Extension of group-valued set functions defined on lattices

by

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Abstract. The extension theorem of Sion for a group-valued function on a ring is generalized to extension theorems for a group-valued function on a lattice.

1. Introduction. We explain the minimal terminology necessary to state, in more general setting, the result of [2] which will be the basis for the generalization of the Sion extension theorem, presented in section 2.

The term *lattice* refers to a lattice $\mathscr{L}$ of subsets of a fixed set T such that $\emptyset \in \mathscr{L}$. Let λ map $\mathscr{L}$ into an abelian group G: it is said to be *strongly additive* if $\lambda(\emptyset) = 0$ and $\lambda(E \cup F)+\lambda(E \cap F) = \lambda(E)+\lambda(F)$ for all $E, F \in \mathscr{L}$. We will use without explicit mention the following result of Pettis ([8, p. 189], [6, p. 327]): "Every strongly additive set function $\lambda\colon \mathscr{L} \to G$ extends uniquely to an additive set function on the ring $\mathscr{R}(\mathscr{L})$ generated by $\mathscr{L}$". This Pettis extension will be denoted by the same symbol λ.

In the rest of the paper, G is assumed to be a complete Hausdorff abelian topological group. It is well known that the topology of G can be generated by a family P of continuous invariant pseudo-metrics on G [4, p. 82]. For $p \in P$ we put $|x|_p = p(x, 0)$, $x \in G$. Then $|x|_p = 0 \Leftrightarrow x = 0$, $|-x|_p = |x|_p$ and $|x+y|_p \leqslant |x|_p+|y|_p$. Henceforth we write $|.|$ for an arbitrary $|.|_p$, $p \in P$.

The term *function* refers to a set function λ mapping a lattice $\mathscr{D}(\lambda)$ into G. We say that λ is σ-*additive* (δ-*additive*) if, for every increasing (decreasing) sequence (L_n) in $\mathscr{D}(\lambda)$ with $\lim L_n \in \mathscr{D}(\lambda)$, we have $\lambda(L_n) \to \lambda(\lim_n L_n)$. If λ is σ-additive and δ-additive we say that λ is (σ, δ)-*additive*. If for every monotone sequence (L_n) in $\mathscr{D}(\lambda)$, $(\lambda(L_n))$ converges, we say that λ is *monotonely convergent*.

Let λ be a function of domain $\mathscr{L} = \mathscr{D}(\lambda)$ and let $E \subseteq T$. The class $\{L \in \mathscr{L}: L \subseteq E\}$ is non-empty and directed by $\supseteq$; it defines the net $(\lambda(L))_{L \subseteq E, L \in \mathscr{L}}$. Similarly, if the class $\{L \in \mathscr{L}: L \supseteq E\}$ is non-empty, it is directed by $\subseteq$ and defines the net $(\lambda(L))_{L \supseteq E, L \in \mathscr{L}}$.

Let λ, μ be functions; we say that (a) λ is μ-*lower regular* if, for every $E \in \mathscr{D}(\lambda)$, $\lim_{L \subseteq E, L \in \mathscr{D}(\mu)} \mu(L) = \lambda(E)$; (b) λ is μ-*upper regular* if, for every $E \in \mathscr{D}(\lambda)$, the class $\{L \in \mathscr{D}(\mu): L \supseteq E\}$ is non-empty and $\lim_{L \supseteq E, L \in \mathscr{D}(\mu)} \mu(L) = \lambda(E)$.

Suppose that λ is strongly additive, (σ,δ)-additive and monotonely convergent. From Lemma 2.4 (Lemma 2.5) and Lemma 2.6 of [2], it follows that the function $\lambda_\sigma(E) = \lim\limits_{L\subseteq E, L\in\mathscr{L}} \lambda(L)$, $E\in\mathscr{L}_\sigma$ $\left(\lambda_\delta(E) = \lim\limits_{L\supseteq E, L\in\mathscr{L}} \lambda(L),\ E\in\mathscr{L}_\delta\right)$ is well defined and is strongly additive, monotonely convergent and σ-additive (δ-additive).

If a lattice $\mathscr{L}$ is closed under countable unions and countable intersections we say that $\mathscr{L}$ is a (σ,δ)-*lattice*.

Taking into account the generalization of Kranz [5], the Theorem 2.10 of [2, p. 104] can be stated:

1.1. THEOREM. *Let λ be a function such that $\mathscr{D}(\lambda) = \mathscr{L}$. If λ is strongly additive and (σ,δ)-additive, then λ extends uniquely to a strongly additive (σ,δ)-additive function on the (σ,δ)-lattice generated by $\mathscr{L}$ if and only if*

(a) *λ is monotonely convergent,*

(b) *λ_σ is λ_δ-lower regular or (equivalently) λ_δ is λ_σ-upper regular.*

In section 3 we prove the σ-additivity of the Pettis extension of a strongly additive function λ, under hypotheses related to tightness. We also study the existence of a unique σ-additive extension of λ on the σ-ring generated by $\mathscr{D}(\lambda)$. This last part of the paper is motivated by the work of Lipecki [7].

If $\mathscr{L}$ is a lattice, a set $L\in\mathscr{L}$ may be called an $\mathscr{L}$-*set*; a sequence in $\mathscr{L}$ may be called an $\mathscr{L}$-*sequence*. The symbol $\exp(T)$ denotes the set of all subsets of T.

