Spaces with a primitive base and perfect mappings

by

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Abstract. The class of topological spaces with a primitive base is shown to be invariant under perfect images. A characterization of quasi-developable spaces is given (using a primitive base) and this is used to finish the perfect mapping question for spaces with a quasi-development (or θ-base).

1. Introduction. The primitive base concept and other topological concepts, defined by means of primitive sequences, have proved to be important topics in general topology as well as providing uniform techniques and a uniform language for the study of other base axioms and related topological structures. Readers not familiar with the language of primitive structures, as studied by H. H. Wicke and J. M. Worrell, Jr., may wish to consult papers such as [Wi], [WW_1], or [WW_3] for a review of the current and historical importance of these notions.

It is known that developable spaces, spaces with a base of countable order, and spaces with a θ-base (quasi-developable spaces) can all be defined using the fundamental base structure known as a primitive base. In [Wo_1] and [Wo_2] respectively, Worrell showed that developable spaces and spaces with a base of countable order were preserved under a perfect mapping. The author has partial results in [Bu] concerning the invariance, under a perfect map, of spaces with a θ-base.

In this paper we show that the class of topological spaces with a primitive base is invariant under perfect images. A characterization of spaces with a θ-base is given (using a primitive base) and this is used to finish the perfect mapping question for spaces with a θ-base. We finish this section with a few of the basic definitions necessary throughout the paper. Other concepts will be reviewed as needed.

If $\mathscr{W}$ is a well-ordered collection of sets the notation $<$ will be used to denote the order. This should cause no confusion even with more than one such well-ordered collection. Whenever $W \in \mathscr{W}$ the *primitive part* of W in $\mathscr{W}$ is given by

$$p(W, \mathscr{W}) = W - \bigcup \{W' : W' \in \mathscr{W}, W' < W\}.$$

For a set A, $F(A, \mathscr{W})$ denotes the first W in $\mathscr{W}$ such that $A \subset W$ (if such a W exists). Otherwise $F(A, \mathscr{W}) = \emptyset$. Note that if $W \in \mathscr{W}$, $W \neq \emptyset$, then $W = F(x, \mathscr{W})$ if and only if $x \in p(W, \mathscr{W})$.

A space X is said to have a *primitive base* ([WW$_2$], [Wi]) if there is a sequence $\{\mathscr{H}_n\}_1^\infty$ of well-ordered open covers of X such that if $x \in X$ then $\{F(x, \mathscr{H}_n): n \in N\}$ is a local base at x. In addition, it may be assumed that $\{\mathscr{H}_n\}_1^\infty$ is an *open primitive sequence* of X ([WW$_2$], [Wi]). That is, for each $n \in N$:

(a) For each $H \in \mathscr{H}_n, p(H, \mathscr{H}_n) \neq \emptyset$.

(b) If $k<n$, $H \in \mathscr{H}_k$, $H' \in \mathscr{H}_n$ with $p(H, \mathscr{H}_k) \cap p(H', \mathscr{H}_n) \neq \emptyset$ then $H' \subset H$ (i.e., if $x \in X$ then $F(x, \mathscr{H}_n) \subset F(x, \mathscr{H}_k)$ whenever $k<n$).

Recall that a perfect mapping $f\colon X \to Y$ is a closed continuous onto map with $f^{-1}(y)$ compact in X for every $y \in Y$.

2. Primitive bases and perfect mappings. Throughout this section X is assumed to be a topological space and $\{\mathscr{W}_n\}_1^\infty$ is an open primitive sequence of covers of X such that if $x \in X$ then $\{F(x, \mathscr{W}_n): n \in N\}$ is a local base at x. The main result of this section is Theorem 2.5; we proceed with a series of lemmas.

