On classification of weakly infinite-dimensional compacta

by

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Abstract. A classification of weakly infinite-dimensional compacta by means of a Suslin-Sierpiński index in the hyperspace of the Hilbert cube is given. This classification is applied to answer two questions of D. W. Henderson concerning the transfinite inductive dimension Ind and essential maps onto "transfinite cubes". For each \( \alpha < \omega_1 \), a countable dimensional compactum is constructed which contains topologically all compacta with transfinite dimension not greater than \( \alpha \).

Introduction

1. Terminology and notation. In this paper we consider only separable metrizable spaces, and a compactum means a compact space. Our terminology concerning analytic set theory follows [K1], and the terminology related to dimension theory follows [A-P] and [El].

A partition \( L \) in a space \( X \) between two disjoint sets \( A \) and \( B \) in \( X \) is a closed set such that \( X \setminus L = U \cup V \), where \( U \) and \( V \) are disjoint open sets with \( A \subseteq U \) and \( B \subseteq V \).

Throughout this paper \( \omega \) denote the set of natural numbers, \( I \) the unit interval \([0, 1]\), \( H \) the Hilbert cube, \( a^* \) the Baire space, i.e. topologically the irrationals, and \( 2^\mathbb{N} \) the Cantor cube \([0, 1]^\mathbb{N}\).

We denote by \( H \) the hyperspace of the Hilbert cube, i.e. the space of all closed subsets of \( H \) endowed with the topology induced by the Hausdorff distance.

Given a linearly ordered set \( M \) we denote the order type of \( M \) by \( \mathcal{O}(M) \). The symbol \( X \approx Y \) means that the spaces \( X \) and \( Y \) are homeomorphic.

2. Countable-dimensional spaces and transfinite dimensions. A space \( X \) is countable-dimensional if \( X = \bigcup X_i \) with \( X_i \) zero-dimenional.

The transfinite dimensions \( ind \) and \( Ind \) are the ordinal-valued functions obtained by the extension of the classical notions of the small and large inductive dimension respectively, by transfinite induction, i.e. for example, \( Ind X \leq \alpha \) if for each pair \( (A, B) \) of closed disjoint sets in \( X \) there is a partition \( L \) in \( X \) between \( A \) and \( B \) such

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that $\text{Ind}.X \leq \alpha$, $\alpha$ being an ordinal [H-W; p. 50], [A-P; Ch. 10, § 1], [N2; Ch. VI]; a comprehensive survey of the topic is [E2].

The transfinite dimension of an ordinal $\alpha$ defined for a complete space $X$ if and only if $X$ is countable-dimensional, and then $\text{Ind}.X \leq \alpha$. These ordinals are countable.

There exists a function $\varphi$ which maps the set of countable ordinals $\alpha_i$ into itself such that for each countable-dimensional compactum $X$, $\text{Ind}.X \leq \alpha$ is $\varphi$ (ind. $\alpha$) and thus, although the exact relations between the transfinite dimensions of ordinals and $\text{Ind}.X$ are very interesting [L2], globally both of the functions provide essentially the same classification of the family of countable-dimensional compacta.

3. Weakly infinite-dimensional compacta. A compactum $X$ is weakly infinite-dimensional if for each infinite sequence $(A_1, B_1, A_2, B_2, \ldots)$ of pairs of closed disjoint sets in $X$, there are partitions $L_i$ in $X$ between $A_i$ and $B_i$ such that $\bigcap L_i = \emptyset$ [A-P; Ch. 10, § 5], [N1], [N2; Ch. VI]. We call compacta which are not weakly infinite-dimensional strongly infinite-dimensional; strongly infinite-dimensional compacta can also be characterized as the compacta which have an essential map onto the Hilbert cube, see footnote (1)

Hurewicz [H-W; Ch. IV, § 6 (A)] (cf. also [A; § 4]) proved that countable-dimensional compacta are weakly infinite-dimensional. There exist, however, quite natural examples of weakly infinite-dimensional compacta which are not countable-dimensional [P].

4. Results. In § 1 we assign in a natural way to each weakly infinite-dimensional compactum $X$ a countable ordinal index $\alpha$ which is a topological invariant. The index can be interpreted as a Lusin-Sierpiński index when one considers compacta as the points in the hyperspace of the Hilbert cube. Thus the classification of the weakly infinite-dimensional compacta by means of the index is quite regular from the point of view of analytic set theory; we do not know how regular the classification of the countable-dimensional compacta by means of the transfinite dimensions is from this point of view. The index is bounded over each family of countable-dimensional compacta with bounded transfinite dimensions. The converse is not true: there exists $\alpha \leq \omega_1$ such that $\sup\{\text{ind}.X: X \leq \alpha\} = \alpha$. This result is obtained by a certain kind of approximation of an arbitrary compactum by compacta which are countable disjoint unions of finite polytopes (we call them "transfinite polytopes"), which we discuss in § 2. In § 3 we use the index and the result we have just mentioned to answer two questions raised by Henderson [H] concerning essential maps onto "transfinite cubes" defined by him. It seems worthwhile to notice that, when answering one of these questions, we prove an existence of a compactum with certain "transfinite dimensional" properties using an existence of an analytic non-Borel set, and we do not know how to do this in more explicit way. In the last paragraph, which is independent of the rest of the paper, we construct for each $\xi < \omega_1$ a countable-dimensional compactum $X_\xi$ containing topologically all compacta $S$ with $\text{ind}.S \leq \xi$ (we do not know what the ordinal $\text{ind}.X_\xi$ is). The construction is based upon a classical idea of a universal function for a given family of sets.

Generally speaking, the common idea underlying this paper is the investigation of the collection of weakly infinite-dimensional compacta, considered as the subset of the hyperspace of the Hilbert cube, using some classical methods and concepts of analytic set theory in the form given by Kuratowski in Topology [K1].

§ 1. The Lusin-Sierpiński index of a weakly infinite-dimensional compactum

1. The Brouwer-Kleene order. We shall consider the set $\mathcal{F}$ of all nonempty finite subsets of $\omega$ with the order $<\alpha$ inverse to the lexicographic order, i.e. $\sigma <\alpha$ means that there is an $n \in \sigma$ such that $\sigma \cap \{1, \ldots, n-1\} = \tau \cap \{1, \ldots, n-1\}$ and $n \in \sigma \setminus \tau$.

In the sequel we use the following well-known property of the order $<\alpha$ [K-M; Ch. X, § 7, Corollary 4].

Lemma 1.1. Given a decreasing sequence $\sigma_1 > \sigma_2 > \cdots$ of elements of $\mathcal{F}$, there exists an increasing sequence $f(1) < f(2) < \cdots$ of natural numbers such that for every $k$ there exists an $m$ with $\{f(1), \ldots, f(m)\} < \sigma_k$.

2. Essential families. We say that a family $\mathcal{A} = \{(A_i, B_i): i \in \omega\}$ of pairs of closed disjoint subsets of a compactum $X$ is essential if for arbitrary partitions $L_i$ in $X$ between $A_i$ and $B_i$ the intersection $\bigcap L_i$ is nonempty (cf. [Z; Definition 7]). Thus strongly infinite-dimensional compacta are the compacta which have an infinite essential family.

If we assign to each pair $(A_i, B_i)$ from the family $\mathcal{A}$ a continuous map $f_i: X \to I$ such that $f_i(A_i) = 0$ and $f_i(B_i) = 1$, then the diagonal map $\bigwedge f_i: X \to I^\omega$ is essential if and only if the family $\mathcal{A}$ is essential [A-P; Ch. 3, § 5, Ch. 10, § 4], [E1; Problem 1.9.(A)].

3. The index. Let $X$ be a compactum. We say that a sequence $\mathcal{A} = \{(A_i, B_i): i \in \omega\}$ of pairs of closed disjoint sets in $X$ is separating if for each pair $(A_i, B_i)$ of disjoint closed sets in $X$ $A_i \subseteq B_i$ and $B_i \subseteq A_i$ for infinitely many indices $i$ (3).

It is easy to construct a separating sequence in $X$ considering finite sums of elements of an arbitrary base of $X$.

