattered subsets are moreover closed in X. Hence it follows that Theorem 1.3 needs to be essentially improved to get a characterization of spaces favorable for Player I (the problem in [7], p. 222).

2. The game $G^{\#}(K, X)$ and paracompactness. Let $G^{\#}(K, X)$ denote the following modification of the game $G(K, X)$: Player I chooses a locally finite closed cover $\{X(t): t \in T\}$ of X and then Player II chooses a t_1 in T. After that Player I chooses a closed subset $Y(t_1)$ of $X(t_1)$ with $Y(t_1) \in \boldsymbol{K}$, and Player II chooses a closed subset $Z(t_1)$ of $X(t_1)$ with $Z(t_1) \cap Y(t_1) = 0$. Again Player I chooses a locally finite closed cover $\{X(t_1, t): t \in T\}$ of $Z(t_1)$ and Player II chooses a t_2 in T. After that Player I chooses a closed subset $Y(t_1, t_2)$ of $X(t_1, t_2)$ with $Y(t_1, t_2) \in \boldsymbol{K}$, and Player II chooses a closed subset $Z(t_1, t_2)$ of $X(t_1, t_2)$ with $Z(t_1, t_2) \cap Y(t_1, t_2) = 0$, and so on. Player I wins the play if

$$\bigcap \{X(t_1, \ldots, t_n): n \in N\} = 0;$$

otherwise Player II wins.

2.1. Lemma. *Let X be a paracompact space and let Y and Z be closed subsets of X, where $0 \neq Y \subset Z$ and $Y \in \boldsymbol{SK}$ (i.e., Y is $\boldsymbol{K}$-scattered). Then there is a locally finite closed cover $\{X(t): t \in T\}$ of Z so that for each $t \in T$ there is an ordinal $\alpha(t)$ for which $0 \neq (Y \cap X(t))^{(\alpha(t))} \in \boldsymbol{K}$ (recall that $A^{(\alpha)}$ in that context denotes the $\boldsymbol{K}$-derivative of A of order α* (cf. [7], p. 205)).

If X, Y, Z and $\{X(t): t \in T\}$ satisfy the requirements of the lemma, we shall say that $\{X(t): t \in T\}$ is a *good cover of* $\langle Z, Y \rangle$.

Proof. Let α be the least ordinal for which $Y^{(\alpha)} = 0$. Proceeding by transfinite induction we may assume that the lemma is true for any $\boldsymbol{K}$-scattered closed set Y_1 with $Y_1^{(\alpha_1)} = 0$ for some $\alpha_1 < \alpha$.

Case 1: $\alpha = \beta + 1$. Then $0 \neq Y^{(\beta)} \in \boldsymbol{LK}$ and thus there is a locally finite family $\{U(t): t \in T\}$ of Z-open sets in Z so that $Y^{(\beta)} \subset \bigcup \{U(t): t \in T\}$ and $0 \neq Y^{(\beta)} \cap \overline{U(t)} \subset Y^{(\beta)} \cap \overline{U(t)} \in \boldsymbol{K}$ for each $t \in T$. Hence we have $0 \neq (Y \cap \overline{U(t)})^{(\beta)} \in \boldsymbol{K}$, because $Y^{(\beta)} \cap U(t) \subset (Y \cap \overline{U(t)})^{(\beta)}$ and $(Y \cap \overline{U(t)})^{(\beta)} \subset Y^{(\beta)} \cap \overline{U(t)}$. Let $Y_1 = Y - \bigcup \{U(t): t \in T\}$ and $Y_2 = \bigcup \{\overline{U(t)}: t \in T\}$. Then $Y = Y_1 \cup Y_2$ and $Y_1^{(\beta)} = 0$. Now, $\langle Z, Y_1 \rangle$ has a good cover $\{X(t): t \in T_1\}$ by the inductive assumption, while $\{\overline{U(t)}: t \in T\}$ is a good cover of $\langle Z, Y_2 \rangle$, where $T_1 \cap T = 0$. Clearly,

$$\{X(t): t \in T_1\} \cup \{\overline{U(t)}: t \in T\}$$

is a good cover of $\langle Z, Y \rangle$.

Case 2: α is a limit ordinal. Then $\{Z - Y^{(\beta)}: \beta < \alpha\}$ is a Z-open cover of Z. Let $\{Y(t): t \in T\}$ be a locally finite closed refinement of $\{Z - Y^{(\beta)}: \beta < \alpha\}$. Then $(Y \cap Y(t))^{(\beta)} = 0$ for some $\beta < \alpha$, whenever $t \in T$. By the inductive assumption, $\langle Z, Y \cap Y(t) \rangle$ has a good cover, say $\{X(t, t_1): t_1 \in T_1\}$ (without loss of generality we may assume that $Y \cap Y(t) \neq 0$ for each $t \in T$). Finally, $\{X(t, t_1) \cap Y(t): t \in T$ and $t_1 \in T_1\}$ is a good cover of $\langle Z, Y \rangle$. The proof is complete.

2.2. Proposition. *If X is paracompact and Player I has a winning strategy in $G(\boldsymbol{SK}, X)$, then Player I has a winning strategy in $G^{\#}(\boldsymbol{K}, X)$.*

Proof. Let s be a winning strategy of Player I in $G(\boldsymbol{SK}, X)$. We describe a winning strategy of Player I in $G^{\#}$ (K, X) as follows. Let, as usual, $E_0 = X$ and $E_1 = s(E_0)$. Let Player I choose a good cover $\{X(t): t \in T\}$ of $\langle E_0, E_1 \rangle$ and let Player II chooses a t_1 in T. After that Player I chooses $Y(t_1) = (E_1 \cap X(t_1))^{(\alpha(t_1))}$, where $\alpha(t_1)$ is choosen for the same purpose as in the lemma, and Player II chooses a closed subset $Z(t_1)$ of $X(t_1)$ with $Z(t_1) \cap Y(t_1) = 0$. Proceeding by induction, assume that for some $n \geqslant 1$ the sets $X(t_1), Y(t_1), Z(t_1), \ldots, X(t_1, \ldots, t_n), Y(t_1, \ldots, t_n), Z(t_1, \ldots, t_n), E_0, E_1, \ldots, E_{2k-1}$ have been chosen, where $E_0 = X$, $E_1 = s(E_0)$, $E_2 = Z(t_1, \ldots, t_{i_1})$, $E_3 = s(E_0, E_1, E_2)$, $E_4 = Z(t_1, \ldots, t_{i_1}, \ldots, t_{i_2}), \ldots, E_{2k-2} = Z(t_1, \ldots, t_{i_k})$, $E_{2k-1} = s(E_0, E_1, \ldots, E_{2k-2})$, and $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. There are two cases to be considered.