2.1. Lemma. *If C is a nonempty compact subset of X and $k \in N$, then $\{W \in \mathscr{W}_k\colon p(W, \mathscr{W}_k) \cap C \neq \emptyset\}$ has a largest element (with respect to the order on $\mathscr{W}_k$).*

Proof. Let $\mathscr{H} = \{W \in \mathscr{W}_k\colon p(W, \mathscr{W}_k) \cap C \neq \emptyset\}$ and for each $H \in \mathscr{H}$ let

$$V(H) = \bigcup \{W \in \mathscr{H}\colon W \leqslant H\}.$$

If $\mathscr{H}$ does not have a largest element it follows that $\{V(H)\colon H \in \mathscr{H}\}$ is an open cover of C with no finite subcover. This is impossible since C is compact.

Notation. For $A \subset X$ and $i \in N$ let $L(A, i)$ denote the largest element of $\mathscr{W}_i$ (if it exists) such that $p(L(A, i), \mathscr{W}_i) \cap A \neq \emptyset$.

2.2. Definition. If $A \subset X$ and $(n_1, \ldots, n_r)$ is a nondecreasing finite sequence of natural numbers, then a collection $\mathscr{H} = (W_1, W_2, \ldots, W_r)$ is said to be a *canonical cover* of A (with respect to the sequence $(n_1, \ldots, n_r)$) if:

(1) $A \subset \bigcup \mathscr{H}$.

(2) $W_i \in \mathscr{W}_{n_i}$ for $i = 1, \ldots, r$.

(3) $W_i = L(A - \bigcup_{j<i} W_j, n_i)$ for $i = 1, \ldots, r$.

It may be useful to remark on a few of the elementary properties of the canonical cover $\mathscr{H}$ given in 2.2. First notice that such a cover is necessarily unique, i.e., if A is given and $(n_1, \ldots, n_r)$ is given, there is at most one canonical cover of A with respect to $(n_1, \ldots, n_r)$. Also, if $1 \leqslant i<j \leqslant r$ there is $W'_j \in \mathscr{W}_{n_i}$ such that $W_j \subset W'_j$ and $W'_j < W_i$. Hence $W_j \cap p(W_i, \mathscr{W}_{n_i}) = \emptyset$ and it follows that $\mathscr{H}$ is a special type of minimal cover of A.

The notion of a canonical cover and its use in Lemma 2.3 is the key to Theorem 2.5. In the remainder of this section, $\{\mathscr{W}_n\}_1^\infty$ is assumed to have *consistent orderings* [Wi], i.e., whenever $W, V \in \mathscr{W}_{n+1}$, $W \leqslant V$, then $F(W, \mathscr{W}_n) \leqslant F(V, \mathscr{W}_n)$.

2.3. Lemma. *If $C \subset U \subset X$, where C is a nonempty compact set and U is open in X, there is a canonical cover $\mathscr{H}$ of C (with respect to some sequence $(n_1, \ldots, n_r)$) such that $\bigcup \mathscr{H} \subset U$.*

Proof. Let $C_1 = C$ and n_1 be the first element of N such that $L(C_1, n_1) \subset U$. (Such an integer exists since $\{p(L(C_1, n), \mathscr{W}_n) \cap C\}_{n=1}^\infty$ is a monotone decreasing sequence of nonempty compact sets, closed in C, and if $x \in \bigcap_{n=1}^\infty p(L(C_1, n), \mathscr{W}_n) \cap C$ then $\{L(C_1, n)\}_1^\infty$ is a local base at x.) Continuing by induction, if C_k and n_k are defined, let $C_{k+1} = C_k - L(C_k, n_k)$ and (if $C_{k+1} \neq \emptyset$) let n_{k+1} be the smallest integer $\geqslant n_k$ such that $L(C_{k+1}, n_{k+1}) \subset U$. If, at any stage, $C_{k+1} = \emptyset$ then

$$\{L(C_i, n_i)\ 1 \leqslant i \leqslant k\}$$

is the desired canonical cover of C.