(1) A map $f: X \to I^\omega$ is essential, where $\mathcal{J} \subseteq \omega$ is infinite, if for each finite $\alpha \in J$ the composition $\pi_\alpha: f: X \to I^\alpha$ of the map with the projection $\pi_\alpha: I^\omega \to I^\alpha$ is essential in the classical sense [A-P; (E)].

(2) Cf. [A-P; Ch. 10. § 7, Lemma 2 and Appendix, Lemma 3].
Given a separating sequence \( S \) in \( X \) let us put

\[
M_S(X) = \{ \sigma \in \text{Fin} \omega : \text{the family } \{(A_i, B_i) : i \in \sigma \text{ is essential} \}.
\]

We shall consider the set \( M_S(X) \) with the order \( < \) defined in section 1.

**Lemma 3.1.** Let \( S = \{(A_i, B_i) : i \in \omega \} \) and \( S' = \{(A_i, B_i) : i \in \omega \} \) be two separating sequences in \( X \). Then the ordered set \( M_S(X) \) is similar to a subset of the ordered set \( M_{S'}(X) \), and vice versa.

**Proof.** Choose, using (3.2), a sequence \( j(1) < j(2) < \cdots \) such that \( A_{j(i)} = A_{j(i)} \) and \( B_{j(i)} = B_{j(i)} \). If \( \sigma \in M_S(X) \) then \( j(\sigma) \in M_{S'}(X) \), for, if \( L_{j(i)} \) is a partition in \( X \) between \( A_{j(i)} \) and \( B_{j(i)} \) then this is also a partition between \( A_i \) and \( B_i \) and hence \( \bigcap \{L_{i} : i \in \sigma \} \neq \emptyset \), since \( \sigma \in M_{S'}(X) \). The map \( \sigma \rightarrow j(\sigma) \) is thus an order-preserving embedding of \( M_S(X) \) into \( M_{S'}(X) \). The symmetric argument proves the converse.

**Lemma 3.2.** A compactum \( X \) is weakly infinite-dimensional if and only if for some (equivalently—for each) separating sequence \( S \) in \( X \) the set \( M_S(X) \) is well-ordered.

**Proof.** Let \( S \) be a separating sequence (3.1). Assume that \( X \) is strongly infinite-dimensional and let \( \{E_i, F_i : i \in \omega \} \) be an infinite essential family in \( X \). By (3.2) one can choose numbers \( j(i) < j(2) < \cdots \) such that \( E_{j(i)} = A_{j(i)} \) and \( F_{j(i)} = B_{j(i)} \). Then \( \sigma_j = \{j(1), \ldots, j(i) \in \sigma \} \in M_S(X) \) and \( \sigma_j \prec \sigma_j \prec \cdots \), i.e., \( M_S(X) \) is not well-ordered.

Conversely, assume that there exist \( \sigma_j \in M_S(X) \) such that \( \sigma_j \prec \sigma_j \prec \cdots \) and let \( j(1) < j(2) < \cdots \) be a sequence such as in Lemma 1.1. The infinite family \( \{A_{j(i)}, B_{j(i)} : i \in \omega \} \) is then essential. Indeed, given partitions \( L_i \) in \( X \) between \( A_{j(i)} \) and \( B_{j(i)} \) and an arbitrary \( \kappa \in \omega \) one can find an \( m \) such that \( j(1), \ldots, j(k) \in \sigma_k \in M_S(X) \) and hence \( \bigcap \{L_{i} : i \in \kappa \} \neq \emptyset \), i.e., \( \bigcap \{L_{i} : i \in \omega \} \neq \emptyset \) by compactness of \( X \).

The statement in parenthesis follows from Lemma 3.1.

The two lemmas justify the following definition.

**Definition 3.3.** Let \( X \) be a weakly infinite-dimensional compactum. Then the order type of the set \( M_S(X) \) is a countable ordinal which is independent of the choice of the separating sequence \( S \) in \( X \); i.e., this is a topological invariant.

We define

\[
\text{index } X = \text{type } M_S(X)
\]

and call index \( X \) the *Lusin-Sierpiński index* of \( X \).

**Remark 3.4.** The index can also be defined by means of maps into \( \mathbb{N} \). Given a continuous map \( f : X \rightarrow \mathbb{N} \) of a compactum \( X \) put \( M(f) = \{ \sigma \in \text{Fin} \omega : \sigma \cup f \text{ is essential} \} \), where \( \eta \cup \ell \) is the projection (cf. footnote (1)'); then index \( X = \sup \{ \text{type } M(f) : f : X \rightarrow \mathbb{N} \} \), provided that \( X \) is weakly infinite-dimensional. This easily follows by a remark in section 2.

(1) The terminology is explained in the next section.

We finish this section with a simple observation which yields monotonicity of index.

**Lemma 3.5.** Let \( S = \{(A_i, B_i) : i \in \omega \} \) be a separating sequence in a compactum \( Y \) and let \( X \) be a compact set in \( Y \). Then \( S' = \{(A_i, B_i) : i \in \omega \} \) is a separating sequence in \( X \) and \( M_S(X) = \{ \sigma \in M_S(Y) : \text{the family } \{(A_i, B_i, i \in \sigma) \text{ is essential in } X \} \} \). In particular, if \( Y \) is weakly infinite-dimensional then index \( X \leq \text{index } Y \).

**Proof.** This follows immediately from a lemma on extension of partitions \( \{A \in \mathbb{P} : \text{Ch.10, Lemma 1.1, E1; Lemma 1.2.9} \} \).

4. The index as a Lusin-Sierpiński index. Let us fix a separating sequence \( S = \{(A_i, B_i) : i \in \omega \} \) in the Hilbert cube. For every \( \sigma \in \text{Fin} \omega \) let us put

\[
W_\sigma = \{x \in \mathbb{H} : \text{the family } \{(A_i \cap X, B_i \cap X) : i \in \sigma \text{ is essential} \} \}
\]

\( \mathbb{H} \) being the hyperspace of \( \mathbb{I} \).

The sets \( W_\sigma \) are closed, for if \( x \notin W_\sigma \) then there are partitions \( L_i \) in \( \mathbb{I} \) between \( A_i \) and \( B_i \), such that \( x \) is disjoint from \( L_i \) in \( \{x \in \mathbb{H} \} \) (cf. [E1; Lemma 1.2.9] and the set \( \{x \in \mathbb{H} : \mathbb{A} \cap X = \emptyset \} \) is a neighbourhood \( \mathbb{X} \) disjoint from \( W_\sigma \).

Thus (cf. sec. 1)

\[
W = \{x \in \mathbb{H} : x \in W_\sigma, x \in \text{Fin} \omega \}
\]

is a closed Lusin sieve in the hyperspace \( \mathbb{H} \).

For each compactum \( X \in \mathbb{H} \) let us put

\[
M(X) = \{x \in \text{Fin} \omega : x \in W_\sigma \}
\]

and let us recall [K1; § 3, XV] that the set \( L(W) \) sifted by the sieve \( W \) is defined by the formula

\[
X \in L(W) = \{M(X) \text{ is not well-ordered by } < \}
\]

and also, that the Lusin-Sierpiński index of \( X \) \( \neq L(W) \) with respect to the sieve \( W \) is the order type of the set \( M(X) \).

Now, it is clear by Lemma 3.5 and Definition 3.3 that \( L(W) \) is exactly the set of strongly infinite-dimensional compacta contained in \( \mathbb{I} \) and if \( X \) is a weakly infinite-dimensional compactum in \( \mathbb{I} \) then the Lusin-Sierpiński index of \( X \) with respect to the sieve \( W \) coincides with the topological invariant index \( X \).

5. Families of compacta with bounded index. In this section we give a few corollaries to the following two basic properties of the Lusin-Sierpiński index [K1; § 39, Theorem 4 and Corollary 5a]:

(A) For every \( \alpha \in \omega \), the set of all points whose Lusin-Sierpiński index is not greater than \( \alpha \) is analytic (in fact Borel).

(*) We show in the sequel that the set \( L(W) \) is non-Borel, or equivalently, that the index is unbounded (see sec. 2 in § 2).
Theorem 5.1. Given a family $\mathcal{E}$ of weakly infinite-dimensional compacta, there exists a weakly infinite-dimensional compactum containing topologically each member of $\mathcal{E}$ if and only if $\sup \{\text{index } X; X \in \mathcal{E}\} < \omega_1$.

