On classification of weakly infinite-dimensional compacta

by

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Abstract. A classification of weakly infinite-dimensional compacta by means of a Lusin–Sierpiński index in the hyperspace of the Hilbert cube is given. This classification is applied to answer two questions of D. W. Henderson concerning the transfinite inductive dimension Ind and essential maps onto "transfinite cubes". For each $\alpha < \omega_1$ a countable dimensional compactum is constructed which contains topologically all compacta with transfinite dimension not greater than α.

Introduction

1. Terminology and notation. In this paper we consider only separable metrizable spaces, and a compactum means a compact space. Our terminology concerning analytic set theory follows [K1], and the terminology related to dimension theory follows [A–P] and [E1].

A *partition* L in a space X between two disjoint sets A and B in X is a closed set such that $X \setminus L = U \cup V$, where U and V are disjoint open sets with $A \subset U$ and $B \subset V$.

Throughout this paper ω denote the set of natural numbers, I the unit interval $[0, 1]$, I^ω the Hilbert cube, ω^ω the Baire space, i.e. topologically the irrationals, and 2^ω the Cantor cube $\{0, 1\}^\omega$.

We denote by $\underline{H}$ the hyperspace of the Hilbert cube, i.e. the space of all closed subsets of I^ω endowed with the topology induced by the Hausdorff distance.

Given a linearly ordered set M we denote the order type of M by type M. The symbol $X \underset{\text{top}}{=} Y$ means that the spaces X and Y are homeomorphic.

2. Countable-dimensional spaces and transfinite dimensions. A space X is *countable-dimensional* if $X = \bigcup_i X_i$ with X_i zero-dimensional.

The *transfinite dimensions* ind and Ind are the ordinal-valued functions obtained by the extension of the classical notions of the small and large inductive dimension respectively, by transfinite induction, i.e. for example, $\operatorname{Ind} X \leqslant \alpha$ if for each pair (A, B) of closed disjoint sets in X there is a partition L in X between A and B such

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that $\operatorname{Ind} L \leqslant \alpha$, α being an ordinal [H–W; p. 50], [A–P; Ch. 10, § 1], [N2; Ch. VI]; a comprehensive survey of the topic is [E2].

The transfinite dimension ind or Ind is defined for a complete space X if and only if X is countable-dimensional, and then $\operatorname{ind} X \leqslant \operatorname{Ind} X$ and these ordinals are countable.

There exists a function φ which maps the set of countable ordinals ω_1 into itself such that for each countable-dimensional compactum X, $\operatorname{ind} X \leqslant \operatorname{Ind} X \leqslant \varphi(\operatorname{ind} X)$ and thus, although the exact relations between the transfinite dimensions ind and Ind are very interesing [Lu], globally both of the functions provide essentially the same classification of the family of countable-dimensional compacta.

3. Weakly infinite-dimensional compacta. A compactum X is *weakly infinite-dimensional* if for each infinite sequence (A_1, B_1), (A_2, B_2), ... of pairs of closed disjoint sets in X there are partitions L_i in X between A_i and B_i such that $\bigcap_i L_i = \emptyset$ [A–P; Ch. 10, § 47], [N1], [N2; Ch. VI]. We call compacta which are not weakly-infinite dimensional *strongly infinite-dimensional*; strongly infinite-dimensional compacta can also be characterized as the compacta which have an essential map onto the Hilbert cube, see footnote (1).

Hurewicz [H–W; Ch. IV, § 6 (A)] (cf. also [A; § 4]) proved that countable-dimensional compacta are weakly infinite-dimensional. There exist, however, quite natural examples of weakly infinite-dimensional compacta which are not countable-dimensional [P].

4. Results. In § 1 we assign in a natural way to each weakly infinite-dimensional compactum X a countable ordinal index X which is a topological invariant. The index can be interpreted as a Lusin–Sierpiński index when one considers compacta as the points in the hyperspace of the Hilbert cube. Thus the classification of the weakly infinite-dimensional compacta by means of the index is quite regular from the point of view of analytic set theory; we do not know how regular the classification of the countable-dimensional compacta by means of the transfinite dimensions is from this point of view. The index is bounded over each family of countable-dimensional compacta with bounded transfinite dimensions. The converse is not true: there exists $\lambda < \omega_1$ such that $\sup\{\operatorname{ind} X\colon X$ is a countable-dimensional compactum with index $X \leqslant \lambda\} = \omega_1$. This result is obtained by a certain kind of approximation of an arbitrary compactum by compacta which are countable disjoint unions of finite polytopes (we call them "transfinite polytopes"), which we discuss in § 2. In § 3 we use the index and the result we have just mentioned to answer two questions raised by Henderson [He] concerning essential maps onto "transfinite cubes" defined by him. It seems worthwhile to notice that, when answering one of these questions, we prove an existence of a compactum with certain "transfinite dimensional" properties using an existence of an analytic non-Borel set, and we do not know how to do this in more explicit way. In the last paragraph, which is independent of the rest of the paper, we construct for each $\xi < \omega_1$ a countable-dimensional compactum X_ξ containing topologically all compacta S with $\operatorname{ind} S \leqslant \xi$ (we do not know what the ordinal $\operatorname{ind} X_\xi$ is). The construction is based upon a classical idea of a universal function for a given family of sets.

Generally speaking, the common idea underlying this paper is the investigation of the collection of weakly infinite-dimensional compacta, considered as the subset of the hyperspace of the Hilbert cube, using some classical methods and concepts of analytic set theory in the form given by Kuratowski in Topology [K1].

§ 1. The Lusin–Sierpiński index of a weakly infinite-dimensional compactum

1. The Brouwer-Kleene order. We shall consider the set $\operatorname{Fin}\omega$ of all nonempty finite subsets of ω with the order $\prec$ inverse to the lexicographic order, i.e. $\sigma \prec \tau$ means that there is an $n \in \omega$ such that $\sigma \cap \{1, ..., n-1\} = \tau \cap \{1, ..., n-1\}$ and $n \in \sigma \setminus \tau$.

In the sequel we use the following well-known property of the order $\prec$ [K–M; Ch. X, § 7, Corollary 4].

LEMMA 1.1. *Given a decreasing sequence $\sigma_1 \succ \sigma_2 \succ ...$ of elements of* $\operatorname{Fin}\omega$, *there exists an increasing sequence $j(1) < j(2) < ...$ of natural numbers such that for every k there exists an m with $\{j(1), ..., j(k)\} \subset \sigma_m$.*

2. Essential families. We say that a family $\mathscr{A} = \{(A_i, B_i)\colon i \in J \subset \omega\}$ of pairs of closed disjoint subsets of a compactum X is essential if for arbitrary partitions L_i in X between A_i and B_i the intersection $\bigcap\{L_i\colon i \in J\}$ is nonempty (cf. [Z; Definition 7]). Thus strongly infinite-dimensional compacta are the compacta which have an infinite essential family.

If we assign to each pair (A_i, B_i) from the family $\mathscr{A}$ a continuous map $f_i\colon X \to I$ such that $f_i(A_i) = 0$ and $f_i(B_i) = 1$, then the diagonal map $(f_i)_{i \in J}\colon X \to I^J$ is essential if and only if the family $\mathscr{A}$ is essential [A–P; Ch. 3, § 5, Ch. 10, § 4], [E1; Problem 1.9.A] (1).

3. The index. Let X be a compactum. We say that a sequence

$$\mathscr{S} = \{(A_i, B_i)\colon i \in \omega\} \tag{3.1}$$

of pairs of closed disjoint sets in X is separating if for each pair (A, B) of disjoint closed sets in X

$$A \subset A_i \quad \text{and} \quad B \subset B_i \quad \text{for infinitely many indices } i \text{ (2)}. \tag{3.2}$$

It is easy to construct a separating sequence in X considering finite sums of elements of an arbitrary base of X.

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(1) A map $f\colon X \to I^J$ is essential, where $J \subset \omega$ is infinite, if for each finite $\sigma \subset J$ the composition $p_\sigma \circ f\colon X \to I^\sigma$ of the map with the projection $p_\sigma\colon I^J \to I^\sigma$ is essential in the classical sense [A–P], [E1].

(2) Cf. [A–P; Ch. 10. § 7, Lemma 2 and Appendix, Lemma 3].

Given a separating sequence $\mathscr{S}$ in X let us put

(3.3) $$M_{\mathscr{S}}(X) = \{\sigma \in \operatorname{Fin}\omega : \text{the family } \{(A_i, B_i): i \in \sigma\} \text{ is essential}\}.$$

We shall consider the set $M_{\mathscr{S}}(X)$ with the order $\prec$ defined in section 1.