So assume $C_i \neq \emptyset$ for each $i \in N$ and note than $n_i \to \infty$ (otherwise, there would exist $m \in N$ with a strictly decreasing infinite sequence in $\mathscr{W}_m$). Now let

$$K = C - \bigcup_{i=1}^\infty L(C_i, n_i)$$

and let

$$K_i = p(L(K, i), \mathscr{W}_i) \cap K,$$

for $i \in N$; then $\{K_i\}_1^\infty$ is a decreasing sequence of nonempty compact sets (closed in K). If $z \in \bigcap_{i=1}^\infty K_i$ then $\{L(K, i)\}_{i=1}^\infty$ is a local base at z so there exists $m \in N$ such that $L(K, m) \subset U$ (we may assume $m = n_k$, for some $k \in N$). Now

$$E = C - \bigcup \{W\colon W \in \mathscr{W}_m, W \leqslant L(K, m)\}$$

is covered by $\{L(C_i, n_i)\}_1^\infty$ (by definition of K and $L(K, m)$) so there exists $n_r \geqslant n_k = m$ such that

$$E \subset \bigcup_{i=1}^r L(C_i, n_i).$$

We may assume $n_{r+1} > n_r$ (otherwise choose appropriate n_r) and n_{r+1} is the smallest integer $\geqslant n_r$ such that $L(C_{r+1}, n_{r+1}) \subset U$. Since $n_{r+1} > n_r$ we know $L(C_{r+1}, n_r) \not\subset U$. Since

$$C_r - \bigcup_{i \leqslant r} L(C_i, n_i) = C_{r+1}\ (\neq \emptyset)$$

we have

$$C_{r+1} \subset \bigcup \{W\colon W \in \mathscr{W}_m, W \leqslant L(K, m)\},$$

which implies

$$p(L(C_{r+1}, n_r), \mathscr{W}_{n_r}) \cap (\bigcup \{W\colon W \in \mathscr{W}_m, W \leqslant L(K, m)\}) \neq \emptyset.$$

Pick y from this intersection and find $V_m \in \mathscr{W}_m$ such that $y \in p(V_m, \mathscr{W}_m)$. Then

$$p(V_m, \mathscr{W}_m) \cap (\bigcup \{W\colon W \in \mathscr{W}_m, W \leqslant L(K, m)\}) \neq \emptyset$$

so $V_m \leqslant L(K, m)$. Now

$$y \in p(V_m, \mathscr{W}_m) \cap p(L(C_{r+1}, n_r), \mathscr{W}_{n_r}) \neq \emptyset$$

implies $V_m = F(L(C_{r+1}, n_r), \mathscr{W}_m)$, and

$$p(L(K, m), \mathscr{W}_m) \cap p(L(K, n_r), \mathscr{W}_{n_r}) \neq \emptyset$$

implies $L(K, m) = F(L(K, n_r), \mathscr{W}_m)$. Notice that $V_m \neq L(K, m)$ for otherwise we would have $L(C_{r+1}, n_r) \subset V_m = L(K, m) \subset U$ which is impossible. Hence $V_m < L(K, m)$ and by using the consistency of the orders on $\{\mathscr{W}_i\}_1^\infty$ it follows that $L(C_{r+1}, n_r) \leqslant L(K, n_r)$. But $K \subset C_{r+1}$ implies $L(K, n_r) \leqslant L(C_{r+1}, n_r)$ and so $L(C_{r+1}, n_r) = L(K, n_r)$. This is again a contradiction since $L(K, n_r) \subset U$ and $L(C_{r+1}, n_r) \not\subset U$. That completes the proof of the lemma.

Now suppose $f: X \to Y$ is a perfect mapping. We are ready to construct a primitive base for the space Y.