This follows from the monotonicity of the index (Lemma 3.5), the property (A) of the Luzin-Sierpiński index, and the following lemma.

Lemma 5.2. Let $E \subset H$. There exists then a weakly infinite-dimensional compactum $C$ containing topologically each member of $\mathcal{E}$ if and only if there exists an analytic set $A \subset H$ containing $E$ and satisfying of weakly infinite-dimensional compacta.

Proof. Assume that the compactum $E$ exists and let $A = \{X \in H; X \text{ can be embedded in } E\}$. The set $A$ is analytic, for, if $\mathcal{F} = C(I^n, \mathcal{I})$ is the space of all continuous maps from $I^n$ into itself endowed with the compact-open topology, then $A = \{ (x, f) \in H \times \mathcal{F}; f(X) \subset E \text{ and } f|X \text{ is an injection} \}$, where we assume that $E$ is embedded into $I^n$; cf. [KI; 44].

Conversely, assume that the analytic set $A$ exists and let $\varphi: \omega^* \to A$ be a continuous surjection of the irrationals onto $A$. Let $G = \{(t, a); a \in \Phi(t)\}$. The projection $\pi: G \to \omega^*$ of a closed map whose fibers $\pi^{-1}(r) = \Phi(r)$ are weakly infinite-dimensional compacta and the range $E$ is zero-dimensional; therefore $G$ is weakly infinite-dimensional [La]. Since $G$ is closed in $\omega^* \times I^n$, this is a complete space, and hence there exists a compactification $E$ with countable-dimensional remainder $E \cap G$ (5). The space $E$ is the compactum we are looking for, see [A-P, Ch. 10, § 5, Theorem 21].

Theorem 5.3. For each $\alpha < \omega_1$ we have

$$\sup \{\text{index } X; X \subset E \} < \omega_1$$

The theorem follows from the next lemma and the property (B) of the Luzin-Sierpiński index (one can also use Lemma 5.2 instead of this property).

Lemma 5.4. (cf. [KI; 45, IV Theorem 4]). For each $\alpha < \omega_1$ the set $E_\alpha = \{X \in H; \text{index } X < \alpha \}$ and $G_\alpha = \{X \in H; \text{index } X < \alpha \}$ are analytic.

Proof. We shall check this only for $G_1$; the case of $H_\alpha$ is similar, and even simpler. The proof is by transfinite induction on $\alpha$; we refer the reader to [KI; 43] for some details we omit.

There is nothing to prove if $\alpha = -1$; assume that for $\beta < \alpha$ the sets $G_\beta$ are analytic. Let $\mathcal{G} = \{(A, B_i); i \in \omega \}$ be a separating sequence in $G_\alpha$. At first let us check that the set $M = \{(A, B, C) \in H \times E \times H; A \text{ is a partition in } I^n \text{ between } A \}$ is analytic. (This follows from a classical theorem of Kuratowski [KI; Théorème 21]; cf. also [EE; 4.15]. In the case of our "graph" $G$ one can construct the compactification $E$ in a particularly simple way such that in addition it extends to a continuous map $\pi: E \to C$ with zero-dimensional range and $\pi^{-1}(V)$ is finite-dimensional for any $V$ of the form $\pi^{-1}(U)$, see Theorem 5.3 in § 2.)

(1) This is a well-known fact following easily from a classical theorem of Kuratowski [KI; Théorème 21]; cf. also [EE; 4.15]. In the case of our "graph" $G" one can construct the compactification $E$ in a particularly simple way such that in addition $E$ extends to a continuous map $\pi: E \to C$ with zero-dimensional range and $\pi^{-1}(V)$ is finite-dimensional for any $V$ of the form $\pi^{-1}(U)$, see Theorem 3.3 in § 2.)

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and $B$) is an $F_{\Sigma}$-set. Indeed, if $(A, B, L) \not\in M$ then either $L \cap (A \cup B) \not\in G$ (and this formula describes a closed set) or there is a continuum in $I^n$ disjoint from $L$ which intersects both sets $A$ and $B$ (and this formula describes an open set). This observation and the inductive assumption yield the analyticity of the set

$$d_{1, ..., n} = \{ (X, L_1, ..., L_n); (A_1, B_1, L_1) \in M \text{ and } X \cap L_i \subset \bigcup_{j \neq i} I_j \text{ for } i = 1, ..., n \},$$

and hence the set

$$A_i = \bigcup_{d_i \in \mathcal{D}_i} \{ A_1 \subset A_1; B \subset \bigcap_{j \neq i} B_j \}$$

is analytic for every $i \in \omega$. It remains to notice that $L_\omega$ is the projection $d_i$.

Theorem 5.5. If $D$ is an upper semicontinuous decomposition of a compactum $X$ into weakly infinite-dimensional compacta then

$$\sup \{\text{index } A; A \in D \} < \omega_1.$$

Proof. One can assume that $X \subset I^n$ and then $D$ is an analytic set in $H$ [KI; 43], and hence the assertion follows from the property (B) of the Luzin-Sierpiński index.

6. Remarks and questions. R. D. Mauldin has kindly pointed out to the author that the transfinite inductive dimensions can be viewed as special cases of the notion of monotone inductive operators investigated in [C-M]. Let us consider for example the transfinite induction Ind. Let $\mathcal{G}$ be a separating sequence in $I^n$ and let $L_\omega$ be the set of all partitions in $I^n$ between $A$ and $B$, and $B_\omega$ be the $\omega$th iteration of $\Gamma$ (i.e. $\Gamma^{\omega+1}(1) = \Gamma(\Gamma(1))$ and $\Gamma(\alpha+1) = \bigcup (\Gamma(\beta)); \beta < \alpha$) for limit ordinals.

Then $\Gamma^{\omega+1}(1) = \{ X \in H; \text{index } X < \omega \}$ and the closure $\Gamma^{\omega+1}(B)$ of the operator $\Gamma [C-M; sec. 1]$ is the set of all countable-dimensional compacta in $H$.

We do not know how regular from the point of view of analytic set theory the operator $\Gamma$ is. For example: is the analytic set $\{ X \in H; \text{index } X < \omega \}$ always a Borel set? If $\alpha < \omega$ this is the case [KI; 45, IV Theorem 4]. The set $\Gamma^{\omega+1}(B)$ is a $\mathcal{L}_\alpha$-set in $H$. One can show that this is a $\mathcal{P}\mathcal{L}_\alpha$-set, using the fact that the subspace of $\mathcal{L}_\alpha$ consisting of the points which have only finitely many rational coordinates is universal for the class of countable-dimensional spaces [N2, Theorem IV.5], or alternatively, using the general results from [C-M]. And finally, does Ind have the property (B) of the Luzin-Sierpiński index formulated in sec. 57?

The last question can also be formulated in the following way (see the proof of Lemma 5.2 and Theorem 5.5):

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QUESTION 6.1. Assume that $D$ is an upper semicontinuous decomposition of a compactum $X$ into countable-dimensional compacta. Is it true that $\{\text{Ind } A : A \in D\} < \omega_1$?

Yet another formulation of this question is given in sec. 2, § 4.

§ 2. Transfinite polytopes

1. A theorem of W. Hurewicz. The line of reasoning in this paragraph is based upon the following lemma, which is a reformulation of a theorem of Hurewicz [Hu] in the spirit of Kuratowski and Szymański [K-S].

LEMMA 1.1. Let $p : L \to L$ be a continuous surjection of the compactum $L$ onto the space $L$, let $A \subseteq \mathbb{R}$ be an analytic set and let $Q$ be a dense-in-itself countable subset of $L$. Then each compactum in $p^{-1}(Q)$ can be embedded into some member of $A$. Then there exists a $t \in L \setminus Q$ such that the compactum $p^{-1}(t)$ can be embedded into some $A \in A$.

Proof. Let $K$ and $L$ be hyperspaces of the compactum $K$ and $L$, respectively. It is a theorem of Hurewicz [K1; § 43, VII, Corollary 3] that the subset of $L$

\[(1.1) \quad Q = \{x \in L : \exists z \in Q\}
\]

is not analytic.

On the other hand (see the proof of Lemma 5.2 in § 1), the subset of $K$

\[(1.2) \quad \mathcal{C} = \{x \in L : p^{-1}(x) \in \mathcal{B}\}
\]

is analytic.