LEMMA 3.1. *Let $\mathscr{S} = \{(A_i, B_i): i \in \omega\}$ and $\mathscr{S}' = \{(A_i', B_i'): i \in \omega\}$ be two separating sequences in X. Then the ordered set $M_{\mathscr{S}'}(X)$ is similar to a subset of the ordered set $M_{\mathscr{S}}(X)$, and vice versa.*

Proof. Choose, using (3.2), a sequence $j(1)<j(2)<\ldots$ such that $A_i' \subset A_{j(i)}$ and $B_i' \subset B_{j(i)}$. If $\sigma \in M_{\mathscr{S}'}(X)$ then $j(\sigma) \in M_{\mathscr{S}}(X)$, for, if $L_{j(i)}$ is a partition in X between $A_{j(i)}$ and $B_{j(i)}$ then this is also a partition between A_i' and B_i' and hence $\bigcap\{L_{j(i)}: i \in \sigma\} \neq \emptyset$, since $\sigma \in M_{\mathscr{S}'}(X)$. The map $\sigma \to j(\sigma)$ is thus an order-preserving embedding of $M_{\mathscr{S}'}(X)$ into $M_{\mathscr{S}}(X)$. The symmetric argument proves the converse.

LEMMA 3.2. *A compactum X is weakly infinite-dimensional if and only if for some (equivalently — for each) separating sequence $\mathscr{S}$ in X the set $M_{\mathscr{S}}(X)$ is well-ordered.*

Proof. Let $\mathscr{S}$ be a separating sequence (3.1). Assume that X is strongly infinite-dimensional and let $\{(E_i, F_i): i \in \omega\}$ be an infinite essential family in X. By (3.2) one can choose numbers $j(i)<j(2)<\ldots$ such that $E_i \subset A_{j(i)}$ and $F_i \subset B_{j(i)}$. Then $\sigma_i = \{j(1), \ldots, j(i)\} \in M_{\mathscr{S}}(X)$ and $\sigma_1 \succ \sigma_2 \succ \ldots$, i.e. $M_{\mathscr{S}}(X)$ is not well-ordered.

Conversely, assume that there exist $\sigma_i \in M_{\mathscr{S}}(X)$ such that $\sigma_1 \succ \sigma_2 \succ \ldots$ and let $j(1)<j(2)<\ldots$ be a sequence such as in Lemma 1.1. The infinite family $\{(A_{j(i)}, B_{j(i)}): i \in \omega\}$ is then essential. Indeed, given partitions L_i in X between $A_{j(i)}$ and $B_{j(i)}$ and an arbitrary $k \in \omega$ one can find an m such that $\{j(1), \ldots, j(k)\} \subset \sigma_m \in M_{\mathscr{S}}(X)$ and hence $\bigcap\{L_{j(i)}: i \leqslant k\} \neq \emptyset$, i.e. $\bigcap\{L_{j(i)}: i \in \omega\} \neq \emptyset$ by compactness of X.

The statement in parenthesis follows from Lemma 3.1.

The two lemmas justify the following definition.

DEFINITION 3.3. Let X be a weakly infinite-dimensional compactum. Then the order type of the set $M_{\mathscr{S}}(X)$ is a countable ordinal which is independent of the choice of the separating sequence $\mathscr{S}$ in X, i.e. this is a topological invariant. We define

(3.4) $$\operatorname{index} X = \operatorname{type} M_{\mathscr{S}}(X)$$

and call index X the *Lusin–Sierpiński index of* X [3].

Remark 3.4. The index can be also defined by means of maps into I^ω. Given a continuous map $f\colon X \to I^\omega$ of a compactum X put $M(f) = \{\sigma \in \operatorname{Fin}\omega : p_\sigma \circ f$ is essential$\}$, where $p_\sigma\colon I^\omega \to I^\sigma$ is the projection (cf. footnote [1]); then $\operatorname{index} X = \sup\{\operatorname{type} M(f): f\colon X \to I^\omega\}$, provided that X is weakly infinite-dimensional. This easily follows by a remark in section 2.

[3] The terminology is explained in the next section.

We finish this section with a simple observation which yields monotonicity of index.

LEMMA 3.5. *Let $\mathscr{S} = \{(A_i, B_i): i \in \omega\}$ be a separating sequence in a compactum Y and let X be a compact set in Y. Then $\mathscr{S}' = \{(A_i \cap X, B_i \cap X): i \in \omega\}$ is a separating sequence in X and $M_{\mathscr{S}'}(X) = \{\sigma \in M_{\mathscr{S}}(X)$: the family $\{(A_i \cap X, B_i \cap X): i \in \sigma\}$ is essential in $X\}$. In particular, if Y is weakly infinite-dimensional then $\operatorname{index} X \leqslant \operatorname{index} Y$.*

Proof. This follows immediately from a lemma on extension of partitions [A–P; Ch.10, 3, Lemma 1], [E1; Lemma 1.2.9].

4. The index as a Lusin-Sierpiński index. Let us fix a separating sequence $\mathscr{S} = \{(A_i, B_i): i \in \omega\}$ in the Hilbert cube. For every $\sigma \in \operatorname{Fin}\omega$ let us put

$$\underline{W}_\sigma = \{X \in \underline{H}: \text{the family } \{(A_i \cap X, B_i \cap X): i \in \sigma\} \text{ is essential}\},$$

$\underline{H}$ being the hyperspace of I^ω.

The sets $\underline{W}_\sigma$ are closed, for if $X \notin \underline{W}_\sigma$ then there are partitions L_i in I^ω between A_i and B_i such that X is disjoint from $L = \bigcap\{L_i: i \in \sigma\}$ (cf. [E1; Lemma 1.2.9]) and the set $\{A \in \underline{H}: A \cap L = \emptyset\}$ is a neighbourhood of X disjoint from $\underline{W}_\sigma$.

Thus (cf. sec. 1)

$$\underline{W} = \{\underline{W}_\sigma: \sigma \in \operatorname{Fin}\omega, \prec\}$$

is a closed Lusin sieve in the hyperspace $\underline{H}$.

For each compactum $X \in \underline{H}$ let us put

$$M(X) = \{\sigma \in \operatorname{Fin}\omega: X \in \underline{W}_\sigma\}$$

and let us recall [K1; § 3, XV] that the set $L(\underline{W})$ sifted by the sieve $\underline{W}$ is defined by the formula

$$(X \in L(\underline{W})) \equiv (M(X) \text{ is not well-ordered by } \prec)$$

and also, that the Lusin–Sierpiński index of an $X \notin L(\underline{W})$ with respect to the sieve $\underline{W}$ is the order type of the set $M(X)$.

Now, it is clear by Lemma 3.5 and Definition 3.3 that $L(\underline{W})$ is exactly the set of strongly infinite-dimensional compacta contained in I^ω and if X is a weakly infinite-dimensional compactum in I^ω then the Lusin–Sierpiński index of X with respect to the sieve $\underline{W}$ coincides with the topological invariant index X [4].

5. Families of compacta with bounded index. In this section we give a few corollaries to the following two basic properties of the Lusin–Sierpiński index [K1; § 39, Theorem 4 and Corollary 5a]:

(A) For every $\alpha < \omega_1$ the set of all points whose Lusin–Sierpiński index is not greater than α is analytic (in fact Borel).

[4] We show in the sequel that the set $L(\underline{W})$ is non-Borel, or equivalently, that the index is unbounded (see sec. 2 in § 2).

(B) The Lusin–Sierpiński index is bounded over every analytic set disjoint from the set sifted by the sieve.

THEOREM 5.1. *Given a family* $\underline{E}$ *of weakly infinite-dimensional compacta, there exists a weakly infinite-dimensional compactum containing topologically each member of* $\underline{E}$ *if and only if* $\sup\{\text{index } X\colon X\in\underline{E}\}<\omega_1$.

This follows from the monotonicity of the index (Lemma 3.5), the property (A) of the Lusin–Sierpiński index, and the following lemma.

LEMMA 5.2. *Let* $\underline{E}\subset\underline{H}$. *There exists then a weakly infinite-dimensional compactum* E *containing topologically each member of* $\underline{E}$ *if and only if there exists an analytic set* $\underline{A}\subset\underline{H}$ *containing* $\underline{E}$ *and consisting of weakly infinite-dimensional compacta.*

Proof. Assume that the compactum E exists and let $\underline{A}=\{X\in\underline{H}\colon X$ can be embedded in $E\}$. The set $\underline{A}$ is analytic, for, if $\mathscr{C}=C(I^\omega,I^\omega)$ is the space of all continuous maps from I^ω into itself endowed with the compact-open topology, then $\underline{A}=\text{projection }\{(X,f)\in\underline{H}\times\mathscr{C}\colon f(X)\subset E$ and $f|X$ is an injection$\}$, where we assume that E is embedded into I^ω; cf. [K1; § 44].