Case 1: $Z(t_1, \ldots, t_n) \cap E_{2k-1} \neq 0$. Then Player I chooses a good cover $\{X(t_1, \ldots, t_n, t)\colon\ t \in T\}$ of $\langle Z(t_1, \ldots, t_n), Z(t_1, \ldots, t_n) \cap E_{2k-1}\rangle$, and Player II chooses a t_{n+1} in T. After that Player I chooses

$$Y(t_1, \ldots, t_n, t_{n+1}) = \big(X(t_1, \ldots, t_n, t_{n+1}) \cap E_{2k-1}\big)^{(\alpha(t_1, \ldots, t_n, t_{n+1}))}$$

in agreement with the lemma, and Player II chooses a closed subset

$$Z(t_1, \ldots, t_n, t_{n+1}) \quad \text{of} \quad X(t_1, \ldots, t_n, t_{n+1})$$

with

$$Z(t_1, \ldots, t_n, t_{n+1}) \cap Y(t_1, \ldots, t_n, t_{n+1}) = 0 .$$

Case 2: $Z(t_1, \ldots, t_n) \cap E_{2k-1} = 0$. Then Player I chooses a good cover $\{X(t_1, \ldots, t_n, t)\colon\ t \in T)$ of $\langle E_{2k}, E_{2k+1}\rangle$, where $E_{2k} = Z(t_1, \ldots, t_n)$ and $E_{2k+1} = s(E_0, E_1, \ldots, E_{2k})$, and Player II chooses a t_{n+1} in T. After that Player I chooses

$$Y(t_1, \ldots, t_{n+1}) = \big(X(t_1, \ldots, t_{n+1}) \cap E_{2k+1}\big)^{(\alpha(t_1, \ldots, t_{n+1}))}$$

in agreement with the lemma, and Player II chooses a closed subset $Z(t_1, \ldots, t_{n+1})$ of $X(t_1, \ldots, t_{n+1})$ with $Z(t_1, \ldots, t_{n+1}) \cap Y(t_1, \ldots, t_{n+1}) = 0$. Finally, it remains to show that the above procedure yields a winning strategy, i.e., $\bigcap \{X(t_1, \ldots, t_n)\colon n \in N\} = 0$. Since

$$\bigcap \{X(t_1, \ldots, t_n)\colon\ n \in N\} = \bigcap \{Z(t_1, \ldots, t_n)\colon\ n \in N\},$$

$$\{E_{2k}\colon k \in N\} \subset \{Z(t_1, \ldots, t_n)\colon\ n \in N\} \quad \text{and} \quad \bigcap \{E_{2k}\colon\ k \in N\} = 0 ,$$

it suffices to point out that Case 2 occurs for infinitely many $n \in N$. For, suppose n is the last number for which Case 2 had the place. Then $Z(t_1, \ldots, t_n) \cap E_{2k-1} = 0$, $E_{2k} = Z(t_1, \ldots, t_n)$, $E_{2k+1} = s(E_0, E_1, \ldots, E_{2k})$, $\{X(t_1, \ldots, t_n, t)\colon\ t \in T\}$ is good for $\langle E_{2k}, E_{2k+1}\rangle$, $Y(t_1, \ldots, t_{n+1}) = (X(t_1, \ldots, t_{n+1}) \cap E_{2k+1})^{(\alpha_{n+1})}$ in agreement with the lemma, where we write α_{n+1} for short, and

$$Z(t_1, \ldots, t_{n+1}) \subset X(t_1, \ldots, t_{n+1}) - Y(t_1, \ldots, t_{n+1}) .$$

However, for each $m > n$ we have $Z(t_1, \ldots, t_m) \cap E_{2k+1} \neq 0$, $\{X(t_1, \ldots, t_m, t)\colon t \in T\}$ is a good cover of $\langle Z(t_1, \ldots, t_m), Z(t_1, \ldots, t_m) \cap E_{2k+1}\rangle$,

$$Y(t_1, \ldots, t_m) = \big(X(t_1, \ldots, t_m) \cap E_{2k+1}\big)^{(\alpha_m)},$$

where $\alpha_m = \alpha(t_1, \ldots, t_m)$, and $Z(t_1, \ldots, t_{m+1}) \subset X(t_1, \ldots, t_{m+1}) - Y(t_1, \ldots, t_{m+1})$. Hence $\alpha_{n+1} > \alpha_{n+2} > \alpha_{n+3} > \ldots$ The contradiction indicates that Case 2 must occur after finitely many moves again. The proof is complete.

2.3. Lemma. *Let* $\{X(t_1, \ldots, t_n)\colon \langle t_1, \ldots, t_n\rangle \in T^n$ *and* $n \in N\}$ *be a family of subsets of a set* X, *where* $\{X(t_1, \ldots, t_n)\colon \langle t_1, \ldots, t_n\rangle \in T^n\}$ *is point-finite for each* $n \in N$ *and* $X(t_1, \ldots, t_n, t_{n+1}) \subset X(t_1, \ldots, t_n)$ *for each* $\langle t_1, \ldots, t_{n+1}\rangle \in T^{n+1}$ *and* $n \in N$. *Then*

$$\bigcap \{\bigcup \{X(t_1, \ldots, t_n)\colon\ \langle t_1, \ldots, t_n\rangle \in T^n\}\colon\ n \in N\} = \bigcup \{\bigcap \{X(t_1, \ldots, t_n)\colon\ n \in N\}\colon\ \langle t_1, t_2, \ldots\rangle \in T^N\} .$$

The proof requires just a standard application of Tihonov product theorem (the product of finite spaces is compact) and thus it is omitted.

2.4. Proposition. *If Player I has a winning strategy in* $G^{\#}(K, X)$, *then he has a winning strategy also in* $G(K', X)$, *where* K' *is the class of all spaces having locally finite closed covers by sets of* K.

Proof. We shall describe a winning strategy of Player I in $G(K', X)$ as follows. Let $E_0 = X$. Having $\{X(t_1)\colon t_1 \in T\}$ and $\{Y(t_1)\colon t_1 \in T\}$, we set

$$E_1 = \bigcup \{Y(t_1)\colon\ t_1 \in T\} .$$

Then E_1 is closed in X and $E_1 \in K'$. Let E_2 be a closed subset of X with $E_2 \cap E_1 = 0$. For each t_1 in T we set $Z(t_1) = X(t_1) \cap E_2$. Since $Z(t_1)$ is a closed subset of $X(t_1)$ and $Z(t_1) \cap Y(t_1) \subset E_2 \cap E_1 = 0$, there are $\{X(t_1, t_2)\colon t_2 \in T\}$ and $\{Y(t_1, t_2)\colon t_2 \in T\}$ such that $\bigcup \{X(t_1, t_2)\colon t_2 \in T\} = Z(t_1)$, $Y(t_1, t_2) \subset X(t_1, t_2)$ and $Y(t_1, t_2) \in K$. We set $E_3 = \bigcup \{Y(t_1, t_2)\colon\ \langle t_1, t_2\rangle \in T^2\}$, and so on. Suppose to contrary that $\bigcap \{E_{2n}\colon\ n \in N\} \neq 0$ and pick $x_\infty \in \bigcap \{E_{2n}\colon\ n \in N\}$. Since

$$E_2 = X \cap E_2 = \bigcup \{X(t_1) \cap E_2\colon\ t_1 \in T\} = \bigcup \{Z(t_1)\colon\ t_1 \in T\},$$

$$E_4 = E_2 \cap E_4 = \bigcup \{Z(t_1) \cap E_4\colon\ t_1 \in T\} = \bigcup \{X(t_1, t_2) \cap E_4\colon\ \langle t_1, t_2\rangle \in T^2\}$$

$$= \bigcup \{Z(t_1, t_2)\colon\ \langle t_1, t_2\rangle \in T^2\},$$

and so on, we have

$$x_\infty \in \bigcap \{\bigcup \{Z(t_1, \ldots, t_n)\colon\ \langle t_1, \ldots, t_n\rangle \in T^n\}\colon\ n \in N\} .$$

Hence, by Lemma 2.3, there is $\langle t_1, t_2, \ldots\rangle \in T^N$ with $x_\infty \in \bigcap \{Z(t_1, \ldots, t_n)\colon n \in N\}$. On the other hand,

$$\bigcap \{Z(t_1, \ldots, t_n)\colon\ n \in N\} = \bigcap \{X(t_1, \ldots, t_n)\colon\ n \in N\} = 0 ,$$

because Player I has used the winning strategy in $G^{\#}(K, X)$. The contradiction indicates that $\bigcap \{E_{2n}\colon\ n \in N\} = 0$. The proof is complete.

2.5. Theorem. *Let* X *be a paracompact space and let* K *be a class of spaces invariant with respect to finite closed unions (i.e., if* $Y = \bigcup \{Y_i\colon 1 \leqslant i \leqslant n\}$, *where* Y_i *is closed in* Y *and* $Y_i \in K$ *for each* $i \leqslant n$, *then* $Y \in K$). *Then the following conditions are equivalent:*

2.5.1. $I^{\#}(K, X)$.