The assumption about $p^{-1}(Q)$ implies that $Q \subseteq \mathcal{C}$ and thus (1.1) and (1.2) together yield an existence of an $x \in \mathcal{C}$ with $x \notin Q = \emptyset$. Now, each $t \in L \setminus Q$ has the required property.

2. The Yu. M. Smirnov's compacta. Smirnov [S], [A-P; Ch. 10, § 1, [E2] defined a compactum $S_1$, $S_2$, ..., $S_\alpha$, ..., $\alpha < \omega_1$, by transfinite induction in the following way:

\[(2.1) \quad S_1 = I, \quad S_{\alpha+1} = S_\alpha \times I, \quad S_\alpha = \bigcup_{\beta < \alpha} S_\beta,
\]

if $\alpha$ is a limit ordinal then $S_\alpha$ is the one-point compactification of the free union $\bigcup_{\beta < \alpha} S_\beta$.

Smirnov proved that $\text{Ind } S_\alpha = \alpha$; the reasoning in the proof of Lemma 2.2 in § 3 shows also that $\text{Ind } S_\alpha \geq \alpha$. Notice that each compactum $S_\alpha$ is a countable disjoint union of finite-dimensional cubes $I^n$. In the sequel we need the following universal property of Smirnov's compacta.

Lemma 2.1. The class of compacta which have at most countably many components and each of which is finite-dimensional coincides with the class of compacta which can be embedded into some Smirnov's compactum.

Proof. Given a countable compactum $C$, let $\gamma(C)$ be the ordinal $\alpha$ such that the $\alpha$th derived set $C^{(\alpha)}$ of $C$ is finite, and let $S_\gamma$ be the class of compacta $X$ which admit a continuous map $p : X \to C$ with finite-dimensional fibres onto a countable compactum $C$ with $\gamma(C) < \alpha$. Clearly, it is enough to check that for each $\alpha < \omega_1$ there is a $

\text{Lemma 2.2.}$ If $\alpha \in \mathbb{R}$ then $\theta(\alpha) = \omega_1$.

Proof. If $\alpha < \omega_1$ then $\alpha$ is a limit ordinal, and hence $\alpha = \omega_1$. Therefore, $\theta(\alpha) = \omega_1$.

\[(2.3) \quad \text{If } t \in \mathbb{R}, \text{ then } p^{-1}(t) = I^n \text{ for some } n,
\]

(3) To derive Theorem 2.2 from Theorem 2.2 assume that $X \subseteq I^n$ and put $A = \{a \in \mathbb{R} : a \in X\}$. Then

\[(2.4) \quad \text{If } t \in \mathbb{R}, \text{ then } p^{-1}(t) = I^n.
\]
If \( C \subset Q \) is a compactum then the components of the compactum \( p^{-1}(C) \) are the fibers \( p^{-1}(t) \), for \( t \in C \), and hence by (2.3) and Lemma 2.1 it follows that \( p^{-1}(C) \) embeds into some compactum \( S \) and hence, by the assumption, into some \( A \in \mathcal{A} \).

Lemma 1.1 yields thus an existence of a \( t \in L \setminus \emptyset \) such that the fiber \( p^{-1}(t) \) embeds into some \( A \in \mathcal{A} \) and this completes the proof by (2.4).

Remark 2.3. (a) Theorem 2.1 can be considered as a generalization of the well-known (and easy to prove) fact that each \( G_\delta \)-set containing the subspace of the Hilbert cube consisting of the points with all but finitely many coordinates equal to zero, contains the Hilbert cube topologically [A–P; Ch. 10, § 7], [E2].

(b) Theorem 2.2 implies that there is no universal space in the class of all weakly infinite-dimensional compacta; cf. Smirnov [S] where the compacta \( S \) were used in a proof that there is no universal space in the class of countable-dimensional compacta.

(c) Theorem 2.3 yields also the following corollary:

Let \( X \) be a complete space such that the cone over each \( \sigma \)-compact subset of \( X \) has a continuous injection into \( X \); then \( X \) contains the Hilbert cube topologically.

Indeed, the assumption allows one to embed in \( X \) step by step, all Smirnov's compacta.

3. Transfinite polytopes. We shall call a compactum \( X \) a transfinite polytope if \( X \) is a disjoint union of at most countably many finite polytopes. In other words, a transfinite polytope is a compactum which has at most countably many components and each of which is a finite polytope.

In particular, each Smirnov's compactum is a transfinite polytope and each transfinite polytope embeds into some Smirnov's compactum by Lemma 2.1.

Now, we shall describe a certain kind of approximation of an arbitrary compactum by transfinite polytopes which we use in the proof of Theorem 3.2, a main result of this section.

Let us fix an enumeration \( q_1, q_2, \ldots \) of the subset \( Q \) of the Cantor cube \( 2^n \) consisting of the points all but finitely many of whose coordinates are equal to zero.

Throughout this section \( X \) denotes a compactum and \( f_j: X \to W_j \) are continuous \( 1/\ell \)-maps (i.e. \( \text{diam}(f_j^{-1}(y)) \leq 1/\ell \) for \( y \in W_j \)) onto finite polyhedra \( W_j \).

We denote by \( X(f_1, f_2, \ldots) \) the compactum obtained from the product \( 2^n \times X \) by attaching to each compactum \( (q_j)_i \times X \) the polyhedron \( W_j \) by the map \( f_j \).

More precisely, we consider in the compactum \( 2^n \times X \) the decomposition into the singletons \( \{(t, x)\} \), where \( t \notin Q \), and into the sets \( (q_j)_i \times f_j^{-1}(y) \), where \( y \in W_j \). This is an upper semicontinuous decomposition and the quotient space is the compactum \( X(f_1, f_2, \ldots) \).

Assigning to the equivalence class of the point \((t, x)\) the point \( t \) one defines a continuous "projection" \( p: X(f_1, f_2, \ldots) \to 2^n \).

such that (cf. (2.3) and (2.4))

\[
(3.2) \quad p^{-1}(q_i) = W_i \quad \text{for} \quad q_i \in Q,
\]

\[
(3.3) \quad p^{-1}(t) = X \quad \text{for} \quad t \in 2^n \setminus Q.
\]

Let

\[
(3.4) \quad Q = \{C \subseteq Q: C \text{ is a compactum}\}.
\]

The objects we are interested in are the transfinite polytopes

\[
(3.5) \quad X(C) = p^{-1}(C), \quad C \subseteq \mathcal{Q}.
\]

**Proposition 3.1.** The space \( X \) is countable-dimensional if and only if \( \lambda(f_1, f_2, \ldots) = \sup \{\text{ind} X(C): C \subseteq \mathcal{Q}\} < \omega_1 \). The same is true for Ind and one can also replace "countable-dimensional" by "weakly infinite-dimensional" and "ind" by "index" in this statement.

**Proof.** If \( X \) is countable-dimensional then so is \( X(f_1, f_2, \ldots) \) and then \( \lambda(f_1, f_2, \ldots) \leq \sup \{\text{ind} X(C): C \subseteq \mathcal{Q}\} < \omega_1 \). To prove the converse, put \( \mathcal{K} = X(f_1, f_2, \ldots), L = 2^n, \mathcal{A} = \{A \in \mathcal{H}: \text{ind} A \leq \lambda(f_1, f_2, \ldots)\} \) and apply Lemma 1.1, using (3.2), (3.3) and Lemma 5.4 in § 1.

If \( X \) is weakly infinite-dimensional then so is \( X(f_1, f_2, \ldots) \) (by the properties of the map \( p \) defined in (3.1) and a theorem in [Le]) and hence \( \lambda(f_1, f_2, \ldots) \leq \sup \{\text{ind} X(f_1, f_2, \ldots)\} \), since the index is monotone (Lemma 3.5 in § 1). To prove the converse one can repeat the argument for Ind, because the set \( \{A \in \mathcal{H}: \text{ind} A \leq \lambda\} \) is analytic (see see sec. 4 in § 1).