Conversely, assume that the analytic set $\underline{A}$ exists and let $\Phi\colon\omega^\omega\to\underline{A}$ be a continuous surjection of the irrationals onto $\underline{A}$. Let $G=\{(t,x)\colon x\in\Phi(t)\}$. The projection $\pi\colon G\to\omega^\omega$ is a closed map whose fibers $\pi^{-1}(t)=\Phi(t)$ are weakly infinite-dimensional compacta and the range is zero-dimensional; therefore G is weakly infinite-dimensional [Le]. Since G is closed in $\omega^\omega\times I^\omega$, this is a complete space, and hence there exists a compactification E of G with countable-dimensional remainder $E\setminus G$ (5). The space E is the compactum we are looking for, see [A–P, Ch. 10, § 5, Theorem 21].

THEOREM 5.3. *For each* $\alpha<\omega_1$ *we have*

$$\sup\{\text{index } X\colon\ \text{ind } X\leqslant\alpha\}<\omega_1\ (^6).$$

The theorem follows from the next lemma and the property (B) of the Lusin–Sierpiński index (one can also use Lemma 5.2 instead of this property).

LEMMA 5.4 (cf. [K1; § 45, IV Theorem 4]). *For each* $\alpha<\omega_1$ *the set* $\underline{I}_\alpha=\{X\in\underline{H}\colon \text{ind } X\leqslant\alpha\}$ *and* $\underline{I}'_\alpha=\{X\in\underline{H}\colon \text{Ind } X\leqslant\alpha\}$ *are analytic.*

Proof. We shall check this only for $\underline{I}_\alpha$; the case of $\underline{I}'_\alpha$ is similar, and even simpler. The proof is by transfinite induction on α; we refer the reader to [K1; § 43] for some details we omit.

There is nothing to prove if $\alpha=-1$; assume that for $\beta<\alpha$ the sets $\underline{I}_\beta$ are analytic. Let $\mathscr{S}=\{(A_i,B_i)\colon i\in\omega\}$ be a separating sequence in I^ω. At first let us check that the set $\underline{M}=\{(A,B,L)\in\underline{H}\times\underline{H}\times\underline{H}\colon L$ is a partition in I^ω between A and $B\}$ is an F_σ-set. Indeed, if $(A,B,L)\notin\underline{M}$ then either $L\cap(A\cup B)\neq\emptyset$ (and this formula describes a closed set) or there is a continuum in I^ω disjoint from L which intersects both sets A and B (and this formula describes an open set). This observation and the inductive assumption yield the analycity of the set

$$\underline{A}_{i_1i_2\ldots i_n}=\{(X,L_{i_1},\ldots,L_{i_n})\colon (A_i,B_i,L_i)\in\underline{M}\text{ and } X\cap L_i\in\bigcup_{\beta<\alpha}\underline{I}_\beta\text{ for } i=i_1,\ldots,i_n\},$$

and hence the set

$$\underline{A}_i=\bigcup\{\underline{A}_{i_1,\ldots,i_n}\colon A_i\subset\bigcup_{j=1}^{n}A_{i_j},\ B\subset\bigcap_{j=1}^{n}B_{i_j}\}$$

is analytic for every $i\in\omega$. It remains to notice that $\underline{I}_\alpha=\bigcap_i\text{projection }\underline{A}_i$.

(5) This is a well-known fact following easily from a classical theorem of Kuratowski [K2; Théorème 2]; cf. also [E2; 4.15]. In the case of our "graph" G one can construct the compactification E in a particularly simple way such that in addition π extends to a continuous map $\tilde{\pi}\colon E\to C$ with zero-dimensional range and $\tilde{\pi}^{-1}(t)$ is finite-dimensional for $t\notin\omega^\omega$ (cf. sec. 3 in § 2).

(6) Cf. Theorem 3.2 in § 2.

THEOREM 5.5. *If* $\underline{D}$ *is an upper semicontinuous decomposition of a compactum* X *into weakly infinite-dimensional compacta then*

$$\sup\{\text{index} A\colon A\in\underline{D}\}<\omega_1.$$

Proof. One can assume that $X\subset I^\omega$ and then $\underline{D}$ is an analytic set in $\underline{H}$ [K1; § 43], and hence the assertion follows from the property (B) of the Lusin–Sierpiński index.

6. Remarks and questions. R. D. Mauldin has kindly pointed out to the author that the transfinite inductive dimensions can be viewed as special cases of the notion of monotone inductive operators investigated in [C–M]. Let us consider for example the transfinite dimension Ind. Let (cf. the proof of Lemma 5.4) $\mathscr{S}=\{(A_i,B_i)\colon i\in\omega\}$ be a separating sequence in I^ω and let $\underline{L}_i$ be the set of all partitions in I^ω between A_i and B_i ($\underline{L}_i$ is a G_δ-set in $\underline{H}$). Let us define a monotone in-inductive analytic operator Γ over the hyperspace $\underline{H}$ (see [C–M; (1.5)]) assuming for $\underline{A}\subset\underline{H}$

$$\Gamma(\underline{A})=\{X\in\underline{H}\colon \text{there exist } L_i\in\underline{L}_i \text{ for } i=1,2,\ldots \text{ such that } X\cap L_i\in\underline{A}\}\cup\underline{A},$$

and let Γ^α be the αth iteration of Γ (i.e. $\Gamma^{\alpha+1}(\underline{A})=\Gamma(\Gamma(\underline{A}))$ and $\Gamma^\lambda(\underline{A})=\bigcup\{\Gamma^\alpha(\underline{A})\colon\alpha<\lambda\}$ for limit ordinals).

Then $\Gamma^{\alpha+1}(\{\emptyset\})=\{X\in\underline{H}\colon \text{Ind } X\leqslant\alpha+1\}$ and the closure $\Gamma^{\omega_1}(\{\emptyset\})$ of the operator Γ [C–M; sec. 1] is the set of all countable-dimensional compacta in $\underline{H}$.

We do not know how regular from the point of view of analytic set theory the operator Γ is. For example: is the analytic set $\{X\in\underline{H}\colon\text{Ind } X\leqslant\alpha\}$ always a Borel set? If $\alpha<\omega$ this is the case [K1; § 45, IV Theorem 4]. Is the set $\Gamma^{\omega_1}(\{\emptyset\})$ a $\underline{CA}$-set in $\underline{H}$? One can show that this is a $\underline{PCA}$-set, using the fact that the subspace of I^ω consisting of the points which have only finitely many rational coordinates is universal for the class of countable-dimensional spaces [N2, Theorem IV.5], or alternatively, using the general results from [C–M]. And finally, does Ind have the property (B) of the Lusin–Sierpiński index formulated in sec. 5?

The last question can also be formulated in the following way (see the proof of Lemma 5.2 and Theorem 5.5):

QUESTION 6.1. Assume that $\underline{D}$ is an upper semicontinuous decomposition of a compactum X into countable-dimensional compacta. Is it true that $\sup\{\operatorname{Ind} A: A \in \underline{D}\} < \omega_1$?

Yet another formulation of this question is given in sec. 2, § 4.

§ 2. Transfinite polytopes

1. A theorem of W. Hurewicz. The line of reasoning in this paragraph is based upon the following lemma, which is a reformulation of a theorem of Hurewicz [Hu] in the spirit of Kuratowski and Szpilrajn [K–S].

LEMMA 1.1. *Let $p: K \to L$ be a continuous surjection of the compactum K onto the compactum L, let $\underline{A} \subset \underline{H}$ be an analytic set and let Q be a dense-in-itself countable subset of L* (7) *such that each compactum contained in $p^{-1}(Q)$ can be embedded into some member of $\underline{A}$. Then there exists a $t \in L \setminus Q$ such that the compactum $p^{-1}(t)$ can be embedded into some $A \in \underline{A}$.*

Proof. Let $\underline{K}$ and $\underline{L}$ be hyperspaces of the compactum K and L respectively. It is a theorem of Hurewicz [K1; § 43, VII, Corollary 3] that the subset of $\underline{L}$

(1.1) $$\underline{Q} = \{X \in \underline{L}: X \subset Q\} \quad \text{is not analytic}.$$

On the other hand (see the proof of Lemma 5.2 in § 1), the subset of $\underline{K}$

$$\underline{B} = \{X \in \underline{K}: X \text{ embeds in some } A \in \underline{A}\} \quad \text{is analytic},$$

and hence, also the subset of $\underline{L}$ (cf. [K1; § 43])

(1.2) $$\underline{C} = \{X \in \underline{L}: p^{-1}(X) \in \underline{B}\} \quad \text{is analytic}.$$

The assumption about $p^{-1}(Q)$ implies that $\underline{Q} \subset \underline{C}$ and thus (1.1) and (1.2) together yield an existence of an $X \in \underline{C}$ with $X \setminus Q \neq \emptyset$. Now, each $t \in X \setminus Q$ has the required property.