2.5.2. $I(LK, X)$.

2.5.3. $I(SK, X)$.

Proof. $I^{\#}(\boldsymbol{K}, X)$ implies $I(\boldsymbol{LK}, X)$ by Proposition 2.4. In order, $I(\boldsymbol{LK}, X)$ implies $I(\boldsymbol{SK}, X)$ because $\boldsymbol{LK} \subset \boldsymbol{SK}$. Finally, $I(\boldsymbol{SK}, X)$ implies $I^{\#}(\boldsymbol{K}, X)$ by Proposition 2.2. The proof is complete.

Since $\boldsymbol{L1} = \boldsymbol{LF} = \boldsymbol{D}$, we get from Theorem 2.5 the following.

2.6. THEOREM. *For a paracompact space X the following conditions are equivalent.*

2.6.1. $I^{\#}(\boldsymbol{1}, X)$.

2.6.2. $I^{\#}(\boldsymbol{F}, X)$.

2.6.3. $I(\boldsymbol{D}, X)$.

2.6.4. $I(\boldsymbol{SF}, X)$.

2.7. LEMMA. *If X is paracompact and locally compact, then $X = X' \cup X''$, where X' and X'' are unions of discrete families of compact sets.*

Proof. Under the assumptions, X has a discrete cover $\{X(t)\colon t \in T\}$ consisting of σ-compact, locally compact sets. Furthermore, for each $t \in T$, there is an open cover $\{U(t, n)\colon n \in N\}$ of $X(t)$ so that $\overline{U(t, n)}$ is compact and $\overline{U(t, n)} \subset U(t, n+1)$ whenever $n \in N$. Finally, we set

$$X'(t) = \overline{U(t, 1)} \cup \bigcup \overline{U(t, 2n+1)} - U(t, 2n)\colon n \in N\},$$

$$X''(t) = \bigcup \{\overline{U(t, 2n)} - U(t, 2n-1)\colon n \in N\},$$

$$X' = \bigcup \{X'(t)\colon t \in T\} \quad \text{and} \quad X'' = \bigcup \{X''(t)\colon t \in T\}.$$

Now it is easy to check that X' and X'' satisfy the requirements of the lemma. The proof is complete.

2.8. PROPOSITION. *If X is paracompact and Player I has a winning strategy in $G(\boldsymbol{LC}, X)$, then he has a winning strategy in $G(\boldsymbol{DC}, X)$. (The converse implication is immediate, because $\boldsymbol{DC} \subset \boldsymbol{LC}$.)*

Proof. If X is paracompact and s is a winning strategy for Player I in $G(\boldsymbol{LC}, X)$, then a winning strategy for that player in $G(\boldsymbol{DC}, X)$ can be defined using Lemma 2.7. Indeed, if $E_0 = X$, we set $E_1 = s(E_0)'$ (by the lemma we have $s(E_0) = s(E_0)' \cup s(E_0)''$). If E_2 is closed in X and $E_2 \cap E_1 = \emptyset$, we set $E_3 = E_2 \cap s(E_0)''$, and so on. However, the proposition follows also from 4.1 of [7], because $\boldsymbol{LC} \cap 2^X \subset \boldsymbol{FDC}$ by Lemma 2.7 (the games $G(\boldsymbol{K}, X)$ and $G(\boldsymbol{K} \cap 2^X, X)$ are equivalent). The proof is complete.

From Theorem 2.5 and Proposition 2.8 we get

2.9. THEOREM. *For a paracompact space X the following conditions are equivalent.*

2.9.1. $I^{\#}(\boldsymbol{C}, X)$.

2.9.2. $I(\boldsymbol{DC}, X)$.

2.9.3. $I(\boldsymbol{LC}, X)$.

2.9.4 $I(\boldsymbol{SC}, X)$.

2.10. Remark. In terms of [7] the last theorem can be restated as follows: Let X be a paracompact space. Then X is $\boldsymbol{DC}$-like iff X is $\boldsymbol{LC}$-like iff X is $\boldsymbol{SC}$-like.

Furthermore, Theorem 2.9 constitutes an improvement of Theorem 11.4 in [7]. Finally, according to Theorem 2.9, the product theorem of Y. Yajima ([13], Thm. 2.1) is in fact a refinement of Theorem 14.6 in [7] (cf. also § 4 of [13] for strongly rectangular product spaces).

3. The game $G^+(\boldsymbol{K}, X)$ and Lindelöf property. This game is a modification of the game $G(\boldsymbol{K}, X)$: Player I chooses a countable closed cover $\{X(k)\colon k \in N\}$ of X and Player II chooses a k_1 in N. After that Player I chooses a closed subset $Y(k_1)$ of $X(k_1)$ with $Y(k_1) \in \boldsymbol{K}$, and Player II chooses a closed subset $Z(k_1)$ of $X(k_1)$ with $Z(k_1) \cap Y(k_1) = 0$. Then Player I chooses a countable closed cover $\{X(k_1, k)\colon k \in N\}$ of $Z(k_1)$ and Player II chooses a k_2 in N. After that Player I chooses a closed subset $Y(k_1, k_2)$ of $X(k_1, k_2)$ with $Y(k_1, k_2) \in \boldsymbol{K}$, and Player II chooses a closed subset $Z(k_1, k_2)$ of $X(k_1, k_2)$ with $Z(k_1, k_2) \cap Y(k_1, k_2) = 0$, and so on. Player I wins the play if $\bigcap \{X(k_1, \ldots, k_n)\colon n \in N\} = 0$; otherwise Player II wins.

3.1. LEMMA. *Let $A = \bigcup \{N^i\colon i \in N\}$, $A_{n,i} = \{\langle k_1, \ldots, k_i\rangle \in N^i\colon k_1 + \ldots + k_i = n\}$, and $A_n = \bigcup \{A_{n,i}\colon i \leqslant n\}$. Then there is the unique one-to-one function f from A onto N so that*

3.1.1. *$f|A_{n,i}$ preserves the lexicographic order of $A_{n,i}$,*

3.1.2. *$\max f(A_{n,i}) < \min f(A_{n,j})$ for $i < j \leqslant n$, and*

3.1.3. *$\max f(A_m) < \min f(A_n)$ for $m < n$.*

Since the sets $A_{n,i}$ and A_n are finite, the lemma easy follows.

3.2. THEOREM. *Player I has a winning strategy in $G^+(\boldsymbol{K}, X)$ iff he has a winning strategy in $G(\boldsymbol{K}, X)$.*

Proof. We shall prove the nontrivial part of the theorem only. Assume that Player I has a winning strategy in $G^+(\boldsymbol{K}, X)$. We define a winning strategy s for the player in $G(\boldsymbol{K}, X)$. Player I starts in $G^+(\boldsymbol{K}, X)$ by choosing $\{X(k)\colon k \in N\}$ and $\{Y(k)\colon k \in N\}$. We set $E_0 = X$ and $E_1 = Y(1)$. Then Player II chooses a closed subset E_2 of X with $E_2 \cap E_1 = 0$. Proceeding by induction, assume that $E_0, E_1, \ldots, E_{2n-2}$, where $n > 1$, are already chosen. We define E_{2n-1} as follows. If $n = f(k)$, then $k > 1$, and we put $E_{2n-1} = Y(k) \cap E_{2n-2}$. If $n = f(k_1, \ldots, k_i, 1)$, then we put $Z(k_1, \ldots, k_i) = X(k_1, \ldots, k_i) \cap E_{2n-2}$. Now, the strategy of Player I in $G^+(\boldsymbol{K}, X)$ provides the families

$$\{X(k_1, \ldots, k_i, k)\colon k \in N\} \quad \text{and} \quad \{Y(k_1, \ldots, k_i, k)\colon k \in N\}.$$