**Theorem 3.2.** There exists a family \( \mathcal{T} \) of transfinite polytopes such that

\[
\sup \{\text{index} T: T \in \mathcal{T}\} < \sup \{\text{ind} T: T \in \mathcal{T}\} = \omega_1.
\]

**Proof.** Let \( X \) be a weakly infinite-dimensional compactum which is not countable-dimensional [F] and let

\[
(3.6) \quad \mathcal{T} = \{X(C): C \subseteq \mathcal{Q}\},
\]

where \( \mathcal{Q} \) is defined by (3.4) and \( X(C) \) by (3.5). The family \( \mathcal{T} \) has the required properties by Proposition 3.1.

Remark 3.3. (a) In the proof of Theorem 3.2 we used a compactum \( X \) which is weakly infinite-dimensional but not countable-dimensional to produce the family \( \mathcal{T} \). Conversely, given such a family \( \mathcal{T} \) one can construct a compactum \( X \) with these properties using Theorem 5.1 in § 1. It seems interesting (but also difficult) to obtain the family \( \mathcal{T} \) in Theorem 3.2 in a more explicit way; cf. Remark 2.4 in § 3.

(b) Theorem 3.2 and Theorem 5.1 in § 1 yield an existence of a weakly infinite-dimensional compactum containing compact subspaces with arbitrarily large transfinite dimensions (one can take for this purpose the compactum \( X(f_1, f_2, \ldots) \) considered in the proof of Theorem 3.2).
EXAMPLE 3.4. (a) Let $X$ be a continuum which is countable-dimensional but not finitely-dimensional (for example, let $X$ be the continuum $\mathcal{H}_t$ defined in sec. 1 §3) and let us consider the family of transfinite polytopes $X(C)$ defined by (3.5). Then $\text{sup} \{\text{Ind}(X(C)): C \in \mathcal{Q}\} < \alpha_0$ and $\text{sup} \{\text{index}(X(C)): C \in \mathcal{Q}\} < \alpha_0$, by Proposition 3.1 but there is no $\alpha < \alpha_0$ such that all compacta $X(C)$ can be embedded into the Smirnov’s compactum $\mathcal{S}_\alpha$, compare with Lemma 2.1.

Indeed, if there were such an $\alpha$ then the reasoning in the proof of Proposition 3.3 with $\alpha = \mathcal{S}_\alpha$ and $a = A \in \mathcal{H}_t$ would lead us to the conclusion that $X$ embeds into $\mathcal{S}_\alpha$, which is false.

(b) Let $X = K \times K \times \ldots$ be the countable product of the pseudo-arc $[K]$; (§ 48, X]. The compactum $X$ does not contain the Hilbert cube topologically and for every $i = 1, 2, \ldots$ there is a 1-1-map $f_i: X \rightarrow I^i$ of $X$ onto the $i$-dimensional cube. Let, as in (a), the transfinite polytopes $X(C)$ be defined by (3.5). The components of each $X(C)$ are finite-dimensional cubes (see (3.2)) and the transfinite dimensions and the index are unbounded over the family $\{X(C): C \in \mathcal{Q}\}$ (by Proposition 3.1, since $X$ is strongly infinite-dimensional). However, the compactum $X(f_1, f_2, \ldots)$ which contains all members of this family does not contain the Hilbert cube topologically; this is a contrast to Theorem 2.2.

§ 3. On two questions of D. W. Henderson

1. The D. W. Henderson’s compacta. Henderson [He] defined AR-compacta $H_{1,1}, H_{2,1}, \ldots, H_{a,1}, a < \alpha_0$, and their boundaries $\partial H_a$ by transfinite induction in the following way (cf. sec. 2 in §2).

Let $H_1 = \emptyset, \partial H_1 = \partial I = \{0,1\}, p_1 = \{0\}$ and assume that for $\beta < \alpha$ the compacta $H_\beta$, their boundaries $\partial H_\beta$ and the points $p_\alpha \in \partial H_\alpha$ are defined. If $\alpha = \beta + 1$ then we let $H_{\beta+1} = H_\beta \times I, \partial H_{\beta+1} = (\partial H_\beta \times I) \cup (H_\beta \times \partial I)$, and $p_{\beta+1} = (p_\beta, p_1)$. If $\alpha$ is limit, let $K_\alpha$ be the union of $H_\beta$ and a half-open arc $A_\beta$ such that $A_\beta \cap H_\beta = \{p_\beta\}$ (end point of $A_\beta$). Define $H_\alpha$ to be the one-point compactification of the free union $\bigcup_\beta K_\beta$, $\partial H_\alpha = H_\alpha \cup \{H_\alpha \setminus \partial H_\alpha\}$ and let $p_\alpha$ be the compactifying point. Henderson called a continuous map $f: X \rightarrow H_\alpha$ essential if each continuous extension over $X$ of the restriction $f|f^{-1}(\partial H_\alpha)$ maps $X$ onto $H_\alpha$. In the case where $\alpha = < \alpha_0, H_\alpha$ is the $i$-dimensional cube, $\partial H_\alpha$ is its boundary and the notion of essential maps coincides with the classical one $\{A, P\}, [E_1].$

It was shown in [He] that if a countable-dimensional compactum $X$ admits an essential map onto $H_\alpha$ then $\text{Ind}(H) \geq \alpha$.

2. Results. The theorem below answers affirmatively the second question raised by Henderson in [He; p. 168].

THEOREM 2.1. A compactum which admits for every $\alpha < \alpha_0$ an essential map onto the Henderson’s compactum $H_\alpha$ is strongly infinite-dimensional.

This result follows immediately from the following lemma which we prove in the next section.

LEMMA 2.2. If a weakly infinite-dimensional compactum $X$ admits an essential map onto $H_\alpha$ then $\text{Ind}(X) \geq \alpha$.

The next theorem answers negatively the first question formulated by Henderson in [He; p. 168].

THEOREM 2.3. There exists a countable-dimensional compactum $X$ such that $\text{Ind}(X) = \alpha$ but $X$ does not admit any essential map onto the Henderson’s compactum $H_\alpha$, $a$ being an arbitrary countable ordinal.

Proof. Let $T$ be the family of countable polytopes described in Theorem 3.2 in § 2 and $\lambda = \text{sup} \{\text{index}(T): T \in T\}$. Then any $X \in T$ with $\text{Ind}(X) = \alpha > \lambda$ has the required property by Lemma 2.2.

REMARK 2.4. (a) The proof of Theorem 2.3 is based on an existence of the family $T$ which, in fact, we derived from an existence of an analytic set which is non-Borel (cf. the proof of the Hurewicz’s theorem given in [K1; § 43, VII]). In effect, we have only a vague idea how the compactum $X$ looks and we have no information on the least $\alpha$ for which such a compactum exists. As was pointed out in Remark 3.3 (a) in § 1, a more constructive method would be of interest.

(b) By virtue of Theorem 2.3 one can ask the following natural extension of the Henderson’s question: does there exist for each $\alpha < \alpha_0$ an AR-compactum $B_\alpha$ and its subcompact $\partial B_\alpha$ such that, with the definition of essential maps $f: X \rightarrow B_\alpha$ analogous to that in sec. 1, a countable-dimensional compactum $X$ admits an essential map onto $B_\alpha$ if and only if $\text{Ind}(X) \geq \alpha$.

We also do not know the situation if one replaces $\text{Ind}$ by index in the Henderson’s question.

3. Proof of Lemma 2.2. We shall define for each $\alpha < \alpha_0$ by transfinite induction a sequence of pairs of closed disjoint sets in $H_\alpha$:

$$
(3.1) \quad (A_1, B_1), (A_2, B_2), \ldots
$$

(3.2) a set $C_\alpha$ of closures of some components of $H_\alpha \setminus \partial H_\alpha$,

and for each $C \in C_\alpha$, a homemorphism

$$
(3.3) \quad h_C: C \rightarrow \mathcal{P}(C)
$$

in such a way that if we put

$$
(3.4) \quad M_\alpha = \{\sigma \in \text{Fin}_0: \text{ there is } C \in C_\alpha \text{ such that the pairs } (h_C(A_i \cap C), h_C(B_i \cap C)) \text{ are distinct opposite faces of the cube } \mathcal{P}(C) \text{ for } i \in \sigma\},
$$

then we have

$$
(3.5) \quad \text{type } M_\alpha \equiv \alpha
$$

(recall, that we consider $\text{Fin}_0$ with the order introduced in sec. 1 in § 1).
Observe, that $M_\alpha$ is well-ordered, since for a separating sequence $\mathcal{Q} = (X_i, Y_i): i \in \omega$ in $H_\alpha$ with $X_i = A_i$ and $Y_i = B_i$ we have $M_\alpha \leq M_\alpha(X)$ and the compactum $M_\alpha$ is weakly infinite-dimensional (see sec. 3 in §1).