2. The Yu. M. Smirnov's compacta. Smirnov [S], [A–P; Ch. 10, § 1], [E2] defined compacta $S_1, S_2, \ldots, S_\alpha, \ldots, \alpha < \omega_1$, by transfinite induction in the following way:

(2.1) $$S_1 = I, \quad S_{\alpha+1} = S_\alpha \times I,$$

(2.2) if α is a limit ordinal then S_α is the one-point compactification of the free union $\bigoplus_{\beta<\alpha} S_\beta$.

Smirnov proved that $\operatorname{Ind} S_\alpha = \alpha$; the reasoning in the proof of Lemma 2.2 in § 3 shows also that index $S_\alpha \geq \alpha$. Notice that each compactum S_α is a countable disjoint union of finite-dimensional cubes I^n.

In the sequel we need the following universal property of Smirnov's compacta.

(7) By virtue of a generalization of the Hurewicz's theorem due to Christensen [Ch], it is enough to assume that Q is not a G_δ-set in L.

LEMMA 2.1. *The class of compacta which have at most countably many components and each of which is finite-dimensional coincides with the class of compacta which can be embedded into some Smirnov's compactum.*

Proof. Given a countable compactum C, let $\gamma(C)$ be the ordinal α such that the αth derived set $C^{(\alpha)}$ of C is finite, and let $\underline{C}_\alpha$ be the class of compacta X which admit a continuous map $p: X \to C$ with finite-dimensional fibres onto a countable compactum C with $\gamma(C) \leq \alpha$. Clearly, it is enough to check that for each $\alpha < \omega_1$ there is $\varphi(\alpha) < \omega_1$ such that each $X \in \underline{C}_\alpha$ can be embedded into $S_{\varphi(\alpha)}$. We shall verify this by induction.

By the classical embedding theorem we can put $\varphi(0) = \omega$. Assume that the required $\varphi(\beta)$ is defined for all $\beta < \alpha$, let $X \in \underline{C}_\alpha$ and let $p: X \to C$ be a map such as in the definition of $\underline{C}_\alpha$. Let $C = C_0 \supset C_1 \supset \ldots$ be open-and-closed sets such that $\bigcap_i C_i = C^{(\alpha)}$, and let $X_i = p^{-1}(C_i \setminus C_{i+1})$, $X_\infty = p^{-1}(C^{(\alpha)})$. Since $X_i \in \underline{C}_{\beta_i}$ for some $\beta_i < \alpha$, there is an embedding $f_i: X_i \to S_{\varphi(\beta_i)}$. Let f be the map of X into the one-point compactification of the free union $\bigoplus_i S_{\varphi(\beta_i)}$ which coincides with f_i on each X_i and which takes X_∞ to the point at infinity. By (2.2) one can assume that the range of f is contained in S_ξ, where ξ is the least ordinal greater than any of $\varphi(\beta_i)$. Now, again by the classical embedding theorem, there exists an embedding $g: X_\infty \to I^n$, and let $h: X \to I^n$ be a continuous extension of g. The diagonal map $(f, g): X \to S_\xi \times I^n = S_{\xi+n}$ is then an embedding and hence one can take $\varphi(\alpha) = \sup\{\varphi(\beta): \beta < \alpha\} + 2\omega$.

THEOREM 2.2. *If a complete space X contains each Smirnov's compactum S_α topologically, then X contains the Hilbert cube topologically.*

In fact, we shall prove a somewhat more general fact (8).

THEOREM 2.2'. *Let $\underline{A} \subset \underline{H}$ be an analytic set. If for each Smirnov's compactum S_α there exists an $A_\alpha \in \underline{A}$ containing S_α topologically, then there exists an $A \in \underline{A}$ which contains the Hilbert cube topologically.*

Proof. Put $J = \{0\} \cup [\frac{1}{2}, 1] \subset I$ and let p be the mapping of $K = J^\omega$ onto $L = 2^\omega$ defined by the formula

$$p((x_i)) = (y_i) \quad \text{where} \quad y_i = \begin{cases} 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i \in [\frac{1}{2}, 1]. \end{cases}$$

Let $Q = \{(y_i) \in L: y_i = 0 \text{ for all but finitely many } i\}$. Then

(2.3) if $t \in Q$, then $p^{-1}(t) \underset{\text{top}}{=} I^n$ for some n,

(2.4) if $t \in L \setminus Q$, then $p^{-1}(t) \underset{\text{top}}{=} I^\omega$.

(8) To derive Theorem 2.2 from Theorem 2.2' assume that $X \subset I^\omega$ and put $\underline{A} = \{A \in \underline{H}: A \subset X\}$.

If $C\subset Q$ is a compactum then the components of the compactum $p^{-1}(C)$ are the fibers $p^{-1}(t)$, for $t\in C$, and hence by (2.3) and Lemma 2.1 it follows that $p^{-1}(C)$ embeds into some compactum S_α and hence, by the assumption, into some $A_\alpha\in \underline{A}$. Lemma 1.1 yields thus an existence of a $t\in L\setminus Q$ such that the fiber $p^{-1}(t)$ embeds into some $A\in\underline{A}$ and this completes the proof by (2.4).

Remark 2.3. (a) Theorem 2.1 can be considered as a generalization of the well-known (and easy to prove) fact that each G_δ-set containing the subspace of the Hilbert cube consisting of the points with all but finitely many coordinates equal to zero, contains the Hilbert cube topologically [A–P; Ch. 10, § 7], [E2].

(b) Theorem 2.2 implies that there is no universal space in the class of all weakly infinite-dimensional compacta; cf. Smirnov [S] where the compacta S_α were used in a proof that there is no universal space in the class of countable-dimensional compacta.

(c) Theorem 2.2 yields also the following corollary:

Let X be a complete space such that the cone over each σ-compact subset of X has a continuous injection into X; then X contains the Hilbert cube topologically.

Indeed, the assumption allows one to embed in X, step by step, all Smirnov's compacta.

3. Transfinite polytopes. We shall call a compactum X a *transfinite polytope* if X is a disjoint union of at most countably many finite polytopes. In other words, a transfinite polytope is a compactum which has at most countably many components and each of which is a finite polytope.

In particular, each Smirnov's compactum is a transfinite polytope and each transfinite polytope embeds into some Smirnov's compactum by Lemma 2.1.

Now, we shall describe a certain kind of approximation of an arbitrary compactum by transfinite polytopes which we use in the proof of Theorem 3.2, a main result of this section.

Let us fix an enumeration $q_1, q_2, \ldots$ of the subset Q of the Cantor cube 2^ω consisting of the points all but finitely many of whose coordinates are equal to zero.

Throughout this section X denotes a compactum and $f_i\colon X\to W_i$ are continuous $1/i$-maps (i.e. $\operatorname{diam} f_i^{-1}(y)\leqslant 1/i$ for $y\in W_i$) onto finite polyhedra W_i.

We denote by $X(f_1, f_2, \ldots)$ the compactum obtained from the product $2^\omega\times X$ by attaching to each compactum $\{q_i\}\times X$ the polyhedron W_i by the map f_i. More precisely, we consider in the compactum $2^\omega\times X$ the decomposition into the singletons $\{(t,x)\}$, where $t\notin Q$, and into the sets $\{q_i\}\times f_i^{-1}(y)$, where $y\in W_i$. This is an upper semicontinuous decomposition and the quotient space is the compactum $X(f_1, f_2, \ldots)$.

Assigning to the equivalence class of the point (t,x) the point t one defines a continuous "projection"

(3.1) $$p\colon X(f_1, f_2, \ldots)\to 2^\omega$$

such that (cf. (2.3) and (2.4))

(3.2) $$p^{-1}(q_i)\underset{\text{top}}{=} W_i \quad\text{for}\quad q_i\in Q,$$

(3.3) $$p^{-1}(t)\underset{\text{top}}{=} X \quad\text{for}\quad t\in 2^\omega\setminus Q.$$

Let

(3.4) $$\underline{Q}=\{C\subset Q\colon C \text{ is a compactum}\}.$$

The objects we are interested in are the transfinite polytopes

(3.5) $$X(C)=p^{-1}(C), \quad\text{where}\quad C\in\underline{Q}.$$

Proposition 3.1. *The space X is countable-dimensional if and only if $\lambda(f_1, f_2, \ldots)=\sup\{\operatorname{ind} X(C)\colon C\in\underline{Q}\}<\omega_1$. The same is true for* Ind *and one can also replace "countable-dimensional" by "weakly infinite-dimensional" and "ind" by "index" in this statement.*

Proof. If X is countable-dimensional then so is $X(f_1, f_2, \ldots)$ and then $\lambda(f_1, f_2, \ldots)\leqslant \operatorname{ind} X(f_1, f_2, \ldots)$. To prove the converse, put $K=X(f_1, f_2, \ldots)$, $L=2^\omega$, $\underline{A}=\{A\in\underline{H}\colon \operatorname{ind} A\leqslant\lambda(f_1, f_2, \ldots)\}$ and apply Lemma 1.1, using (3.2), (3.3) and Lemma 5.4 in § 1.