For each nondecreasing finite sequence $(n_1, \ldots, n_r)$ of natural numbers let $\mathscr{G}(n_1, \ldots, n_r)$ be the collection of all sequences $\mathscr{H} = (H_1, \ldots, H_r)$ where $\mathscr{H}$ is canonical cover of $f^{-1}(y)$ (with respect to $(n_1, \ldots, n_r)$) for some $y \in Y$. For each $\mathscr{H} \in \mathscr{G}(n_1, \ldots, n_r)$ let

$$U(\mathscr{H}) = Y - f(X - \bigcup \mathscr{H})$$

and

$$\mathscr{U}(n_1, \ldots, n_r) = \{U(\mathscr{H}) \colon \mathscr{H} \in \mathscr{G}(n_1, \ldots, n_r)\} \cup \{Y\}.$$

2.4. Lemma. *If $\mathscr{H} = (H_1, \ldots, H_r)$ and $\mathscr{K} = (K_1, \ldots, K_r)$ are distinct elements of $\mathscr{G}(n_1, \ldots, n_r)$ then $U(\mathscr{H}) \neq U(\mathscr{K})$.*

Proof. Suppose $\mathscr{H}$ is less than $\mathscr{K}$ relative to the lexicographic order on $\mathscr{W}_{n_1} \times \ldots \times \mathscr{W}_{n_r}$ and let n_j be the first integer in $(n_1, \ldots, n_r)$ such that $H_j < K_j$ (relative to the order on $\mathscr{W}_{n_j}$). Now $\mathscr{K} \in \mathscr{G}(n_1, \ldots, n_r)$ implies there is some $y \in Y$ such that $\mathscr{K}$ is a canonical cover of $f^{-1}(y)$. Pick

$$z \in p(K_j, \mathscr{W}_{n_j}) \cap \big(f^{-1}(y) - \bigcup_{i<j} K_i\big).$$

Clearly $z \notin H_i$ if $1 \leqslant i < j$ and $z \notin H_j$ since $H_j < K_j$ and $z \in p(K_j, \mathscr{W}_{n_j})$. If $j < i \leqslant r$ it follows by the definition of $\mathscr{H}$ being a canonical cover of some set A that there is some $H_i' \in \mathscr{W}_{n_j}$ with $H_i \subset H_i'$ and $H_i' \leqslant H_j < K_j$. Again we have $z \notin H_i$ so $z \notin \bigcup \mathscr{H}$ and $y = f(z) \notin U(\mathscr{H})$. However $y \in U(\mathscr{K})$ so $U(\mathscr{K}) \neq U(\mathscr{H})$ and the lemma is proved.

The above lemma indicates there is a one-to-one correspondence between $\mathscr{U}(n_1, \ldots, n_r) - \{Y\}$ and a subset of $\mathscr{W}_{n_1} \times \mathscr{W}_{n_2} \times \ldots \times \mathscr{W}_{n_r}$. Using the lexicographic order on $\mathscr{W}_{n_1} \times \ldots \times \mathscr{W}_{n_r}$ induce a well-order of $\mathscr{U}(n_1, \ldots, n_r)$ in the obvious manner (making Y the largest element).

The family

$$\{\mathscr{U}(n_1, \ldots, n_r) \colon (n_1, \ldots, n_r) \text{ is a finite nondecreasing sequence from } N\}$$

is a countable collection of well-ordered open covers of Y, and to show Y has a primitive base it suffices to show that if $y \in V \subset Y$, with V open, there is a nondecreasing sequence $(n_1, \ldots, n_r)$ from N such that the first element of $\mathscr{U}(n_1, \ldots, n_r)$ containing y is a subset of V. To this end note that Lemma 2.3 says there is a canonical cover $\mathscr{H}$ of $f^{-1}(y)$ (with respect to some $(n_1, \ldots, n_r)$) such that $\bigcup \mathscr{H} \subset f^{-1}(V)$. Hence $U(\mathscr{H}) \in \mathscr{U}(n_1, \ldots, n_r)$, $y \in U(\mathscr{H}) \subset f(\bigcup \mathscr{H}) \subset V$, and the proof of Lemma 2.4 shows that $U(\mathscr{H})$ is the first element of $\mathscr{U}(n_1, \ldots, n_r)$ containing y. That concludes the proof of the main result of this section.