The construction is evident for $\alpha = 1$. Assume that the objects are defined for $\alpha$ and put (cf. the definition of $H_{\alpha+1}$ in sec. 1):

- $C_{\alpha+1} = \{C \times I: C \in C_\alpha\}$ and $h_{\alpha+1} = h_{\alpha} \times \text{id}_I$,
- $A_{\alpha+1}^1 = H_\alpha \times \{0\}$, $B_{\alpha+1}^1 = H_\alpha \times \{1\}$,
- $A_{\alpha+1}^0 = A_\alpha \times I$, $B_{\alpha+1}^0 = B_\alpha \times I$, $i = 1, 2, ...$

Let us check that type $M_{\alpha+1} \geq \alpha + 1$, where $M_{\alpha+1}$ is defined by (3.4).

For every $\sigma \in M_\alpha$ let $\sigma^* = \{i + 1: i \in \sigma \} \cup \{1\}$ and $M_\sigma^* = \{\sigma^*: \sigma \in M_\alpha\}$. Observe, that $\sigma^* \in M_{\alpha+1}$ (see (3.4)) and that the injection $\sigma \mapsto \sigma^*$ preserves the order, and hence type $M_{\alpha+1}^\sigma = \text{type } M_\sigma$, by the inductive assumption (see (3.5)). Now, $\tau = \{1\}$ is an element of $M_{\alpha+1}$ greater than each of the elements of $M_\alpha$ and therefore type $M_{\alpha+1} > \text{type } M_\sigma^\tau + 1 = \alpha + 1$, as required.

Assume now that $\alpha$ is a limit ordinal and that the construction is performed for each $\beta < \alpha$. To make the 4th step we need only enumerate in an appropriate way the objects we have constructed. Let us split $\omega$ into disjoint infinite sets $N_i$, for $\beta < \alpha$ let us fix for each $\beta$ an order-preserving enumeration $f_\beta: \omega \to N_i$, and let us put $(A_i, B_i) = (A_{\beta}^f, B_{\beta}^f)(0)$ for $i \in N_i$, and $C = \bigcup C_\alpha$ (cf. the definition of $H_\alpha$ in sec. 1). Then $M_\alpha$ contains for each $\beta \leq \alpha$ the set $D_\beta = \{j(\sigma): \sigma \in M_\beta\}$ similar to $M_\beta$ and hence type $M_\alpha \geq \text{sup}\{\text{type } M_\beta: \beta < \alpha\}$.

Having completed the construction of the objects (3.1)–(3.5), we are now in a position to prove the lemma.

Let $f: X \to H_\alpha$ be an essential map of the weakly infinite-dimensional compactum $X$ and let $\mathcal{Q} = (A_i, B_i): i \in \omega$ be a separating sequence in $X$ (see sec. 3 in §1). By (3.5) it is enough to find an order-preserving injection of $M_\alpha$ into $M_\alpha(X)$ (see (3.3) in §1). Let us choose (see (3.2) in §1) a sequence $j(\sigma) < (j(\bar{\sigma})) < c$ of natural numbers such that (see (3.3))

\[ f^{-1}(A_\alpha^f) = A_{\alpha_0} \quad \text{and} \quad f^{-1}(B_\alpha^f) = B_{\alpha_0}. \]

Since the correspondence $\sigma \mapsto j(\sigma)$ is an order-preserving injection we have only to verify that

\[ j(\sigma) \in M_\alpha(X), \quad \text{whenever} \quad \sigma \in M_\alpha. \]

If $\sigma \in M_\alpha$ then there exists a $C \in C_\alpha$ such as in (3.4). Now, Henderson [Hc; Proposition 3] has shown that $f$ is also an essential map when restricted to the set $Y = f^{-1}(C)$ and so the map $g = h + f^{-1}(Y): Y \to H_\alpha$ (see (3.3)). By the choice of the set $C$, the pairs $(f^{-1}(A_\beta \cap C), f^{-1}(B_\beta \cap C))$, where $i \in \sigma$, are the inverse images by $g$ of distinct pairs of the opposite faces of the cube $\mathbb{I}^\alpha$. Hence they form an essential family in the space $Y$, since $g$ is essential. It follows by (3.6) that the family $(A_{\beta_0}, B_{\beta_0}): i \in \omega$ is essential which completes the proof of (3.7).

§ 4. Universal functions for the families of compacta with transfinite dimension $\leq \alpha$

1. Results.

**Theorem 1.1.** For each $\alpha < \omega_1$ there exists a continuous function $\Phi: \omega^\alpha \to H$ of the irrationals into the hyperspace of the Hilbert cube such that:

1. $X \in H$ and $\text{Ind } X \leq \alpha$ then $X = \Phi(f_i)$ for some $i \in \omega^\alpha$,
2. $\text{Ind } X = \alpha$, where $G_\alpha = \{(x, t): x \in \Phi(f_i) \times \alpha^\alpha \times \mathbb{I}^\alpha\}$.

**Theorem 1.2.** For each $\alpha < \omega_1$ there exists a continuous function $\Phi^*: \omega_1 \to H$ such that:

1. $X \in H$ and $\text{Ind } X \leq \alpha$ then $X = \Phi^*(f_i)$ for some $i \in \omega_1$,
2. If $C \in C_\alpha = \{(t, x): x \in \Phi(t)\}$ is a compactum then $\text{Ind } C \leq \alpha$.

Since each of the complete spaces $G_\alpha$ and $G_\alpha^*$ has a countable-dimensional compactification $X_\alpha$ and $X_\alpha^*$ respectively (see footnote [*]), we also have the following corollary.

**Corollary 1.3.** For each $\alpha < \omega_1$ there exist countable-dimensional compacta $X_\alpha$ and $X_\alpha^*$ such that $X_\alpha$ contains topologically all compacta $S$ with $\text{Ind } S \leq \alpha$ and $X_\alpha^*$ contains topologically all compacta $S$ with $\text{Ind } S < \alpha$.

We do not know the least possible transfinite dimension $\text{Ind } X_\alpha$ and $\text{Ind } X_\alpha^*$ described in Corollary 1.3.

2. Remarks and questions. Given a continuous map $\Psi: \omega^\alpha \to H$ let us put

\[ G(\Psi) = \{(t, x): x \in \Psi(t)\}. \]

It is a reformulation of Question 6.1 whether always $\Psi(\Psi(i)) < \omega_1$?

**Question 2.1.** Assume that $\text{Ind } \Psi(i) < \omega_1$ for every $i \in \omega$. Does the space $G(\Psi)$ need to be countable-dimensional? (See: Added in proof).

**Remark 2.2.** There exists a map $\Psi$ such that $G(\Psi)$ is not countable-dimensional.

In fact, one can show that:

there exists an open map $g: Z \to C$ of a compact space $Z$ onto the Cantor set $C$ such that each fiber $g^{-1}(t)$ is countable-dimensional but $Z$ is not countable-dimensional.

Indeed, there exists a compact space $X$ and a continuous map $f: X \to C$ onto the Cantor set such that each fiber $f^{-1}(t)$ is countable-dimensional but $X$ is not countable-dimensional [P; Comment B], and one can adopt easily a construction of Michael and Stone [M-St; Proof of Theorem 1.1] to define a compact space $Z = X \times \text{dim}(2^\omega, X) = 0$ and an open extension $g: Z \to C$ of the map $f$ (cf. also [E1; Problem 1.12, 1]). Now, given $g$ as above one can take $\Psi(t) = g^{-1}(t)$, since $\Psi$ is continuous and $G(\Psi)$ is homeomorphic with $Z$.

We do not know whether the map $\Psi$ provides a counterexample to Question 2.1 or whether the construction can be modified to produce such a counterexample.