If X is weakly infinite-dimensional then so is $X(f_1, f_2, \ldots)$ (by the properties of the map p defined in (3.1) and a theorem in [Le]) and hence $\lambda(f_1, f_2, \ldots)\leqslant \operatorname{index} X(f_1, f_2, \ldots)$, since the index is monotone (Lemma 3.5 in § 1). To prove the converse one can repeat the argument for ind, because the set $\{A\in\underline{H}\colon \operatorname{index} A\leqslant\lambda\}$ is analytic (see sec. 4 in § 1).

Theorem 3.2. *There exists a family $\underline{T}$ of transfinite polytopes such that*

$$\sup\{\operatorname{index} T\colon T\in\underline{T}\}<\sup\{\operatorname{ind} T\colon T\in\underline{T}\}=\omega_1.$$

Proof. Let X be a weakly infinite-dimensional compactum which is not countable-dimensional [P] and let

(3.6) $$\underline{T}=\{X(C)\colon C\in\underline{Q}\},$$

where $\underline{Q}$ is defined by (3.4) and $X(C)$ by (3.5). The family $\underline{T}$ has the required properties by Proposition 3.1.

Remark 3.3. (a) In the proof of Theorem 3.2 we used a compactum X which is weakly infinite-dimensional but not countable-dimensional to produce the family $\underline{T}$. Conversely, given such a family $\underline{T}$ one can construct a compactum X with these properties using Theorem 5.1 in § 1. It seems interesting (but also difficult) to obtain the family $\underline{T}$ in Theorem 3.2 in a more explicit way; cf. Remark 2.4 in § 3.

(b) Theorem 3.2 and Theorem 5.1 in § 1 yield an existence of a weakly infinite-dimensional compactum containing compact subspaces with arbitrarily large transfinite dimensions (one can take for this purpose the compactum $X(f_1, f_2, \ldots)$ considered in the proof of Theorem 3.2).

Example 3.4. (a) Let X be a continuum which is countable-dimensional but not finitely-dimensional (for example, let X be the continuum H_ω defined in sec. 1 § 3) and let us consider the family of transfinite polytopes $X(C)$ defined by (3.5). Then $\sup\{\operatorname{Ind} X(C)\colon C \in \underline{Q}\} < \omega_1$ and $\sup\{\operatorname{index} X(C)\colon C \in \underline{Q}\} < \omega_1$, by Proposition 3.1 but there is no $\alpha < \omega_1$ such that all compacta $X(C)$ can be embedded into the Smirnov's compactum S_α-compare with Lemma 2.1.

Indeed, if there were such an α then the reasoning in the proof of Proposition 3 with $\underline{A} = \{A \in \underline{H}\colon A \text{ embeds into } S_\alpha\}$ would lead us to the conclusion that X embeds into S_α, which is false.

(b) Let $X = K \times K \times \ldots$ be the countable product of the pseudo-arc [K1; § 48, X]. The compactum X does not contain the Hilbert cube topologically and for every $i = 1, 2, \ldots$ there is a $1/i$-map $f_i\colon X \to I^i$ of X onto the i-dimensional cube. Let, as in (a), the transfinite polytopes $X(C)$ be defined by (3.5). The components of each $X(C)$ are finite-dimensional cubes (see (3.2)) and the transfinite dimensions and the index are unbounded over the family $\{X(C)\colon C \in \underline{Q}\}$ (by Proposition 3.1, since X is strongly infinite-dimensional). However, the compactum $X(f_1, f_2, \ldots)$ which contains all members of this family does not contain the Hilbert cube topologically; this is a contrast to Theorem 2.2.

§ 3. On two questions of D. W. Henderson

1. The D. W. Henderson's compacta. Henderson [He] defined AR-compacta $H_1, H_2, \ldots, H_\alpha, \ldots, \alpha < \omega_1$, and their boundaries ∂H_α by transfinite induction in the following way (cf. sec. 2 in § 2).

Let $H_1 = I$, $\partial H_1 = \partial I = \{0, 1\}$, $p_1 = \{0\}$ and assume that for $\beta < \alpha$ the compacta H_β, their boundaries ∂H_β and the points $p_\beta \in \partial H_\beta$ are defined. If $\alpha = \beta+1$ then we let $H_{\beta+1} = H_\beta \times I$, $\partial H_{\beta+1} = (\partial H_\beta \times I) \cup (H_\beta \times \partial I)$, and $p_{\beta+1} = (p_\beta, p_1)$. If α is limit, let K_β be the union of H_β and a half-open arc A_β such that $A_\beta \cap H_\beta = \{p_\beta\} =$ (the end point of A_β). Define H_α to be the one-point compactification of the free union $\bigoplus_{\beta<\alpha} K_\beta$, $\partial H_\alpha = H_\alpha \setminus \bigcup_{\beta<\alpha} (H_\beta \setminus \partial H_\beta)$ and let p_α be the compactifying point. Henderson called a *continuous map* $f\colon X \to H_\alpha$ *essential* if each continuous extension over X of the restriction $f|f^{-1}(\partial H_\alpha)$ maps X onto H_α. In the case where $\alpha = i < \omega$, H_i is the i-dimensional cube, ∂H_i is its boundary and the notion of essential maps coincides with the classical one [A–P], [E1].

It was shown in [He] that if a countable-dimensional compactum X admits an essential map onto H_α then $\operatorname{Ind} H \geqslant \alpha$.

2. Results. The theorem below answers affirmatively the second question raised by Henderson in [He; p. 168].

Theorem 2.1. *A compactum which admits for every $\alpha < \omega_1$ an essential map onto the Henderson's compactum H_α is strongly infinite-dimensional.*

This result follows immediately from the following lemma which we prove in the next section.

Lemma 2.2. *If a weakly infinite-dimensional compactum X admits an essential map onto H_α then $\operatorname{index} X \geqslant \alpha$.*

The next theorem answers negatively the first question formulated by Henderson in [He; p. 168].

Theorem 2.3. *There exists a countable-dimensional compactum X such that $\operatorname{Ind} X = \alpha$ but X does not admit any essential map onto the Henderson's compactum H_α, α being an arbitrary countable ordinal.*

Proof. Let $\underline{T}$ be the family of transfinite polytopes described in Theorem 3.2 in § 2 and let $\lambda = \sup\{\operatorname{index} T\colon T \in \underline{T}\}$. Then any $X \in \underline{T}$ with $\operatorname{Ind} X = \alpha > \lambda$ has the required property by Lemma 2.2.

Remark 2.4. (a) The proof of Theorem 2.3 is based on an existence of the family $\underline{T}$ which, in fact, we derived from an existence of an analytic set which is non-Borel (cf. the proof of the Hurewicz's theorem given in [K1; § 43, VII]). In effect, we have only a vague idea how the compactum X looks and we have no information on the least α for which such a compactum exists. As was pointed out in Remark 3.3 (a) in § 1, a more constructive method would be of interest.

(b) By virtue of Theorem 2.3 one can ask the following natural extension of the Henderson's question: does there exist for each $\alpha < \omega_1$ an AR-compactum B_α and its subcompactum ∂B_α such that, with the definition of essential maps $f\colon X \to B_\alpha$ analogous to that in sec. 1, a countable-dimensional compactum X admits an essential map onto B_α if and only if $\operatorname{Ind} X \geqslant \alpha$.

We also do not know the situation if one replaces Ind by index in the Henderson's question.

3. Proof of Lemma 2.2. We shall define for each $\alpha < \omega_1$ by transfinite induction a sequence of pairs of closed disjoint sets in H_α

(3.1) $$(A_1^\alpha, B_1^\alpha),\ (A_2^\alpha, B_2^\alpha),\ \ldots,$$

(3.2) a set $\underline{C}_\alpha$ of closures of some components of $H_\alpha \setminus \partial H_\alpha$,

and for each $C \in \underline{C}_\alpha$ a homeomorphism

(3.3) $$h_C\colon C \to I^{n(C)} \quad \text{such that} \quad h_C(C \cap \partial H_\alpha) = \partial I^{n(C)},$$

in such a way that if we put

(3.4) $$M_\alpha = \{\sigma \in \operatorname{Fin}\omega\colon \text{there is } C \in \underline{C}_\alpha \text{ such that the pairs } (h_C(A_i^\alpha \cap C), h_C(B_i^\alpha \cap C)) \text{ are distinct opposite faces of the cube } I^{n(C)} \text{ for } i \in \sigma\},$$

then we have

(3.5) $$\operatorname{type} M_\alpha \geqslant \alpha$$

(recall, that we consider $\operatorname{Fin}\omega$ with the order introduced in sec. 1 in § 1).

