Alternative rings in which every proper right ideal is maximal

by

A. Widiger

Abstract. The structure of the rings in the title is described. This generalizes results of Luh and Perticani obtained for the appropriate classes of commutative and associative rings respectively.

In [3] Perticani has studied the structure of commutative rings with identity in which every proper ideal is maximal. In [2] Luh has generalized these results for non-commutative associative idempotent rings. Precisely Luh has determined all associative rings A with $A^2 = A$ in which every proper right ideal is maximal.

In this note we shall follow this line and consider the class of all alternative rings in which every proper right ideal is maximal. Moreover, using methods of the theory of artinian rings we can give a short proof of our main result. Throughout this paper A will denote an alternative ring, $(A, +)$ its underlying additive group, and $J(A)$ its Jacobson–Kleinfeld radical.

The symbol $\oplus$ stands for a group theoretic and the symbol $\boxplus$ for a ring theoretic direct sum respectively. For any prime p $Z(p^k)$ denotes the cyclic group of order p^k and also the zero-ring on this group. A field means always an associative commutative field.

For basic facts on alternative rings the reader is referred to [4].

THEOREM. *A is a ring in which every proper right ideal is maximal if and only if A is one of the following types:*

(1) *A is the ring direct sum of two rings which are division rings or splitting Cayley algebras over fields.*

(2) *A is a division ring or the splitting Cayley algebra over a field.*

(3) *A is the 2×2 matrix ring over an associative division ring.*

(4) *A has an identity and an ideal $I \neq (0)$ such that $I^2 = (0)$, A/I is a division ring or the splitting Cayley algebra over a field, and I is a minimal right ideal of A.*

(5) *A is the ring direct sum of a ring $Z(p)$ and a ring which is a division ring or the splitting Cayley algebra over a field.*

(6) $A = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}$, *$D$ an associative division ring.*

(7) $A = \begin{bmatrix} GF(p) & GF(p) \\ 0 & 0 \end{bmatrix}$, *$GF(p)$ the prime field of characteristic $p \neq 0$.*

(8) $A = Z(p)\boxed{+}Z(q)$ *for primes* p *and* q.

(9) $A = Z(p^2)$ *or* $A = Z(p)$ *for some prime* p.

(10) A *is generated by one element* a *with defining relations* $pa = a^2, p^2a = 0$, p *a prime*.

(11) A *is generated by one element* a *with defining relations* $pa = a^3 = 0$, p *a prime*.

Proof. Let A be an alternative ring in which every proper right ideal is maximal. Trivially A is an artinian (and noetherian) ring. Then by [5] and [6] $J(A)$ is nilpotent and if $A \neq J(A)$ then $A/J(A)$ is the ring direct sum of simple rings which are either full matrix rings over associative division rings or Cayley algebras over fields.

Case 1. $A \neq J(A)$.

a) If $J(A) = (0)$ then A is the ring direct sum of full matrix rings over associative division rings and Cayley algebras over fields. Since every proper right ideal is maximal, at most two direct summands can appear. If exactly two summands appear, both have only the trivial right ideals. Hence (1) holds.

If A is simple then clearly either (2) or (3) holds.

(b) Assume $J(A) \neq (0)$. Then $J(A)^2 \neq J(A)$ since $J(A)$ is nilpotent. Therefore $J(A)^2$ is not a maximal ideal, hence $J(A)^2 = (0)$. $A/J(A)$ has an identity $\bar{e}$. Then there exist an idempotent element $e \in A$ with $e \in \bar{e}$ (e is a so called principal idempotent element). We consider the two-sided Peirce decomposition of A relative to e:

$$A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00} \tag{12}$$

(cf. [4]). Since e is a principal idempotent one has $A_{10}, A_{01}, A_{00} \subseteq J(A)$ and because of $J(A)^2 = (0)$ and the properties of the Peirce decomposition we have that A_{10}, A_{01}, and A_{00} are ideals of A. Since $A_{11} \neq (0)$ ($A \neq J(A)$!) our assumption forces two of them to be zero.

Now if $A_{00} = A_{10} = A_{01} = (0)$ then $A = A_{11}$ has an identity. $A/J(A)$ has only the trivial right ideals, and $J(A)$ is a minimal right ideal of A. Thus (4) holds.