**Remark 2.3.** One can define for each $\alpha < \omega_1$ a map $\Psi = \Psi_\alpha$ such that $\Psi_\alpha(i)$
is finite-dimensional for each $i$ but $\text{Ind} G(V_p) \supseteq \alpha$. Indeed, let $C_i$ be the space of the components of the Smirnov's compactum $S \subseteq F^*$ (see § 2 sec. 1). The quotient map $\pi: S \to C_i$ is open and hence the map $V_p: C_i \to H$ defined by the formula $V_p(t) = \pi^{-1}(t)$ is continuous. One can assume that $C_i \subseteq \omega^\alpha$ and then, if we put $V_p = V_p^* \circ r: C_i \to C_i$ being a retraction, the function $V_p$ has the desired property, because $G(V_p) \cap (C_i \times F^*) = S_i$ and $\text{Ind} S_i \supseteq \alpha$.

3. Proof. We shall prove only the first theorem; the second one can be proved in a similar way with some obvious modifications.

The proof is by transfinite induction on $\alpha$. For $\alpha = -1$ we put $\Phi(i) = \emptyset$ and let us assume that we have defined for every $\beta < \alpha$ a continuous map $\Phi: \omega^\alpha \to H$ satisfying the conditions (i) and (ii) in Theorem 1.1. The main step in the construction of $\Phi$ is the proof of the following statement which is a little bit weaker than what we need (compare the conditions (i) in this lemma and in Theorem 1.1).

LEMMA 3.1. There is a continuous function $\Phi: \omega^\alpha \to H$ such that:

(i) if $X \in H$ and $\text{ind} X \leq \alpha$ then $X \subseteq \Phi(i)$ for some $i \in \omega^\alpha$,

(ii) $\text{Ind} G = \alpha$, where $G = \{(x, t): x \in \Phi(i)\} \subseteq \omega^\alpha \times F$.

Proof. Let $p_i: I_i \to I_i$ be the projection of the Hilbert cube onto the $i$th axis and let

$$E_i = p_i^{-1}(0), \quad F_i = p_i^{-1}(1).$$

Let us split $\omega$ into disjoint infinite sets $N_1, N_2, \ldots, N_\alpha$, for $\beta < \alpha$ and let $\beta$ be the set of all $X \in H$ satisfying the following conditions:

(i) if $x \in X \cap F$, $F \subseteq X$ being closed, then $x \in E_i$ and $E_i \subseteq F_i$ for some $i \in \omega$,

(ii) if $i \in N_\alpha$ then there exists a partition $L$ in $\omega^\alpha$ between $E_i$ and $F_i$ such that $\text{ind}(X \cap L) \leq \beta$.

We shall verify that

(i) if $E \subseteq \omega^\alpha$ then $\text{Ind} E \leq \alpha$,

(ii) if $K$ is a compactum with $\text{ind} K \leq \alpha$ then $K$ is homeomorphic to some $X \subseteq E$.

The assertion (3.4) follows from the observation that $\{X \in H: X$ satisfies (3.3)\} $= H \setminus \bigcup_j \{X \in H \setminus C_j\}$, see proof of Lemma 5.4 in § 1. (3.3) follows from (3.3) in § 1.

The assertion (3.5) is obvious.

To prove the last assertion (3.6) let us fix in a compactum $K$ with $\text{ind} K \leq \alpha$ a separating sequence $\{(A_i, B_i): i \in \omega\}$, see (3.1) in § 1. Let $\omega^\alpha$ be the set of all $i \in \omega$ such that there is a partition $L$ in $K$ between $A_i$ and $B_i$ with $\text{ind} L = \beta < \alpha$.

Let $\beta(i)$ be the smallest such $\beta$ for $i \in \omega$. Let $\varphi: \omega^\alpha \to \omega$ be an injection such that $\varphi(i) \in N_{\alpha(i)}$ and let, for every $i \in \varphi^\alpha$, $F_i: K \to I_{\alpha(i)}$ be a continuous map such that $A_i = f_i^\alpha(0)$ and $B_i = f_i^\alpha(1)$. The diagonal map $f = \{(i, \varphi(i))\}$ maps the compactum $K$ onto the subspace $X = f(K)$ of the product $\prod I_{\alpha(i)}$ which we identify with the subspace of the Hilbert cube $\prod I_i$ consisting of the points $(s_i)$ all whose coordinates $x_i$ with $i \neq \varphi(i)$ are zero. Let us verify that $X \subseteq E$. For every $x \in K \cap F_i$ we have the neighborhoods $x \in A_i$, and $x \in B_i$, and we have then $h(s) = f_i^\alpha(A_i) = X \cap I_{\alpha(i)}$ and $f_i^\alpha(B_i) = \varphi_i \cap \alpha(i)$, Hence (3.2) holds and $f$ is a homomorphism onto $X$. To check (3.3), let $j = \varphi(i) \in N_{\alpha(i)}$ (if $j \neq \varphi(i)$ then $E_j \cap X = \emptyset$ and we have nothing to do) and let $L = A_i$ be a partition in $K$ between $A_i$ and $B_i$ such that $\text{ind} L \leq \beta(i)$. The partition $f_i^\alpha(L)$ in $X$ between $A_i$ and $B_i$ (see above) can be extended to a partition $L$ in $I_i$ between $E_i$ and $F_i$, and since $\text{Ind}(X \cap L) = \text{Ind} F_i = \alpha$, we are done.

Let us consider now the space $S$ of all sequences

$$(3.7) \quad (X, Y_{-1}, Z_{-1}, Y_{0}, Z_{0}, Y_{1}, Z_{1}, \ldots, t_{-1}, t_{0}, t_{1}, \ldots) \in H \times H \times \ldots \times \omega^\alpha \times \omega^\alpha \times \ldots$$

satisfying the following three conditions:

$$(3.8) \quad X \subseteq E,$$

$$(3.9) \quad Y_i \cup Z_i = I_i^\alpha, \quad E_i \subseteq Y_i, \quad F_i \subseteq Z_i, \quad (E_i \cup F_i) \cap (Y_i \cap Z_i) = \emptyset,$$

$$(3.10) \quad i \in \omega^\alpha, \quad X \cap Y_i \cap Z_i = \Phi(i),$$

Since the set $S$ is analytic and the maps $\Phi(i)$ are continuous, it is routine to check that the set $S$ is analytic, cf. [K1; § 43]. Thus there exists a continuous parametrization of the set $S$ by irrationals.

$$(3.11) \quad x \mapsto (X(s), Y_{-1}(s), Z_{-1}(s), Y_{0}(s), Z_{0}(s), Y_{1}(s), Z_{1}(s), \ldots, t_{-1}(s), t_{0}(s), t_{1}(s), \ldots)$$

We shall verify that the function we are looking for is

$$(3.12) \quad \Phi(s) = X(s).$$

The property (i) follows from (3.6) and the fact that $\Phi(\omega^\alpha) = E$ which is a consequence of the inductive assumption about $\Phi$.

Thus, it remains to show that

$$(3.13) \quad \text{Ind} G = \alpha, \quad \text{where} \quad G = \{(x, t): x \in X(s)\}.$$

At first let us show that

$$(3.14) \quad i \in \omega^\alpha, \quad i \neq \varphi(i) \text{ then there exists a partition } L \text{ in } \omega^\alpha \times I_i \text{ between } \omega^\alpha \times E_i \text{ and } \omega^\alpha \times F_i = D_i \text{ such that } \text{ind}(G \cap L) \leq \beta.$$

For this purpose fix an $i \in \omega^\alpha$ and put

$$(3.15) \quad Y = \{(x, t): x \in Y_i(s)\}, \quad Z = \{(x, t): x \in Z_i(s)\}.$$

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Both sets $Y$ and $Z$ are closed subsets of $T$ such that (see (3.9)) $Y \cup Z = T$, $C_I = Y$, $D_I = Z$, $(C_I \cup D_I) \cap (Y \cap Z) = \emptyset$ and hence $L = Y \cap Z$ is a partition in $T$ between $C_I$ and $D_I$. To check that $\text{ind}(G \cap T) \leq \beta$ observe that by (3.10)-(3.12) we have

$$G \cap L = \{(x, x) : x \in \Phi_I(x_I)\},$$

and therefore, having in mind the properties of $\Phi_I$, we need only the following simple fact:

**Sublemma 3.2.** Let $M \subset \omega^\omega \times X$ be a closed set, let $t: \omega^\omega \to \omega^\omega$ be a continuous map, and let $\bar{M} = \{(t, x) : (t, x) \in M\}$. Then $\text{ind} \bar{M} \leq \text{ind} M$.