We set $E_{2n-1} = Y(k_1, \ldots, k_i, 1)$. Finally, if $n = f(k_1, \ldots, k_i, k_{i+1})$, where $k_{i+1} > 1$, then we set $E_{2n-1} = Y(k_1, \ldots, k_i, k_{i+1}) \cap E_{2n-2}$. We claim that the strategy s, defined by setting $s(E_0, \ldots, E_{2n-2}) = E_{2n-1}$, is winning. For, suppose that

$$\bigcap \{E_{2n}\colon n \in N\} \neq 0,$$

and pick $x_\infty \in \bigcap \{E_{2n}\colon n \in N\}$. Since $E_0 = X = \bigcup \{X(k)\colon k \in N\}$, there is k_1 in N such that $x_\infty \in X(k_1)$. Let us put $n_1 = f(k_1, 1)$. Then $Z(k_1) = X(k_1) \cap E_{2n_1-2}$ and thus $x_\infty \in Z(k_1)$. Since $Z(k_1) = \bigcup \{X(k_1, k)\colon k \in N\}$, there is k_2 in N such that $x_\infty \in X(k_1, k_2)$. Let $n_2 = f(k_1, k_2, 1)$. Then $Z(k_1, k_2) = X(k_1, k_2) \cap E_{2n_2-2}$ and thus $x_\infty \in Z(k_1, k_2)$. Since $Z(k_1, k_2) = \bigcup \{X(k_1, k_2, k): k \in N\}$, there is k_3 in N such that $x_\infty \in X(k_1, k_2, k_3)$. Continuing in that manner we get an infinite sequence $\langle k_1, k_2, \ldots\rangle \in N^N$ such that $x_\infty \in \bigcap \{X(k_1, \ldots, k_n): n \in N\}$ and this is a contradiction. The proof is complete.

3.3. LEMMA. *If X is a Lindelöf space and Player I has a winning strategy in $G^{\#}(K, X)$, then he has a winning strategy in $G^{+}(K, X)$.*

Proof. If X is a Lindelöf space, then each locally finite family in X is countable. Thus each winning strategy of Player I in $G^{\#}(K, X)$ is a winning strategy for that player in $G^{+}(K, X)$ as well. The proof is complete.

By 2.5, 3.2 and 3.3 we get

3.4. THEOREM. *Let X be a Lindelöf space and let K be a class of spaces invariant with respect to finite closed unions. Then the following conditions are equivalent.*

3.4.1. $I(K, X)$.

3.4.2. $I^{+}(K, X)$.

3.4.3. $I^{\#}(K, X)$.

3.4.4. $I(LK, X)$.

3.4.5. $I(SK, X)$.

By 2.6, 3.4 and by 4.1 of [7] we get

3.5. THEOREM. *For a Lindelöf space X the following conditions are equivalent.*

3.5.1. $I(I, X)$.

3.5.2. $I(F, X)$.

3.5.3. $I^{+}(F, X)$.

3.5.4. $I^{\#}(F, X)$.

3.5.5. $I(D, X)$.

3.5.6. $I(SF, X)$.

By 2.9 and 3.4 we get

3.6. THEOREM. *For a Lindelöf space X the following conditions are equivalent.*

3.6.1. $I(C, X)$.

3.6.2. $I^{+}(C, X)$.

3.6.3. $I^{\#}(C, X)$.

3.6.4. $I(DC, X)$.

3.6.5. $I(LC, X)$.

3.6.6. $I(SC, X)$.

3.7. Remark. Since each one of the conditions 3.5.1, 3.5.2, 3.5.3, 3.6.1, and 3.6.2 implies the Lindelöf property of X, we may use that to rephrase slightly Theorems 3.5 and 3.6.

3.8. Remark. It is easy to check that $II^{+}(K, X)$ implies $II(K, X)$. We conjecture that the converse implication holds as well. Its proof, however, requires a new proof-technique, which is not intended to be developed here.

4. The game $G'(K, X)$. The game $G'(K, X)$ is defined as follows. Player I chooses a closed subset E_1 of X with $E_1 \in K$, and after that Player II chooses an open subset U_1 of X with $U_1 \supset E_1$. Again Player I chooses a closed subset E_2 of X with $E_2 \in K$ and Player II chooses an open subset U_2 of X with $U_2 \supset E_2$, and so on. Player I wins the play $\langle E_1, U_1, E_2, U_2, \ldots\rangle$ of $G'(K, X)$ if $\bigcup \{U_n: n \in N\} = X$; otherwise Player II wins.

If $K = \{0\} \cup \{\{x\}: x \in X\}$, then the game $G'(K, X)$ coincides with the point-open game $G(X)$ introduced and studied by F. Galvin [3].

For the proof-technique it is convenient to consider the strategies in $G'(K, X)$ dependent on the opponent's moves only, i.e., a strategy s of Player I is defined for $\emptyset$ and $\langle U_1, \ldots, U_n\rangle$, while a strategy t of Player II is defined for $\langle E_1, \ldots, E_n\rangle$.

Let us note that the scheme of playing $G'(K, X)$ can be generalized in various ways, e.g., to consider X as bitopological space, or to eliminate the topology completely replacing K and the family of open sets in X by certain families of subsets of X.

It is an easy exercise to point out the following.

4.1. THEOREM. *The games $G'(K, X)$ and $G(K, X)$ are equivalent, i.e., Player I (Player II) has a winning strategy in $G'(K, X)$ iff Player I (Player II, resp.) has a winning strategy in $G(K, X)$.*

5. The game $G^{\delta}(K, X)$ and G_δ sets. This game is an easy modification of $G'(K, X)$: for each $n \in N$, Player II at his nth move chooses a G_δ-set U_n in X with $U_n \supset E_n$.

5.1. THEOREM. *Player I has a winning strategy in $G^{\delta}(K, X)$ iff he has a winning strategy in $G'(K, X)$.*

Proof. We prove the nontrivial part of the theorem only. Let s be a winning strategy of Player I in $G'(K, X)$; we shall define a winning strategy t for the player in $G^{\delta}(K, X)$. We set $t(\emptyset) = s(\emptyset)$. To define $t(U_1, \ldots, U_k)$, where $U_1, \ldots, U_k$ are G_δ subsets of X, we need two auxiliary functions: the function which assigns to each G_δ set U a fixed sequence $\langle U(1), U(2), \ldots\rangle$ of open sets so that $U = \bigcap \{U(n): n \in N\}$, and the function $f\colon N \times N \to N$ of Cantor, i.e., $f(1, 1) = 1, f(1, 2) = 2, f(2, 1) = 3$, $f(1, 3) = 4, f(2, 2) = 5, f(3, 1) = 6$, and so on. Note that $f(m, n) = k > 1$ implies $m < k$. Now, for $k = 1$, we set $t(U_1) = s(U_1(1))$. Let $k > 1$. Then k determines the unique sequence

$$\langle\langle m(k, 1), n(k, 1)\rangle, \langle m(k, 2), n(k, 2)\rangle, \ldots, \langle m(k, p(k)), n(k, p(k))\rangle\rangle,$$

where

$$f(m(k, 1), n(k, 1)) = k-1,\ f(m(k, 2), n(k, 2)) = m(k, 1)-1,\ \ldots$$
$$\ldots,\ f(m(k, p(k)), n(k, p(k))) = m(k, p(k)-1)-1, \text{ and } m(k, p(k)) = 1.$$

Clearly,

$$k > m(k, 1) > m(k, 2) > \ldots > m(k, p(k)-1) > m(k, p(k)) = 1.$$

Now, given G_δ sets $U_1, \ldots, U_k$, where $k>1$, we set

$$t(U_1, \ldots, U_k) = s\big(U_{m(k,p(k))}(n(k,p(k))), \ldots, U_{m(k,1)}(n(k,1))\big).$$