Observe, that M_α is well-ordered, since for a separating sequence $\mathcal{S} = \{(X_i, Y_i): i \in \omega\}$ in H_α with $X_{2i} = A_i$ and $Y_{2i} = B_i$ we have $M_\alpha \subset M_{\mathcal{S}}(X)$ and the compactum H_α is weakly infinite-dimensional (see sec. 3 in § 1).

The construction is evident for $\alpha = 1$. Assume that the objects are defined for an α and put (cf. the definition of $H_{\alpha+1}$ in sec. 1):

$$\underline{C}_{\alpha+1} = \{C \times I: C \in \underline{C}_\alpha\} \quad \text{and} \quad h_{C\times I} = h_C \times \mathrm{id}_I,$$
$$A_1^{\alpha+1} = H_\alpha \times \{0\}, \quad B_1^{\alpha+1} = H_\alpha \times \{1\},$$
$$A_{i+1}^{\alpha+1} = A_i^\alpha \times I, \quad B_{i+1}^{\alpha+1} = B_i^\alpha \times I, \quad i = 1, 2, \ldots$$

Let us check that type $M_{\alpha+1} \geqslant \alpha+1$, where $M_{\alpha+1}$ is defined by (3.4).

For every $\sigma \in M_\alpha$ let $\sigma^* = \{i+1: i \in \sigma\} \cup \{1\}$ and $M_\alpha^* = \{\sigma^*: \sigma \in M_\alpha\}$. Observe, that $\sigma^* \in M_{\alpha+1}$ (see (3.4)) and that the injection $\sigma \to \sigma^*$ preserves the order, and hence type $M_\alpha^* =$ type $M_\alpha \geqslant \alpha$, by the inductive assumption (see (3.5)). Now, $\tau = \{1\}$ is an element of $M_{\alpha+1}$ greater than each of the elements of M_α^* and therefore type $M_{\alpha+1} \geqslant$ type $M_\alpha^* + 1 \geqslant \alpha+1$, as required.

Assume now that α is a limit ordinal and that the construction is performed for each $\beta < \alpha$. To make the αth step we need only enumerate in an appropriate way the objects we have constructed. Let us split ω into disjoint infinite sets N_β, for $\beta < \alpha$ let us fix for each β an order-preserving enumeration $j_\beta: \omega \to N_\beta$ and let us put $(A_i^\beta, B_i^\beta) = (A^\beta_{j_\beta^{-1}(i)}, B^\beta_{j_\beta^{-1}(i)})$ for $i \in N_\beta$, and $\underline{C}_\alpha = \bigcup_{\beta<\alpha} \underline{C}_\beta$ (cf. the definition of H_α in sec. 1). Then M_α contains for each $\beta < \alpha$ the set $\{j_\beta(\sigma): \sigma \in M_\beta\}$ similar to M_β and hence type $M_\alpha \geqslant \sup\{\text{type } M_\beta: \beta < \alpha\} \geqslant \alpha$.

Having completed the construction of the objects (3.1)–(3.5), we are now in a position to prove the lemma.

Let $f: X \to H_\alpha$ be an essential map of the weakly infinite-dimensional compactum X and let $\mathcal{S} = \{(A_i, B_i): i \in \omega\}$ be a separating sequence in X (see sec. 3 in § 1). By (3.5) it is enough to find an order-preserving injection of M_α into $M_{\mathcal{S}}(X)$ (see (3.3) in § 1). Let us choose (see (3.2) in § 1) a sequence $j(1) < j(2) < \ldots$ of natural numbers such that (see (3.1))

(3.6) $$f^{-1}(A_i^\alpha) \subset A_{j(i)} \quad \text{and} \quad f^{-1}(B_i^\alpha) \subset B_{j(i)}.$$

Since the correspondence $\sigma \to j(\sigma)$ is an order-preserving injection we have only to verify that

(3.7) $$j(\sigma) \in M_{\mathcal{S}}(X), \quad \text{whenever} \quad \sigma \in M_\alpha.$$

If $\sigma \in M_\alpha$ then there exists a $C \in \underline{C}_\alpha$ such as in (3.4). Now, Henderson [He; Proposition 3] has shown that f is also an essential map when restricted to the set $Y = f^{-1}(C)$ and so is the map $g = h_C \circ f|Y: Y \to I^{n(C)}$ (see (3.3)). By the choice of the set C, the pairs $(f^{-1}(A_i^\alpha \cap C), f^{-1}(B_i^\alpha \cap C))$, where $i \in \sigma$, are the inverse images by g of distinct pairs of the opposite faces of the cube $I^{n(C)}$. Hence they form an essential family in the space Y, since g is essential. It follows by (3.6) that the family $\{(A_{j(i)}, B_{j(i)}): i \in \sigma\}$ is essential which completes the proof of (3.7).

§ 4. Universal functions for the families of compacta with transfinite dimension $\leqslant \alpha$

1. Results.

THEOREM 1.1. *For each $\alpha < \omega_1$ there exists a continuous function $\Phi_\alpha: \omega^\omega \to \underline{H}$ of the irrationals into the hyperspace of the Hilbert cube such that:*

(i) *if $X \in \underline{H}$ and* $\operatorname{ind} X \leqslant \alpha$ *then $X = \Phi_\alpha(t)$ for some $t \in \omega^\omega$,*

(ii) *ind $G_\alpha = \alpha$, where $G_\alpha = \{(t, x): x \in \Phi_\alpha(t)\} \subset \omega^\omega \times I^\omega$.*

THEOREM 1.2. *For each $\alpha < \omega_1$ there exists a continuous function $\Phi'_\alpha: \omega_1 \to \underline{H}$ such that:*

(i) *if $X \in \underline{H}$ and* $\operatorname{Ind} X \leqslant \alpha$ *then $X = \Phi'_\alpha(t)$ for some $t \in \omega_1$,*

(ii) *If $C \subset G'_\alpha = \{(t, x): x \in \Phi'_\alpha(t)\}$ is a compactum then* $\operatorname{Ind} C \leqslant \alpha$.

Since each of the complete spaces G_α and G'_α has a countable-dimensional compactification X_α and X'_α respectively (see footnote ([5])), we also have the following corollary.

COROLLARY 1.3. *For each $\alpha < \omega_1$ there exist countable-dimensional compacta X_α and X'_α such that X_α contains topologically all compacta S with* $\operatorname{ind} S \leqslant \alpha$ *and X'_α contains topologically all compacta S with* $\operatorname{Ind} S \leqslant \alpha$.

We do not know the least possible transfinite dimension $\operatorname{ind} X_\alpha$ and $\operatorname{Ind} X'_\alpha$ of the compacta X_α and X'_α described in Corollary 1.3.

2. Remarks and questions. Given a continuous map $\Psi: \omega^\omega \to \underline{H}$ let us put

$$G(\Psi) = \{(t, x): x \in \Psi(t)\}.$$

It is a reformulation of Question 6.1 whether always $\sup\{\operatorname{Ind} \Psi(t): t \in \omega^\omega\} < \omega_1$?

QUESTION 2.1. Assume that $\operatorname{Ind} \Psi(t) \leqslant \alpha$ for every $t \in \omega_1$. Does the space $G(\Psi)$ need to be countable-dimensional? (See: Added in proof).

Remark 2.2. There exists a map Ψ such that $G(\Psi)$ is not countable-dimensional. In fact, one can show that:

there exists an open map $g: Z \to C$ of a compact space Z onto the Cantor set C such that each fiber $g^{-1}(t)$ is countable-dimensional but Z is not countable-dimensional.

Indeed, there exists a compact space X and a continuous map $f: X \to C$ onto the Cantor set such that each fiber $f^{-1}(t)$ is countable-dimensional but X is not countable-dimensional [P; Comment B], and one can adopt easily a construction of Michael and Stone [M–S; Proof of Theorem 1.1] to define a compact space $Z \supset X$ with $\dim(Z \setminus X) = 0$ and an open extension $g: Z \to C$ of the map f (cf. also [E1; Problem 1.12. G]). Now, given g as above one can take $\Psi(t) = g^{-1}(t)$, since Ψ is continuous and $G(\Psi)$ is homeomorphic with Z.

We do not know whether the map Ψ provides a counterexample to Question 2.1 or whether the construction can be modified to produce such a counterexample.