This can be easily verified by induction on $\text{ind} M$, and so (3.14) follows.

Let us show how one obtains (3.13) from (3.14). Let $F = \omega^\omega \times X$ be a closed set, and let $p = (s, a) \in G \setminus F$. Since $X(s) \in F$ there is by (3.2) an $i \in a$ such that $a \in E_i$ and $(i) \times X(s(i)) \cap F \subset E_i$, and let $L$ be a partition such as in (3.14). Then for a sufficiently small open-and-closed neighbourhood $V$ of $s$ the set $L \cap (V \times F)$ is a partition in $\omega^\omega \times X$ separating the point $p$ from $F$. This completes the proof of Lemma 3.1.

We shall show now how one can modify the function $\Phi$ in Lemma 3.1 to construct the function $\Phi_I$ we are looking for.

Let $J = [-1, 2]$ and let $\Gamma$ be the space of all automorphisms of the cube $J^\omega$ endowed with the compact-open topology. Since the space $\Gamma$ is complete there is a continuous surjection $\gamma: \Gamma \to T$. Let us put

$$\Psi(t, s) = \gamma(t)(\Phi(s)), \quad \text{where } (t, s) \in \omega^\omega \times \omega^\omega.\tag{3.16}$$

Notice that

$$\text{ind} \Psi \leq \text{ind} \Phi_I \leq \text{ind} \Phi.$$

Indeed, by Lemma 3.1(i) there is $t \in \omega^\omega$ such that $X = \Phi(t)$, and since both compacta $X$ and $\Phi(t)$ are contained in the "pseudoexterior" ($-1, 2)\omega$ of the cube $J^\omega$ there is (see [BP; Lemma 1.3, p. 150]) an automorphism $h$ such that $X = h(\Phi(t))$ and hence $X = \Psi(t, s)$, where $\gamma(t) = h$.

Let us verify that

$$\text{ind} \Psi = \alpha \quad \text{where } G(\alpha) = \{(t, s, x) : x \in \Psi(t, s)\} \subset \omega^\omega \times \omega^\omega \times \omega^\omega.\tag{3.17}$$

This follows immediately from the following easy fact:

**Sublemma 3.3.** Assume that $M \subset \omega^\omega \times X$ is a closed set and let $\bar{M} = \{(t, x, s) : (t, x) \in M\}$. Then $\text{ind} \bar{M} \leq \text{ind} M$.

We shall prove this assertion by transfinite induction on $\text{ind} M$. The case $\text{ind} M = 0$ is evident. Assume that we have verified the assertion for all $\beta < \alpha$, and let $\text{ind} M = \alpha$. Let $p = (u, v, a) \in \bar{M} \setminus F$, $F \subset \bar{M}$ being closed. Let us put $H = F \cap \{(u, 0) \times X\}$ and let $\lambda(1, x, c) = (t, \gamma(t)(x))$. The map $\lambda: \bar{M} \to M$ is continuous and is injective with respect to the variable $x$. Thus $\lambda(p) \in \lambda(H)$ and hence

there exists a partition $L$ in $M$ between $\lambda(p)$ and the compact set $\lambda(H)$ such that $\text{ind} L < \alpha$. The set $\bar{M} = \lambda^{-1}(L) = \{(t, x, s) : t, \gamma(t)(x) \in L\}$ is a partition in $\bar{M}$ between $p$ and $H$, and by the inductive assumption $\text{ind} L < \alpha$. Since the set $\bar{M}$ is closed in the product $\omega^\omega \times \omega^\omega \times X$ there exists an open-and-closed neighbourhood $U$ of the point $(u, 0) \in \omega^\omega \times \omega^\omega$ such that $\bar{M} \cap (U \times F)$ is a partition in $\bar{M}$ between $p$ and $F$ and this ends the proof.

Now, to complete the proof of Theorem 1.1 it is enough to modify the function $\Psi$ slightly in the following way. Let us put $A = \Psi^{-1}(H)$ and let $r: \omega^\omega \times \omega^\omega \to A$ be a retraction onto the closed set $A$. Since one can identify the space $\omega^\omega \times \omega^\omega$ with $\omega^\omega$, it is easy to see that the function $\Phi_A = \Psi \circ r$ has the required properties (cf. (3.17), (3.18) and Sublemma 3.2).

**Remark 3.4.** The idea of the proof has points in common with the idea of the proof of a factorization theorem of Pasyk and given in [A-P; Appendix] (although formally, the subjects seem far). The idea of continuous parametization of families of compact sets in the construction of special sets goes back to Mazurkiewicz [M].

**Added in proofs.** 1. P. Borst solved independently the first question of [B] constructing a space $X$ with $\text{Ind} X = \omega+1$ and without any essential map onto $\mathbb{F}_{\omega+1}$ and J. Dijkstra modified Borst's construction to obtain a compactum with these properties; see J. Dijkstra, Topics in dimension theory, Proc. Fifth Prague Top. Symp. 1981, Berlin 1982, Theorem 2. Notice that in Theorem 2.3 in § 3 the gap between $\text{Ind} X$ and a for which $X$ has no essential map onto $\mathbb{F}_{\omega+1}$ can be arbitrarily large.

2. Question 2.1 in § 4 has a negative answer; in fact the map $\Psi$ considered in Remark 2.2 § 4 provides a counterexample, see Proc. Fifth Prague Top. Symp. 1981, p. 556.

3. One can prove that if $\Omega$ is an upper semicontinuous decomposition of a compactum $X$ into compacta which are countable unions of finite-dimensional compacta, then $\sup(\text{Ind} A: A \subset X) < \omega_0$; cf. Question 6.1 in § 1.

**References**


Spaces defined by topological games, II

by

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Abstract. The paper reports some results on the game \( G(K, X) \) introduced in [7]. The main results contain the following: 1. 'The space favorable for Player I' is the union of countably many \( K \)-scattered subsets. 2. Reduction theorems for actions of Player I. 3. Covering characterization of the spaces favorable for Player II. 4. Indeterminacy of the game in ZFC.

The main object of this work is the topological game \( G(K, X) \), so the present paper is a continuation of [7]. Some of the results included here were announced earlier in [8] and [9]. The game \( G(K, X) \) was used recently for proving general sum theorems for the dimension dim by the author and Y. Yajima [10] and for the dimension ind by Y. Yajima [12]. Furthermore, a general product theorem for paracompact spaces involving that game was established by Y. Yajima in [13]. Section 1 contains the following: if Player I has a winning strategy in \( G(K, X) \), then \( X \) is the union of countably many \( K \)-scattered subsets. In sections 2 and 3 there are introduced auxiliary games \( G^4(K, X) \) and \( G^5(K, X) \) in order to prove reduction theorems concerning the actions of Player I. Section 4 introduces a convenient equivalent form of the game \( G(K, X) \), denoted by \( G^6(K, X) \). A modification of that game involving \( G_2 \)-sets and thus denoted by \( G^7(K, X) \) is studied in section 5. The dual game \( G^8(K, X) \) to the game \( G^9(K, X) \) is introduced in section 6; it provides, as a by-product, a covering characterization of spaces favorable for Player II. Finally, in section 7, the indeterminacy of \( G(K, X) \) in ZFC is established.

For the topological background and undefined notions we refer to R. Engelking's monograph [1]. Each space considered here is assumed to be completely regular. \( N \) denotes the set of positive integers. \( 2^X \) denotes the family of closed subsets of the space \( X \). \( K \) denotes a class of spaces such that (i) \( K \) contains all singletons, and (ii) \( K \) is invariant with respect to closed subspaces, i.e., \( X \in K \) implies \( X \in K \). \( I, C, F \) and \( D \) denote the classes of all singletons, finite spaces, compact spaces, and discrete spaces respectively. \( DK, LK \) and \( SK \) denote the classes of spaces being free unions of spaces from \( K \), locally \( K \), and \( K \)-scattered, respectively. Despite the notation used in [7], \( J(K, X) \) denotes the following statement: Player I (Player II, resp.) has a winning strategy in \( G(K, X) \). For the modifications

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