2.5. Theorem. *If X has a primitive base and $f: X \to Y$ is a perfect mapping then Y has a primitive base.*

3. Spaces with a θ-base and perfect mappings. A base $\mathscr{B}$ for a space X is said to be a *θ-base* [WoW] if $\mathscr{B}$ can be written as $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$ where if $x \in U \subset X$, U open in X, there exists some n and $B \in \mathscr{B}_n$ such that $x \in B \subset U$ and $\operatorname{ord}(x, \mathscr{B}_n)$ is finite. A *quasi-development* [B] for a space X is a base $\mathscr{G} = \bigcup_{i=1}^{\infty} \mathscr{G}_n$ where for any $x \in X$ the collection $\{\operatorname{st}(x, \mathscr{G}_n) \colon n \in N,\ x \in \operatorname{St}(x, \mathscr{G}_n)\}$ is a local base at x.

Bennett and Lutzer have shown [BL] that a space X has a θ-base if and only if it is quasi-developable. In fact, a quasi-developable space X has a base $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ where if $x \in U$, with U open in X, there is some n with $\operatorname{st}(x, \mathscr{H}_n) \subset U$ and $\operatorname{ord}(x, \mathscr{H}_n) = 1$. This base, which is simultaneously a θ-base and a quasi-development, is quite often the easiest form of a θ-base to work with.

To characterize quasi-developable spaces, using spaces with a primitive base, we consider the following property on a topological space.

(*) If $\mathscr{U}$ is any well-ordered open cover of X there is an open refinement $\mathscr{G} = \bigcup_{n=1}^{\infty} \mathscr{G}_n$ of $\mathscr{U}$ such that if $x \in X$ there is some $n \in N$ with $\operatorname{st}(x, \mathscr{G}_n) \subset F(x, \mathscr{U})$ and $\operatorname{ord}(x, \mathscr{G}_n) = 1$.

The proof of the next result is straightforward and is left to the reader.

3.1. Theorem. *A space X is quasi-developable if and only if X satisfies* (*) *and has a primitive base.*

The main reason for Theorem 3.1 is to use Theorem 2.5 to solve the corresponding problem concerning the invariance of quasi-developable spaces under perfect mappings. This will follow once condition (*) above is shown to be preserved under perfect mappings. We begin with a preliminary lemma.

3.2. Lemma. *If X satisfies condition* (*) *and $\mathscr{U}$ is any well-ordered open cover of X there is an open refinement $\mathscr{G} = \bigcup_{n=1}^{\infty} \mathscr{G}_n$ of $\mathscr{U}$ such that whenever $C \subset X$ $(C \neq \emptyset)$ is compact and $U \in \mathscr{U}$ such that $C \subset p(U, \mathscr{U})$ there is some $n \in N$ such that $\mathscr{G}_n$ covers C and $\operatorname{st}(x, \mathscr{G}_n) \subset U$ for some $x \in C$ with $\operatorname{ord}(x, \mathscr{G}_n) = 1$.*

Proof. If $\mathscr{U}$ is any well-ordered open cover of X let $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ be the open refinement as given in (*). Let $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \ldots$ be an enumeration of the open collections obtained by taking unions of a finite number of families from $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$. We show $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ satisfies the desired condition. Suppose C is a nonempty compact set in X and $U \in \mathcal{U}$ such that $C \subset p(U, \mathcal{U})$. Let n_1 be the first integer such that there is some $x_1 \in C$ with $\mathrm{st}(x_1, \mathcal{H}_{n_1}) \subset U$ and $\mathrm{ord}(x_1, \mathcal{H}_{n_1}) = 1$. If $\mathcal{H}_{n_1}$ covers C then $\mathcal{H}_{n_1}$ is one of the collections $\mathcal{G}_j$ and we are through. Otherwise, there is a finite sequence $n_1, n_2, \ldots, n_k$ (nondecreasing) of positive integers such that $\bigcup_{i=1}^{k} \mathcal{H}_{n_i}$ covers C and for each i, $1 < i \leqslant k$, n_i is the smallest integer such that there is