Remark 2.3. One can define for each $\alpha < \omega_1$ a map $\Psi = \Psi_\alpha$ such that $\Psi_\alpha(t)$

is finite-dimensional for each t but $\operatorname{Ind} G(\Psi_\alpha) \geqslant \alpha$. Indeed, let C_α be the space of the components of the Smirnov's compactum $S_\alpha \subset I^\omega$ (see § 2 sec. 1). The quotient map $\pi\colon S_\alpha \to C_\alpha$ is open and hence the map $\Psi'_\alpha\colon C_\alpha \to \underline{H}$ defined by the formula $\Psi'_\alpha(t) = \pi^{-1}(t)$ is continuous. One can assume that $C_\alpha \subset \omega^\omega$ and then, if we put $\Psi_\alpha = \Psi'_\alpha \circ r$, $r\colon \omega^\omega \to C_\alpha$ being a retraction, the function Ψ_α has the desired property, because $G(\Psi_\alpha) \cap (C_\alpha \times I^\omega) \underset{\text{top}}{=} S_\alpha$ and $\operatorname{Ind} S_\alpha \geqslant \alpha$.

3. Proof. We shall prove only the first theorem; the second one can be proved in a similar way with some obvious modifications.

The proof is by transfinite induction on α. For $\alpha = -1$ we put $\Phi_{-1}(t) \equiv \emptyset$ and let us assume that we have defined for every $\beta<\alpha$ a continuous map $\Phi_\beta\colon \omega^\omega \to \underline{H}$ satisfying the conditions (i) and (ii) in Theorem 1.1. The main step in the construction of Φ_α is the proof of the following statement which is a little bit weaker than we need (compare the conditions (i) in this lemma and in Theorem 1.1).

LEMMA 3.1. *There is a continuous function $\Phi\colon \omega^\omega \to \underline{H}$ such that:*

(i) *if $X \in \underline{H}$ and $\operatorname{ind} X \leqslant \alpha$ then $X \underset{\text{top}}{=} \Phi(t)$ for some $t \in \omega^\omega$,*

(ii) $\operatorname{ind} G = \alpha$, *where* $G = \{(t, x)\colon x \in \Phi(t)\} \subset \omega^\omega \times I^\omega$.

Proof. Let $p_i\colon \prod_i I_i \to I_i$ be the projection of the Hilbert cube onto the ith axis and let

(3.1) $$E_i = p_i^{-1}(0), \quad F_i = p_i^{-1}(1).$$

Let us split ω into disjoint infinite sets $N_1, N_2, \ldots, N_\beta, \ldots$, for $\beta<\alpha$ and let $\underline{E}$ be the set of all $X \in \underline{H}$ satisfying the following conditions:

(3.2) if $x \in X \setminus F$, $F \subset X$ being closed, then $x \in E_i$ and $F \subset F_i$ for some $i \in \omega$,

(3.3) if $j \in N_\beta$ then there exists a partition L in I^ω between E_j and F_j such that $\operatorname{ind}(X \cap L) \leqslant \beta$.

We shall verify that

(3.4) $\underline{E}$ is an analytic set,

(3.5) if $X \in \underline{E}$ then $\operatorname{ind} X \leqslant \alpha$,

(3.6) if K is a compactum with $\operatorname{ind} K \leqslant \alpha$ then K is homeomorphic to some $X \in \underline{E}$.

The assertion (3.4) follows from the observation that $\{X \in \underline{H}\colon X$ satisfies (3.2)$\}$ $= \underline{H} \setminus \bigcap_i$ projection $\{(X, x, F)\colon x \notin F \subset X$ and either $x \notin E_i$ or $F \not\subset F_i\}$ and that, in notation of Lemma 5.4 in § 1, $\{X \in \underline{H}\colon X$ satisfies (3.3)$\} = \bigcap_{\beta<\alpha} \bigcap_{i \in N_\beta}$ projection $\{(X, L)\colon (E_i, F_i, L) \in \underline{M}$ and $X \cap L \in \underline{I}_\beta\}$, see proof of Lemma 5.4 in § 1.

The assertion (3.5) is obvious.

To prove the last assertion (3.6) let us fix in a compactum K with $\operatorname{ind} K \leqslant \alpha$ a separating sequence $\{(A_i, B_i)\colon i \in \omega\}$, see (3.1) in § 1. Let ω' be the set of all $i \in \omega$ such that there is a partition L in K between A_i and B_i with $\operatorname{ind} L = \beta<\alpha$.

Let $\beta(i)$ be the smallest such β for $i \in \omega'$. Let $\varphi\colon \omega' \to \omega$ be an injection such that $\varphi(i) \in N_{\beta(i)}$ and let, for every $i \in \omega'$, $f_i\colon K \to I_{\varphi(i)}$ be a continuous map such that $A_i = f_i^{-1}(0)$ and $B_i = f_i^{-1}(1)$. The diagonal map $f = (f_i)_{i\in\omega'}$ maps the compactum K onto the subspace $X = f(K)$ of the product $\prod_{i\in\omega'} I_{\varphi(i)}$ which we identify with the subspace of the Hilbert cube $\prod_{i\in\omega} I_i$ consisting of the points (x_i) all whose coordinates x_i with $i \notin \varphi(\omega')$ are zero. Let us verify that $X \in \underline{E}$. For every $x \in K \setminus F$, $F \subset K$ being closed, there is an $i \in \omega'$ such that $x \in A_i$ and $F \subset B_i$, and we have then $f(x) \in f(A_i) = X \cap E_{\varphi(i)}$ and $f(F) \subset f(B_i) = X \cap F_{\varphi(i)}$. Hence (3.2) holds and f is a homeomorphism onto X. To check (3.3), let $j = \varphi(i) \in N_\beta$ (if $j \notin \varphi(\omega')$ then $F_j \cap X = \emptyset$ and we have nothing to do) and let L' be a partition in K between A_i and B_i such that $\operatorname{ind} L' \leqslant \beta(i) = \beta$. The partition $f(L')$ in X between $X \cap E_j$ and $X \cap F_j$ (see above) can be extended to a partition L in I^ω between E_j and F_j, and since $\operatorname{ind}(X \cap L) = \operatorname{ind} f(L') = \operatorname{ind} L' = \beta$, we are done.

Let us consider now the space $\underline{S}$ of all sequences

(3.7) $(X, Y_{-1}, Z_{-1}, Y_0, Z_0, Y_1, Z_1, \ldots, t_{-1}, t_0, t_1, \ldots) \in \underline{H} \times \underline{H} \times \ldots \times \omega^\omega \times \omega^\omega \times \ldots$

satysfying the following three conditions:

(3.8) $X \in \underline{E}$,

(3.9) $Y_i \cup Z_i = I^\omega, \quad E_i \subset Y_i, \quad F_i \subset Z_i, \quad (E_i \cup F_i) \cap (Y_i \cap Z_i) = \emptyset,$

(3.10) if $i \in N_\beta$, then $X \cap Y_i \cap Z_i = \Phi_\beta(t_i)$, for $\beta<\alpha$.

Since the set $\underline{E}$ is analytic and the maps Φ_β are continuous, it is routine to check that the set $\underline{S}$ is analytic, cf. [K1; § 43]. Thus there exists a continuous parametrization of the set $\underline{S}$ by irrationals

(3.11) $s \mapsto (X(s), Y_{-1}(s), Z_{-1}(s), Y_0(s), Z_0(s), Y_1(s), Z_1(s), \ldots, t_{-1}(s), t_0(s), t_1(s), \ldots)$

We shall verify that the function we are looking for is

(3.12) $$\Phi(s) = X(s).$$

The property (i) follows from (3.6) and the fact that $\Phi(\omega^\omega) = \underline{E}$ which is a consequence of the inductive assumption about Φ_β.

Thus, it remains to show that

(3.13) $$\operatorname{ind} G = \alpha, \quad \text{where} \quad G = \{(s, x)\colon x \in X(s)\}.$$

At first let us show that

(3.14) if $i \in N_\beta$ then there exists a partition L in $\omega^\omega \times I^\omega = T$ between $\omega^\omega \times E_i = C_i$ and $\omega^\omega \times F_i = D_i$ such that $\operatorname{ind}(G \cap L) \leqslant \beta$.

For this purpose fix an $i \in N_\beta$ and put

(3.15) $$Y = \{(s, x)\colon x \in Y_i(s)\}, \quad \text{and} \quad Z = \{(s, x)\colon x \in Z_i(s)\}.$$

Both sets Y and Z are closed subsets of T such that (see (3.9)) $Y \cup Z = T$, $C_i \subset Y$, $D_i \subset Z$, $(C_i \cup D_i) \cap (Y \cap Z) = \emptyset$ and hence $L = Y \cap Z$ is a partition in T between C_i and D_i. To check that $\operatorname{ind}(G \cap T) \leqslant \beta$ observe that by (3.10)–(3.12) we have

$$(3.16) \qquad G \cap L = \{(s, x)\colon x \in \Phi_\beta(t_i(s))\},$$

and therefore, having in mind the properties of Φ_β, we need only the following simple fact:

SUBLEMMA 3.2. *Let $M \subset \omega^\omega \times I^\omega$ be a closed set, let $t\colon \omega^\omega \to \omega^\omega$ be a continuous map, and let $\tilde{M} = \{(s, x)\colon (t(s), x) \in M\}$.* Then $\operatorname{ind}\tilde{M} \leqslant \operatorname{ind} M$.