We claim that t is a winning strategy. For, suppose in contrary, there is a point, say x_0, which is not covered by $\{U_n: n\in N\}$. Let us set $m(1)=1$. Since $x_0 \notin U_1$, there is $n(1)\in N$ such that $x_0 \notin U_{m(1)}(n(1))$. Let us set $m(2) = 1+f(m(1), n(1))$. Since $x_0\notin U_{m(2)}$, there is $n(2)\in N$ such that $x_0 \notin U_{m(2)}(n(2))$. Let us set $m(3) = 1+f(m(2), n(2))$, and so on. Finally, we get the play

$$\langle E_{m(1)}, U_{m(1)}(n(1)), E_{m(2)}, U_{m(2)}(n(2)), \ldots\rangle$$

of $G'(K, X)$, where $E_{m(1)} = E_1 = s(\emptyset)$, $E_{m(k+1)} = s\big(U_{m(1)}(n(1)), \ldots, U_{m(k)}(n(k))\big)$, and $f(m(k), n(k)) = m(k+1)-1$ for each $k\in N$. Hence $\bigcup\{U_{m(k)}(n(k)): k\in N\} = X$, and thus $x_0 \in U_{m(k)}(n(k))$ for some $k\in N$. We get a contradiction and therefore our claim is true. The proof is complete.

5.2. Remark. Clearly, $II'(K, X) \Rightarrow II^\delta(K, X)$. The converse implication, however, cannot be proved in ZFC. For, let us consider the games $G'(I, X)$ and $G^\delta(I, X)$, where X is a subset of the real line R. It is easy to verify that $I^\delta(I, X) \Leftrightarrow$ $\Leftrightarrow \operatorname{card} X \leqslant \aleph_0$, and $II^\delta(I, X) \Leftrightarrow \operatorname{card} X > \aleph_0$. Assuming MA, F. Galvin [3] has constructed $X\subset R$ such that $G'(I, X)$ is undetermined. Hence $II^\delta(I, X) \not\Rightarrow II'(I, X)$ for $X\subset R$ under MA. On the other hand, assuming the Borel Conjecture holds (and this is consistent with ZFC as was established by R. Laver [4]), we have $II'(I, X) \Leftrightarrow \operatorname{card} X>\aleph_0$ for each $X\subset R$ (cf. [3]). Hence $II'(I, X) \Leftrightarrow II^\delta(I, X)$ for $X\subset R$ also is consistent with ZFC.

For any space X, let X_δ denote the set X endowed with the topology generated by the G_δ subsets of the space X.

5.3. Theorem. *Player I has a winning strategy in $G(F, X_\delta)$ iff he has a winning strategy in $G(F, X)$.*

Proof. We shall prove the nontrivial part of the theorem. Assume that Player I has a winning strategy in $G(F, X)$. By Theorem 4.1, Player I has also a winning strategy, say s, in $G^\delta(F, X)$. We shall define a winning strategy t for the player in $G(F, X_\delta)$ as follows. For each finite subset E of X_δ and each open subset of U of X_δ there is a G_δ subset $V(E, U)$ of X so that $E\subset V(E,U)\subset U$. We set

$$t(\emptyset) = s(\emptyset) \quad\text{and}\quad t(U_1, \ldots, U_n) = s\big(V(E_1, U_1), \ldots, V(E_n, U_n)\big),$$

where

$$E_1 = s(\emptyset),\ E_2 = s\big(V(E_1, U_1)\big),\ \ldots,\ E_n = s\big(V(E_1, U_1), \ldots, V(E_{n-1}, U_{n-1})\big).$$

Let $\langle E_1, U_1, E_2, U_2, \ldots\rangle$ be a play of $G(F, X_\delta)$ where $E_1 = t(\emptyset)$ and $E_{n+1} = t(U_1, \ldots, U_n)$ for each $n\in N$. Then $\langle E_1, V(E_1, U_1), E_2, V(E_2, U_2), \ldots\rangle$ is a play of $G^\delta(F, X)$, where $E_1 = s(\emptyset)$ and $E_{n+1} = s(V(E_1, U_1), \ldots, V(E_n, U_n))$ for each $n\in N$. Therefore $\bigcup\{V(E_n U_n): n\in N\} = X$, and also $\bigcup\{U_n: n\in N\} = X$. Hence t is a winning strategy. The proof is complete.

Let us recall that X is said to be a *P-space* if each G_δ set is open in X, i.e., if $X = X_\delta$.

5.4. Lemma. *If X is a Lindelöf subspace of a P-space Y, then X is closed in Y.*

The proof of 5.4 is standard and thus omitted.

5.5. Theorem. *If X is a Lindelöf subspace of Y and Player I has a winning strategy in $G(F, Y)$, then he has a winning strategy in $G(F, X)$.*

Proof. The theorem easily follows by Theorem 5.3, Lemma 5.4 and, moreover, 2.4 of [7].

6. The game $G^*(K, X)$. Let us recall that a family $\mathcal{U}$ of open sets in X is said to be a *K-cover of X* if for each closed subset E of X with $E\in K$ there is an $U\in\mathcal{U}$ so that $E\subset U$ ([7], p. 200).

The dual game $G^*(K, X)$ is defined similarly as by F. Galvin [3]. Now, Player II "opens the game" by choosing a K-cover $\mathcal{U}_1$ of X. After that Player I chooses a $U_1\in\mathcal{U}_1$. Again Player II chooses a K-cover $\mathcal{U}_2$ of X and Player I chooses a $U_2\in\mathcal{U}_2$, and so on. Player I wins the play $\langle\mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \ldots\rangle$ of $G^*(K, X)$ if $\bigcup\{U_n: n\in N\} = X$; otherwise Player II wins.

The next lemma and theorem are a generalization of the corresponding statements of F. Galvin [3].

6.1. Lemma. *Let $\langle\mathcal{U}_1, \ldots, \mathcal{U}_n\rangle$ be a sequence (possibly void) of K-covers of X and let s be a strategy of Player I in $G^*(K, X)$. Then there is a closed subset E of X with $E\in K$ so that for each open set U containing E there is a K-cover $\mathcal{U}$ of X so that $U = s(\mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{U})$.*

Proof. Let $\mathcal{V}$ be the family of all open sets in X that are not of the form $s(\mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{U})$ for some K-cover $\mathcal{U}$ of X. Then it follows that $\mathcal{V}$ is not a K-cover of X, because otherwise $s(\mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{V})\in\mathcal{V}$ what contradicts the definition of $\mathcal{V}$. Now, suppose each closed subset E of X with $E\in K$ has an open nbhd V from $\mathcal{V}$. Then, again, $\mathcal{V}$ is a K-cover of X and this is a contradiction. The proof is complete.