$$x_i \in C - \bigcup \{H\colon H \in \mathcal{H}_{n_j},\ 1 \leqslant j < i\}$$

with $\mathrm{st}(x_i, \mathcal{H}_{n_i}) \subset U$ and $\mathrm{ord}(x_i, \mathcal{H}_{n_i}) = 1$. It follows that $\bigcup_{i=1}^{k} \mathcal{H}_{n_i} = \mathcal{G}_r$, for some $r \in N$, and for this choice of $\mathcal{G}_r$ we have $\mathrm{st}(x_k, \mathcal{G}_r) \subset U$ and $\mathrm{ord}(x_k, \mathcal{G}_r) = 1$.

The author wishes to thank the referee for suggestions which substantially shortened the proof of the next theorem.

3.3. THEOREM. *If $f\colon X \to Y$ is a perfect mapping and X satisfies condition* $(*)$ *then Y also satisfies condition* $(*)$.

Proof. Let $\mathcal{W}$ be any well-ordered open cover of Y and let $\mathcal{U} = \{f^{-1}(W)\colon W \in \mathcal{W}\}$ have the obvious order induced by $\mathcal{W}$. Let $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ be the refinement of $\mathcal{U}$ given by Lemma 3.2.

For $n \in N$ and $W \in \mathcal{W}$ let

$$A_n(W) = \{x\colon x \in \mathrm{st}(x, \mathcal{G}_n) \subset \bigcup \{U \in \mathcal{U}\colon U < f^{-1}(W)\}\}.$$

Note that

$$\overline{A_n(W)} \cap (\bigcup \mathcal{G}_n) \subset \bigcup \{U \in \mathcal{U}\colon U < f^{-1}(W)\}.$$

Let

$$H_n(W) = W - \overline{f(A_n(W))} \quad \text{and} \quad \mathcal{H}_n = \{H_n(W)\colon W \in \mathcal{W}\}.$$

To prove that $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ satisfies $(*)$ with respect to $\mathcal{W}$ take $y \in Y$ and let $W = F(y, \mathcal{W})$. Let $n \in N$ and $x \in f^{-1}(y)$ be chosen such that

$$f^{-1}(y) \subset \bigcup \mathcal{G}_n \quad \text{and} \quad \mathrm{st}(x, \mathcal{G}_n) \subset f^{-1}(W).$$

Then $f^{-1}(y) \cap \overline{A_n(W)} = \emptyset$ and consequently $y \in H_n(W) \subset W$. If $W' < W$ then $y \notin H_n(W')$ because $y \notin W'$. If $W' > W$ then $x \in A_n(W')$ and therefore $y \notin H_n(W')$. Thus $\mathrm{ord}(y, \mathcal{H}_n) = 1$ and $\mathrm{st}(y, \mathcal{H}) \subset W$.

Combining the results of Theorem 2.5, Theorem 3.1, and Theorem 3.3 we have the main result of this section.

3.4. THEOREM. *If $f\colon X \to Y$ is a perfect mapping and X is a quasi-developable space then Y is also quasi-developable.*

It should be remarked that condition $(*)$ has equivalent formulations. The proof of Remark 3.5, which uses standard techniques, is left to the reader. The proof can also be recovered from techniques used for Lemma 4 and Proposition 7 of [BL].

3.5. Remark. For any space X the following are equivalent.

(a) X satisfies $(*)$.

(b) If $\mathcal{U}$ is any well-ordered open cover of X there is an open refinement $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ of $\mathcal{U}$ such that if $x \in X$ there is some $n \in N$ with $x \in \mathrm{st}(x, \mathcal{G}_n) \subset F(x, \mathcal{U})$.

(c) If $\mathcal{U}$ is any well-ordered open cover of X there is a refinement $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ such that each $\mathcal{P}_n$ is discrete relative to $\bigcup \mathcal{P}_n$ and if $x \in X$ there is some $n \in N$ and $P \in \mathcal{P}_n$ with $x \in P \subset F(x, \mathcal{U})$.

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