This can be easily verified by induction on $\operatorname{ind} M$, and so (3.14) follows.

Let us show how one obtains (3.13) from (3.14). Let $F \subset \omega^\omega \times I^\omega$ be a closed set, and let $p = (s, a) \in G \setminus F$. Since $X(s) \in \underline{E}$ there is by (3.2) an $i \in \omega$ such that $a \in E_i$ and $(\{s\} \times X(s)) \cap F \subset F_i$, and let L be a partition such as in (3.14). Then for a sufficiently small open-and-closed neighbourhood V of s the set $L \cap (V \times I^\omega)$ is a partition in $\omega^\omega \times I^\omega$ separating the point p from the set F. This completes the proof of Lemma 3.1.

We shall show now how one can modify the function Φ in Lemma 3.1 to construct the function Φ_α we are looking for.

Let $J = [-1, 2]$ and let Γ be the space of all autohomeomorphisms of the cube J^ω endowed with the compact-open topology. Since the space Γ is complete there is a continuous surjection $\gamma\colon \omega^\omega \to \Gamma$. Let us put

$$(3.16) \qquad \Psi(s, t) = \gamma(s)(\Phi(t)), \quad \text{where} \quad (s, t) \in \omega^\omega \times \omega^\omega.$$

Notice that

$$(3.17) \quad \text{if } X \in \underline{H} \text{ and } \operatorname{ind} X \leqslant \alpha \text{ then } X = \Psi(s, t) \text{ for some } (s, t) \in \omega^\omega \times \omega^\omega.$$

Indeed, by Lemma 3.1(i) there is $t \in \omega^\omega$ such that $X \underset{\text{top}}{=} \Phi(t)$, and since both compacta X and $\Phi(t)$ are contained in the "pseudointerior" $(-1, 2)^\omega$ of the cube J^ω there is (see [B–P; Lemma 1.3, p. 150]) an autohomeomorphism $h \in \Gamma$ such that $X = h(\Phi(t))$ and hence $X = \Psi(s, t)$, where $\gamma(s) = h$.

Let us verify that

$$(3.18) \quad \operatorname{ind} G(\Psi) = \alpha \quad \text{where} \quad G(\Psi) = \{(s, t, x)\colon x \in \Psi(s, t)\} \subset \omega^\omega \times \omega^\omega \times I^\omega.$$

This follows immediately from the following easy fact:

SUBLEMMA 3.3. *Assume that $M \subset \omega^\omega \times I^\omega$ is a closed set and let $\tilde{M} = \{(s, t, x)\colon (t, \gamma(s)^{-1}(x)) \in M\}$. Then* $\operatorname{ind}\tilde{M} \leqslant \operatorname{ind} M$.

We shall prove this assertion by transfinite induction on $\operatorname{ind} M$. The case $\operatorname{ind} M = -1$ is evident. Assume that we have verified the assertion for all $\beta < \alpha$, and let $\operatorname{ind} M = \alpha$. Let $p = (u, v, a) \in \tilde{M} \setminus F$, $F \subset \tilde{M}$ being closed. Let us put $H = F \cap \{(u, v)\} \times I^\omega$ and let $\lambda(s, t, x) = (t, \gamma(s)^{-1}(x))$. The map $\lambda\colon \tilde{M} \to M$ is continuous and it is injective with respect to the variable x. Thus $\lambda(p) \notin \lambda(H)$ and hence there exists a partition L in M between $\lambda(p)$ and the compact set $\lambda(H)$ such that $\operatorname{ind} L < \alpha$. The set $\tilde{L} = \lambda^{-1}(L) = \{(s, t, x)\colon (t, \gamma(s)^{-1}(x)) \in L\}$ is a partition in $\tilde{M}$ between p and H, and by the inductive assumption $\operatorname{ind}\tilde{L} \leqslant \operatorname{ind} L < \alpha$. Since the set $\tilde{M}$ is closed in the product $\omega^\omega \times \omega^\omega \times I^\omega$ there exists an open-and-closed neighbourhood U of the point (u, v) in $\omega^\omega \times \omega^\omega$ such that $\tilde{L} \cap (U \times I^\omega)$ is a partition in $\tilde{M}$ between p and F and this ends the proof.

Now, to complete the proof of Theorem 1.1 it is enough to modify the function Ψ slightly in the following way. Let us put $A = \Psi^{-1}(\underline{H})$ and let $r\colon \omega^\omega \times \omega^\omega \to A$ be a retraction onto the closed set A. Since one can identify the space $\omega^\omega \times \omega^\omega$ with ω^ω, it is easy to see that the function $\Phi_\alpha = \Psi \circ r$ has the required properties (cf. (3.17), (3.18) and Sublemma 3.2).

Remark 3.4. The idea of the proof has points in common with the idea of the proof of a factorization theorem of Pasynkov given in [A–P; Appendix] (although formally, the subjects seem far). The idea of continuous parametrization of families of compact sets in the construction of special sets goes back to Mazurkiewicz [M].

Added in proof. 1. P. Borst solved independently the first question of Henderson [He] constructing a space X with $\operatorname{Ind} X = \omega + 1$ and without any essential map onto $H_{\omega+1}$, and J. Dijkstra modified Borst's construction to obtain a compactum with these properties; see J. Nagata, *Topics in dimension theory*, Proc. Fifth Prague Top. Symp. 1981, Berlin 1982, Theorem 2. Notice that in Theorem 2.3 in § 3 the gap between $\operatorname{Ind} X$ and α for which X has no essential map onto H_α can be arbitrarily large.

2. Question 2.1 in § 4 has a negative answer; in fact the map Ψ considered in Remark 2.2 § 4 provides a counterexample, see Proc. Fifth Prague Top. Symp. 1981, p. 556.

3. One can prove that if $\underline{D}$ is an upper semicontinuous decomposition of a compactum X into compacta which are countable unions of finite-dimensional compacta, then $\sup\{\operatorname{Ind} A\colon A \in \underline{D}\} < \omega_1$; cf. Question 6.1 in § 1.

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Spaces defined by topological games, II *

by

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Abstract. The paper reports some results on the game $G(K, X)$ introduced in [7]. The main results: 1. The space favorable for Player I is the union of countably many K-scattered subsets. 2. Reduction theorems for actions of Player I. 3. Covering characterization of the spaces favorable for Player II. 4. Indeterminacy of the game in ZFC.

The main object of this work is the topological game $G(K, X)$, so the present paper is a continuation of [7]. Some of the results included here were announced earlier in [8] and [9]. The game $G(K, X)$ was used recently for proving general sum theorems for the dimension dim by the author and Y. Yajima [10] and for the dimension Ind by Y. Yajima [12]. Furthermore, a general product theorem for paracompact spaces involving that game was established by Y. Yajima in [13].

Section 1 contains the following: if Player I has a winning strategy in $G(K, X)$, then X is the union of countably many K-scattered subsets. In sections 2 and 3 there are introduced auxiliary games $G^{\#}(K, X)$ and $G^{+}(K, X)$ in order to prove reduction theorems concerning the actions of Player I. Section 4 introduces a convenient equivalent form of the game $G(K, X)$, denoted by $G'(K, X)$. A modification of that game involving G_δ sets and thus denoted by $G^{\delta}(K, X)$ is studied in section 5. The dual game $G^*(K, X)$ to the game $G'(K, X)$ is introduced in section 6; it provides, as a by-product, a covering characterization of spaces favorable for Player II. Finally, in section 7, the indeterminacy of $G(K, X)$ in ZFC is established.

For the topological background and undefined notions we refer to R. Engelking's monograph [1]. Each space considered here is assumed to be completely regular. N denotes the set of positive integers. 2^X denotes the family of closed subsets of the space X. K denotes a class of spaces such that (i) K contains all singletons, and (ii) K is invariant with respect to closed subspaces, i.e., $X \in K$ implies $2^X \subset K$. $\boldsymbol{1}$, $\boldsymbol{F}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ denote the classes of all singletons, finite spaces, compact spaces, and discrete spaces respectively. $\boldsymbol{D}K$, $\boldsymbol{L}K$ and $\boldsymbol{S}K$ denote the classes of spaces being free unions of spaces from K, locally K, and K-scattered, respectively. Inspite of the notation used in [7], $I(K, X)$ ($II(K, X)$) denotes the following statement: Player I (Player II, resp.) has a winning strategy in $G(K, X)$. For the modifications

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