6.2. Theorem. *The games $G^*(K, X)$ and $G'(K, X)$ are equivalent, i.e., Player I (Player II) has a winning strategy in $G^*(K, X)$ iff Player I (Player II, resp.) has a winning strategy in $G'(K, X)$.*

Proof. Let s be a winning strategy of Player I in $G^*(K, X)$. A winning strategy t of the player in $G'(K, X)$ is defined as follows. We set $t(\emptyset) = E_1$ where E_1 is any closed subset of X with $E_1\in K$ and with the property: each nbhd of E_1 has form $s(\mathcal{U})$ for some K-cover $\mathcal{U}$ of X. Lemma 6.1 assures the existence of E_1. Let U_1 be any open set in X containing E_1. Then there is a K-cover $\mathcal{U}_1$ of X so that $U_1 = s(\mathcal{U}_1)$. Again by Lemma 5.1 there is a closed subset E_2 in X with $E_2\in K$ and with the property: each nbhd of E_2 has form $s(\mathcal{U}_1, \mathcal{U})$ for some K-cover $\mathcal{U}$ of X. We set $t(\mathcal{U}_1) = E_2$, and so on. Finally, we get the play $\langle E_1, \mathcal{U}_1, E_2, \mathcal{U}_2, \ldots\rangle$ of $G'(K, X)$ and the play $\langle\mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \ldots\rangle$ of $G^*(K, X)$, where $U_n = s(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ for each $n\in N$. Thus t is a winning strategy of Player I in $G'(K, X)$. Conversely, let s be a winning strategy of Player I in $G'(K, X)$. We define a winning strategy t of the player in $G^*(\boldsymbol{K}, X)$ as follows. Let $\mathscr{U}_1$ be a $\boldsymbol{K}$-cover of X. Then we set $t(\mathscr{U}_1) = U_1$, where U_1 is any set from $\mathscr{U}_1$ containing $s(\emptyset)$. Let $\mathscr{U}_2$ be a $\boldsymbol{K}$-cover of X. Then we set $t(\mathscr{U}_1, \mathscr{U}_2) = U_2$, where U_2 is any set from $\mathscr{U}_2$ containing $s(U_1)$, and so on. It is easy to see that this procedure provides t to be winning. Now, let s be a winning strategy of Player *II* in $G^*(\boldsymbol{K}, X)$. Then a winning strategy for the player in $G'(\boldsymbol{K}, X)$ is defined as follows. Let E_1 be a closed subset of X with $E_1 \in \boldsymbol{K}$. Then the $\boldsymbol{K}$-cover $s(\emptyset)$ of X contains an open set U_1 containing E_1. We set $t(E_1) = U_1$. Let E_2 be a closed subset of X with $E_2 \in \boldsymbol{K}$. Then the $\boldsymbol{K}$-cover $s(U_1)$ contains an open set U_2 containing E_2. We set $t(E_1, E_2) = U_2$, and so on. Since $\bigcup \{U_n : n \in N\} \neq X$, it follows that t is a winning strategy. Finally, let s be a winning strategy of Player *II* in $G'(\boldsymbol{K}, X)$. We define a winning strategy of the player in $G^*(\boldsymbol{K}, X)$ as follows. We set $\mathscr{U}_1 = \{s(E):\ E \in 2^X \cap \boldsymbol{K}\}$ and $t(\emptyset) = \mathscr{U}_1$. Let $U_1 \in \mathscr{U}_1$. Then there is a closed subset E_1 of X with $E_1 \in \boldsymbol{K}$ and $s(E_1) = U_1$. Now we set $\mathscr{U}_2 = \{s(E_1, E):\ E \in 2^X \cap \boldsymbol{K}\}$ and $t(U_1) = \mathscr{U}_2$, and so on. As is easy to verify, we have $\bigcup \{U_n:\ n \in N\} \neq X$. Thus t is a winning strategy. The proof is complete.

An analysis of the above proof leads to the following characterization.

6.3. THEOREM. *Player II has a winning strategy in $G^*(\boldsymbol{K}, X)$ iff there is an indexed family*

$$\{U(t_1, \ldots, t_n):\ \langle t_1, \ldots, t_n\rangle \in T^n \text{ and } n \in N\}$$

of open sets in X so that

6.3.1. *$\{U(t):\ t \in T\}$ is a $\boldsymbol{K}$-cover of X,*

6.3.2. *$\{U(t_1, \ldots, t_n, t):\ t \in T)$ is a $\boldsymbol{K}$-cover of X for each $\langle t_1, \ldots, t_n\rangle \in T^n$ and $n \in N$, and*

6.3.3. *$\bigcup \{U(t_1, \ldots, t_n):\ n \in N\} \neq X$ for each $\langle t_1, t_2, \ldots\rangle \in T^N$.*

In particular, for $\boldsymbol{K} = \boldsymbol{1}$, we get

6.4. COROLLARY. *Player II has a winning strategy in $G^*(\boldsymbol{1}, X)$ iff there is an indexed family*

$$\{U(t_1, \ldots, t_n):\ \langle t_1, \ldots, t_n\rangle \in T^n \text{ and } n \in N\}$$

of open sets in X so that

6.4.1. *$\{U(t):\ t \in T\}$ is a cover of X,*

6.4.2. *$\{U(t_1, \ldots, t_n, t):\ t \in T\}$ is a cover of X for each $\langle t_1, \ldots, t_n\rangle \in T^n$ and $n \in N$, and*

6.4.3. *$\bigcup \{U(t_1, \ldots, t_n):\ n \in N\} \neq X$ for each $\langle t_1, t_2, \ldots\rangle \in T^N$.*

Another characterization of a space favorable for Player *II* has been established by F. Galvin (unpublished):

6.5. THEOREM. *Player II has a winning strategy in $G^*(\boldsymbol{1}, X)$ iff there is an open cover $\mathscr{U}$ of X such that*

6.5.1. *If $U \in \mathscr{U}$ and $x \in X$, then there is $V \in \mathscr{U}$ with $U \cup \{x\} \subset V$, and*

6.5.2. *if $U_1 \subset U_2 \subset \ldots$, where $\{U_n:\ n \in N\} \subset \mathscr{U}$, then $\bigcup \{U_n:\ n \in N\} \neq X$.*

6.6. Remark. If X is not a Lindelöf space, then Player *II* has a winning strategy in $G^*(\boldsymbol{1}, X)$. Indeed, taking always $\mathscr{U}_n = \mathscr{U}$, where $\mathscr{U}$ is a fixed open cover of X without countable subcover, he wins every play. On the other hand, if X is a Lindelöf space, then we may put $T = N$ in Corollary 6.4 (and assume $\mathscr{U}$ is countable in Theorem 6.5).

6.7. Remark. If $\operatorname{ind} X > 0$, then Player *II* has a winning strategy in $G'(\boldsymbol{1}, X)$. For, let x be any point of X with $\operatorname{ind}_x X > 0$ and let U be an open nbhd of x so that there is no closed-open nbhd of x contained in U. Since X is completely regular, there is a continuous map $f:\ X \to [0, 1]$ such that $f(x) = 1$ and $f(X - U) = 0$. However, $f(X) = [0, 1]$. Hence Player *II* can use sets $f^{-1}(a, b)$ for his win (cf. [7], 3.6.2 and 5.11).

6.8. Remark. By preceding remarks we may assume in Corollary 6.4 that X is a zerodimensional Lindelöf space and replace "cover" by "countable closed-open partition". This version of 6.4 suggests we can define a natural map from X into N^N. However, in a different way, a more general result can be derived.

6.9. THEOREM. *Let X be a Lindelöf space. Then Player II has a winning strategy in $G^*(\boldsymbol{1}, X)$ iff there is a continuous map f from X onto a metric separable space Y such that Player II has a winning strategy in $G^*(\boldsymbol{1}, Y)$.*

Proof. Assume that Player *II* has a winning strategy in $G^*(\boldsymbol{1}, X)$, where X is a Lindelöf space. Then, by 6.4, we have a family of open sets

$$\{U(k_1, \ldots, k_n):\ \langle k_1, \ldots, k_n\rangle \in N^n \quad\text{and}\quad n \in N\}$$

in X satisfying the conditions 6.4.1, 6.4.2 and 6.4.3 ($T = N$). Using the Lindelöf property again we may assume without loss of generality that each $U(k_1, \ldots, k_n)$ is a cozero set, i.e., $U(k_1, \ldots, k_n) = f^{-1}_{k_1, \ldots, k_n} R - \{0\})$. Since the family $\{f_{k_1, \ldots, k_n}:\ \langle k_1, \ldots, k_n\rangle \in N^n$ and $n \in N\}$ is countable, the weak topology generated by that family is pseudometric and separable. Finally, the natural quotient map yields the required metric separable space Y. Clearly, the resulting map f from X onto Y is continuous and, moreover, all sets $f(U(k_1, \ldots, k_n))$ are open in Y. Since each map preserves set-theoretical unions, we infer by 6.4, that Player *II* has a winning strategy in $G^*(\boldsymbol{1}, X)$. The converse implication was proved in [7], Theorem 3.5. The proof is complete.