If $A_{00} \neq (0)$ then $A = A_{11}\boxed{+}A_{00}$. Hence $A_{00} \cong Z(p)$, and A_{11} has only the trivial right ideals, i.e. (5) holds. If $A_{01} \neq (0)$ then $A = A_{11} \oplus A_{01}$, and e is a right identity of A. A_{11} is then a division ring or the splitting Cayley algebra over a field. We want to prove that A is associative. First let $a \in A_{01}$, $x, y \in A_{11}$. Then

$$(ax)y - a(xy) = (a, x, y) = -(x, a, y) = -(xa)y + x(ay) = 0$$

because of $A_{01}A_{11} \subseteq A_{01}$ and $A_{11}A_{01} = (0)$. Hence

$$(ax)y = a(xy)\,. \tag{13}$$

Then for $z \in A_{11}$ we have $((ax)y)z = (a(xy))z$ and using again (13) we obtain

$$a(x(yz)) = (ax)(yz) = a((xy)z)\,. \tag{14}$$

Now if $au = 0$, $a \neq 0$, $u \in A_{11}$ then from (13) one has that $aR = (0)$ where R is the right ideal of A_{11} generated by u. Since A_{11} has only the trivial right ideals and $ae = a$, $au = 0$ forces $u = 0$. Hence (14) leads to $x(yz) = (xy)z$, i.e. A_{11} is associative. This together with (13) yields the associativity of A. Hence A_{01} is a right vector space over the division ring $A_{11} = D$ and must have dimension one since every subspace is a (right) ideal of A. Therefore (6) holds.

If $A_{10} \neq (0)$ similary A is associative, and

$$A = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$$

with a division ring D. Now for every subgroup H of $(D, +)$ $\begin{bmatrix} 0 & H \\ 0 & 0 \end{bmatrix}$ is a right ideal of A. Hence $(D, +)$ is a simple group, and (7) holds.

Case 2. $A = J(A)$, i.e. A is nilpotent.

a) $A^2 = (0)$. Then clearly (8) or (9) hold.

b) $A^2 \neq (0)$. Then $A^3 = (0)$. A/A^2 is simple, hence $A/A^2 \cong Z(p)$ for some prime p. Since every subgroup of $(A^2, +)$ is an ideal of A, $(A^2, +)$ is a simple group, $(A^2, +) \cong Z(q)$. Now $p = q$ for otherwise $A = Z(p)\boxed{+}Z(q)$ and $A^2 = (0)$. Thus $|A| = p^2$. If $(A, +) \cong Z(p^2)$ then $A^2 = pA$ and one can choose $a \in A$ with $a^2 = pa$. Hence (10) holds.

If $(A, +) = Z(p) \oplus Z(p)$ then $(A, +) = (A^2, +) \oplus Z(p) = (A^2, +) \oplus (a)$ and (11) holds.

To finish the proof let us remark that the splitting Cayley algebra over a field contains only the trivial right ideals (cf. [1]). Therefore it suffices to prove that the rings of type (4) have the property that every proper right ideal is maximal. Let A be a ring of type (4) We show that I is the only proper right ideal of A. Let $R \neq (0)$ be a right ideal of A. If $R \supseteq I$ then $R = A$ or $R = I$ since A/I has only the trivial right ideals. If $R \not\supseteq I$ then $R \cap I = (0)$ since I is a minimal right ideal of A. Hence $R + I = A$. Let 1 be the identity of A, $1 = r + i$, $r \in R$, $i \in I$. Then $i = ri + i^2 = ri \in R$ since $i^2 = 0$. Because of $R \cap I = (0)$, $i = 0$, hence $1 \in R$ and $R = A$ follows contradicting $R \not\supseteq I$.

References

[1] I. Kaplansky, *Semisimple alternative rings*, Port. Math. 10 (1951), pp. 37–50.

[2] J. Luh, *Rings in which every proper right ideal is maximal*, Fund. Math. 91 (1976), pp. 183–188.

[3] F. I. Perticani, *Commutative rings in which every proper ideal is maximal*, Fund. Math. 71 (1971), pp. 193–198.

[4] R. D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York 1966.

[5] M. Slater, *Alternative rings with d.c.c. I*, J. of Algebra 11 (1969), pp. 102–110.

[6] — *Alternative rings with d.c.c. III*, J. of Algebra 18 (1971), pp. 179–200.

Accepté par la Rédaction le 26. 7. 1980