The next theorem also makes use of 6.4. It has been established by E. K. van Douwen (cf. 6.11 below).

6.10. THEOREM. *If X is a Lindelöf P-space, then Player II has no winning strategy in $G^*(\boldsymbol{1}, X)$.*

Proof. Let X be a Lindelöf P-space. Suppose Player *II* has a winning strategy in $G^*(\boldsymbol{1}, X)$. By 6.4 and 6.6 there is a family $\{U(k_1, \ldots, k_n):\ \langle k_1, \ldots, k_n\rangle \in N^n$ and $n \in N\}$ of open subsets of X satisfying the conditions 6.4.1, 6.4.2, and 6.4.3, where $T = N$. Let $\mathscr{U}(\emptyset) = \{U(k):\ k \in N\}$ and

$$\mathscr{U}(k_1, \ldots, k_n) = \{U(k_1, \ldots, k_n, k):\ k \in N\}$$

for each $\langle k_1, \ldots, k_n\rangle \in N^n$ and $n \in N$. Then $\{\mathscr{U}(k_1, \ldots, k_n): \langle k_1, \ldots, k_n\rangle \in N^n$ and $n \geqslant 0\}$ is a countable family of countable open covers of X. Since X is a Lindelöf P-space, there is an open cover $\mathscr{V} = \{V_n: n \in N\}$ of X which is a common refinement of all covers $\mathscr{U}(k_1, \ldots, k_n)$. Hence, there is $k_1 \in N$ such that $V_1 \subset U(k_1)$, there is $k_2 \in N$ such that $V_2 \subset U(k_1, k_2)$, and so on. Therefore we get $\langle k_1, k_2, \ldots\rangle \in N^N$ such that $V_n \subset U(k_1, \ldots, k_n)$ for each $n \in N$. Since $\bigcup\{V_n: n \in N\} = X$, we have $\bigcup\{U(k_1, \ldots, k_n): n \in N\} = X$ and this is a contradiction. The proof is complete.

6.11. Remark. The preceding theorem can also be derived from 6.9. For, it suffices to verify the following: Let f be a continuous map from a Lindelöf P-space X into a metric space Y. Then $f(X)$ is countable.

7. Indeterminacy of $G(1, X)$ and $G(C, X)$ in ZFC. Assuming MA, F. Galvin [3] has constructed a subset X of the real line so that the game $G(1, X)$ is undetermined, and asked whether there is a topological space Y for which $G(1, Y)$ is undetermined in ZFC (Problem 2). We describe here the space Y constructed by R. Pol. [5] (the space was constructed for another purpose) and prove some additional properties of Y to get the following.

7.1. THEOREM. *The games $G(1, Y)$ and $G(C, Y)$ are undetermined in* ZFC.

There are several authors whose contribution to this section is essential. I asked R. Pol if a Lindelöf P-space must be the countable union of its scattered closed subspaces. After that K. Alster and R. Pol have verified that the space constructed earlier by R. Pol [5] provides a counterexample to my conjecture. Finally, E. K. van Douwen has observed the property stated in 6.10 (cf. [9], Note added).

The rest of this section is occupied by the proof of 7.1.

The space Y is a subspace of $(\{0,1\}^{\omega_1})_\delta$, or equivalently, a subspace of the $\aleph_1$-box product of $\aleph_1$, copies of $\{0,1\}$, and is defined as follows. For any $y \in \{0,1\}^{\omega_1}$ we define $\operatorname{car} y = \{\alpha < \omega_1: y(\alpha) = 1\}$ (the carrier of y). Let Λ denote the set of all limit ordinals $\lambda < \omega_1$. For each $\lambda \in \Lambda$ we pick a fixed increasing sequence $\langle \alpha_1(\lambda), \alpha_2(\lambda), \ldots\rangle$ of ordinals with $\lim_{n\to\infty} \alpha_n(\lambda) = \lambda$, and define a point y_λ to be the unique point of $\{0,1\}^{\omega_1}$ with $\operatorname{car} y_\lambda = \{\alpha_n(\lambda): n \in N\}$. Finally, we set $Y_0 = \{y_\lambda: \lambda \in \Lambda\}$, $Y_1 = \{y \in \{0,1\}^{\omega_1}: \operatorname{card} \operatorname{car} y < \aleph_0\}$, and $Y = Y_0 \cup Y_1$. The (subspace) topology of Y is determined by the basic closed-open neighborhoods of the form $U_\alpha(y) = \{y' \in Y: y'(\xi) = y(\xi)$ for each $\xi < \alpha\}$, where $y \in Y$ and $\alpha < \omega_1$.

From the construction of Y we have immediately

7.2. *Y is a P-space of cardinality $\aleph_1$.*

From 7.2 it follows that

7.3. *Y is hereditarily paracompact.*

If A is a countable subset of $\{\alpha: \alpha < \omega_1\}$, then the set $\{y \in Y: y(\alpha) = 1$ for each $\alpha \in A\}$ is closed-open in Y. Using this observation we get

7.4. *Y_0 is an open discrete subset of Y and $\{y \in Y: \operatorname{card} \operatorname{car} y = n\}$ is a discrete subset of Y. Hence Y is σ-discrete.*

7.5. *Y is a Lindelöf space.*

Proof. Let $\mathscr{A}$ be an open cover of Y. Without loss of generality we may assume that $\mathscr{A}$ consists of basic open sets, i.e., $A = U_{\alpha(A)}(y_A)$ for each $A \in \mathscr{A}$. For each $A \in \mathscr{A}$ we put $A^+ = \{y \in Y: \operatorname{car} y \subset \alpha(A)\}$. Clearly, the set A^+ is countable for each $A \in \mathscr{A}$. Let A_0 be any set from $\mathscr{A}$ for which $\bar{0} \in A_0$ ($\bar{0}(\alpha) = 0$ for each $\alpha < \omega_1$). We set $\mathscr{A}_0 = \{A_0\}$. Since A_0^+ is countable, there is a countable subfamily $\mathscr{A}_1$ of $\mathscr{A}$ such that $A_0^+ \subset \bigcup \mathscr{A}_1$. The set $\bigcup\{A^+: A \in \mathscr{A}_1\}$ also is countable. Thus there is a countable subfamily $\mathscr{A}_2$ of $\mathscr{A}$ so that $\bigcup\{A^+: A \in \mathscr{A}_1\} \subset \bigcup \mathscr{A}_2$. The set $\bigcup\{A^+: A \in \mathscr{A}_2\}$ is again countable and thus it is covered by a countable subfamily $\mathscr{A}_3$ of $\mathscr{A}$, and so on. Finally, we put $\beta = \sup\{\alpha(A): A \in \mathscr{A}_n$ and $n \geqslant 0\}$, and consider two cases.

Case 1: $\beta \notin \Lambda$. Then $\bigcup\{\mathscr{A}_n: n \geqslant 0\}$ covers Y. For, let $y \in Y$. There are again two cases.

Case 1a: $\beta \cap \operatorname{car} y$ is finite. Then there is $y' \in Y_1$ with $\operatorname{car} y' = \beta \cap \operatorname{car} y$. Since $\beta \notin \Lambda$, we have $\beta = \alpha(A)$ for some $A \in \mathscr{A}_n$ and $n \geqslant 0$. Hence $y' \in A^+$ and thus $y' \in A'$ for some $A' \in \mathscr{A}_{n+1}$. Since $y(\xi) = y'(\xi)$ for each $\xi < \beta$ and $\alpha(A') \leqslant \beta$, we have $y \in A'$.

Case 1b: $\beta \cap \operatorname{car} y$ is infinite. Then $y = y_\lambda$ for some $\lambda < \beta$ and therefore $y \in A^+$, where $A \in \mathscr{A}_n$ and $\alpha(A) = \beta$. Again, there is $A' \in \mathscr{A}_{n+1}$ with $y \in \mathscr{A}'$. Hence $\bigcup\{\mathscr{A}_n: n \geqslant 0\}$ covers Y.

Case 2: $\beta \in \Lambda$. Then $\{A_1\} \cup \bigcup\{\mathscr{A}_n: n \geqslant 0\}$ covers Y, where A_1 is any set from $\mathscr{A}$ containing y_β. For, let $y \in Y - \{y_\beta\}$. Then there are two cases according to whether $\beta \cap \operatorname{car} y$ is finite or infinite. Since this case is similar to the former, we shall not repeat the reasoning. In both cases we have found a countable subcover of A. The proof is complete.

A subset S of ω_1 is said to be stationary if it meets each closed unbounded subset of ω_1. For the lemma on regressive functions (Pressing Down Lemma) used below we refer to G. Fodor [2].

7.6. *Y is not the union of a countable family of its scattered closed subsets.*

Proof. Suppose $Y = \bigcup\{S_n: n \in N\}$, where each S_n is a scattered closed subset of Y. Then there is $k \in N$ such that $\{\lambda \in \Lambda: y_\lambda \in S_k\}$ is stationary (in ω_1). Since S_k has the Lindelöf property, there is a point y in S_k such that

$$\{\lambda \in \Lambda: y_\lambda \in S_k \cap U_\alpha(y)\}$$

is stationary for each $\alpha < \omega_1$. Since S_k is scattered, we may choose such a point y with the lowest rank and a basic nbhd $U_\alpha(y)$ of y so that for each $y' \in S_k \cap U_\alpha(y) - \{y\}$ there is $\alpha' < \omega_1$ for which $\{\lambda \in \Lambda: y_\lambda \in S_k \cap U_\alpha(y) \cap U_{\alpha'}(y')\}$ is not stationary. Thus we may assume, without loss of generality, that y is the unique point of S_k such that $\{\lambda \in \Lambda: y_\lambda \in S_k \cap U_\alpha(y)\}$ is stationary for each $\alpha < \omega_1$. Let α be any ordinal with $\alpha>\sup\operatorname{car} y$. Since the set $\Sigma=\{\lambda\in\Lambda:\ \lambda>\alpha$ and $y_\lambda\in S_k\}$ is stationary and the function $f\colon\Sigma\to\omega_1$ defined by $f(\lambda)\in\{\alpha_n(\lambda):\ n\in N\}-\alpha$ is regressive (pressing down), there is a stationary set $\Sigma_0\subset\Sigma$ such that $f(\Sigma_0)=\{\beta\}$. (Clearly, $\beta>\alpha$). Hence $\beta\in\operatorname{car} y_\lambda$ for each $\lambda\in\Sigma_0$, and therefore the set $\{\lambda\in\Lambda:\ y_\lambda\in S_k$ and $\beta\in\operatorname{car} y_\lambda\}$ is stationary. Now, the set $S=S_k\cap\{y\in Y:\ \beta\in\operatorname{car} y\}$ is closed and the set $\{\lambda\in\Lambda:\ y_\lambda\in S\}$ is stationary, because

$$\{\lambda\in\Lambda:\ y_\lambda\in S_k \text{ and } \beta\in\operatorname{car} y_\lambda\}=\{\lambda\in\Lambda:\ y_\lambda\in S\}.$$

However, S has the Lindelöf property, because it is closed in Y. Hence there is a point y' in S such that $\{\lambda\in\Lambda:\ y_\lambda\in S\cap U_\eta(y')\}$ is stationary for each $\eta<\omega_1$. But S is closed-open in S_k, $y\notin S$ and the existence of y' contradicts to the choice of y. The proof is complete.

7.7. *Player I has no winning strategy in $G(I, Y)$.*

Proof. Suppose Player I has a winning strategy in $G(I, Y)$. By 7.3, Y is hereditarily paracompact. Hence, by Theorem 11.1 of [7], the space Y is the union of a countable family of its scattered closed subsets. That, however, contradicts to 7.6 above. The proof is complete.

7.8. *Player II has no winning strategy in $G(I, Y)$.*

The last statement follows immediately from 7.2, 7.5 and 6.10.

Since each compact subset of Y is finite, the games $G(C, Y)$ and $G(F, Y)$ coincide. Moreover, the games $G(F, Y)$ and $G(I, Y)$ are equivalent by 4.1 and 4.2 of [7]. Hence both games $G(I, Y)$ and $G(C, Y)$ are undetermined.

Note added on May 13, 1981. I asked Fred Galvin if the stationary sets can be eliminated from the proof of 7.7. Answering to that question he has given such a proof and, moreover, his proof refers to the definition of the space Y only. Here it is. Let s be a strategy of Player I in $G(I, Y)$. Without loss of generality we may assume that Player II chooses basic open sets only, i.e., the sets $U_\alpha(y)=\{y'\in Y:\ y'(\xi)=y(\xi)$ for each $\xi<\alpha\}$, where $y\in Y$ and $\alpha<\omega_1$. Moreover, we may assume that Player II responds to a choice of a point y by giving an ordinal $\alpha<\omega_1$ (because y and α determine $U_\alpha(y)$). So we may assume that s is defined for any finite (possibly void) sequence $\langle\alpha_1,\ldots,\alpha_n\rangle$ of countable ordinals.

Claim 1. *There is a countable limit ordinal $\lambda=\lambda(s)$ such that*

$$\sup\operatorname{car} s(\emptyset)<\lambda \text{ and } \sup\operatorname{car} s(\alpha_1,\ldots,\alpha_n)<\lambda \text{ whenever } \alpha_1<\lambda,\ldots,\alpha_n<\lambda \text{ and } n\in N.$$

For, let $\lambda_0=\sup\operatorname{car} s(\emptyset)$ and let λ_{k+1} be the least limit ordinal $\geqslant\lambda_k$ so that $\sup\operatorname{car} s(\alpha_1,\ldots,\alpha_n)<\lambda_{k+1}$ whenever $\alpha_1,\ldots,\alpha_n$ are $<\lambda_k$. Then $\lambda=\lim_k\lambda_k$ has the required property, because the definition of each λ_k involves countably many sequences only.

Claim 2. *Player II can avoid covering the point y_λ, and therefore s is not a winning strategy.*

For, let $x_1=s(\emptyset)$. Since $\sup\operatorname{car} x_1<\lambda$ and $\alpha_n(\lambda)\nearrow\lambda$, there is $n_1\in N$ such that $\sup\operatorname{car} x_1<\alpha_{n_1}(\lambda)$. Now Player II picks an α_1 such that $\alpha_{n_1}(\lambda)<\alpha_1<\lambda$. Since $x_1(\alpha_{n_1}(\lambda))=0$, we have $y_\lambda\notin U_{\alpha_1}(x_1)$. Let $x_2=s(\alpha_1)$. Then $\sup\operatorname{car} x_2<\lambda$, so there is $n_2\in N$ such that $\sup\operatorname{car} x_2<\alpha_{n_2}(\lambda)$. Taking an α_2 with $\alpha_{n_2}(\lambda)<\alpha_2<\lambda$, we have $x_2(\alpha_{n_2}(\lambda))=0$, so $y_\lambda\notin U_{\alpha_2}(x_2)$, and so on. Finally, $y_\lambda\notin\bigcup\{U_{\alpha_n}(x_n):\ n\in N\}$, thus s is not a winning strategy.

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