2. Lattice extension. We will need the following generalization of Lemma 2.3 of [2, p. 100]:

2.1. LEMMA. *Let λ be a function and let μ be a strongly additive function:*

(a) *If λ is μ-lower regular, then for every decreasing $\mathscr{D}(\lambda)$-sequence $\{E_n\}$ and every $\varepsilon>0$, there is a decreasing $\mathscr{D}(\mu)$-sequence $\{F_n\}$ such that $F_n\subseteq E_n$ and $|\mu(L)-\lambda(E_n)|<\varepsilon$ whenever $L\in\mathscr{D}(\mu)$ and $F_n\subseteq L\subseteq E_n$.*

(b) *If λ is μ-upper regular, then for every increasing $\mathscr{D}(\lambda)$-sequence $\{E_n\}$ and every $\varepsilon>0$, there is an increasing $\mathscr{D}(\mu)$-sequence $\{F_n\}$ such that $F_n\supseteq E_n$ and $|\mu(L)-\lambda(E_n)|<\varepsilon$ whenever $L\in\mathscr{D}(\mu)$ and $F_n\supseteq L\supseteq E_n$.*

Proof. (a) We note the identity for any finite $\exp(T)$-sequence $\{A_k\}_1^n$, $n>1$:

(i) $A_n-\bigcap\limits_{k=1}^{n} A_k = \bigcup\limits_{k=1}^{n-1}\left[\{A_n\cap\bigcap\limits_{i=1}^{k-1}A_i\}-\{A_n\cap\bigcap\limits_{i=1}^{k}A_i\}\right]$ where $\bigcap\limits_{i=1}^{k-1}A_i = T$ for $k=1$.

We note also the following property of μ:

(ii) Let $E\in\exp(T)$, $\varepsilon>0$: If F is a $\mathscr{D}(\mu)$-set contained in E such that $F\subseteq F'\subseteq E$, $F'\in\mathscr{D}(\mu)$ imply $|\mu(F')-\mu(F)|<\varepsilon$, then $A, B\subseteq E$, $A-B\subseteq E-F$ and $A, B\in\mathscr{D}(\mu)$ imply $|\mu(A-B)|<2\varepsilon$.

In fact,

$$|\mu(A-B)| = |\mu(A\cup B\cup F)-\mu(B\cup F)|\leqslant|\mu(A\cup B\cup F)-\mu(F)|+ \\ +|\mu(F)-\mu(B\cup F)<\varepsilon+\varepsilon.$$

By the lower regularity there is a $\mathscr{D}(\mu)$-sequence $\{B_n\}_1^\infty$ such that $B_n\subseteq E_n$ and $|\mu(F)-\lambda(E_n)|$, $|\mu(F)-\mu(B_n)|<\varepsilon/2^{n+2}$ whenever $B_n\subseteq F\subseteq E_n$ and $F\in\mathscr{D}(\mu)$. Setting

$F_n = \bigcap\limits_{i=1}^{n} B_i$, $n = 1, 2, \ldots$, we have the decreasing $\mathscr{D}(\mu)$-sequence $\{F_n\}$ such that $F_n\subseteq E_n$. It remains to show that $F_n\subseteq L\subseteq E_n$, $L\in\mathscr{D}(\mu)$ imply $|\mu(L)-\lambda(E_n)|<\varepsilon$. This is clear for $n=1$ so we suppose $n>1$. Since

$$|\mu(L)-\lambda(E_n)|\leqslant|\lambda(E_n)-\mu(B_n\cup L)|+|\mu(B_n\cup L)-\mu(L)|<\frac{\varepsilon}{2^{n+2}}+|\mu(B_n-L)|$$

it will suffice to show that $|\mu(B_n-L)|<\frac{1}{2}\varepsilon$.

By (i) we have

$$B_n-L = (L\cup B_n)-(L\cup F_n) = (L\cup B_n)-\bigcap_{k=1}^{n}(L\cup B_k) \\ = \bigcup_{k=1}^{n-1}\left[\{(L\cup B_n)\cap\bigcap_{i=1}^{k-1}(L\cup B_i)\}-\{(L\cup B_n)\cap\bigcap_{i=1}^{k}(L\cup B_i)\}\right].$$

The kth term of this partition is a difference, contained in E_k-B_k, of $\mathscr{D}(\mu)$-sets contained in E_k, so by (ii),

$$|\mu[\{(L\cup B_n)\cap\bigcap_{i=1}^{k-1}(L\cup B_i)\}-\{(L\cup B_n)\cap\bigcap_{i=1}^{k}(L\cup B_i)\}]|<\frac{\varepsilon}{2^{k+1}}$$

and therefore $|\mu(B_n-L)|<\frac{1}{2}\varepsilon$.

(b) We note the identity for any finite $\exp(T)$-sequence $\{A_k\}_1^n$, $n>1$:

(iii) $\bigcup\limits_{k=1}^{n} A_k-A_n = \bigcup\limits_{k=1}^{n-1}\left(\bigcup\limits_{i=k}^{n}A_i-\bigcup\limits_{i=k+1}^{n}A_i\right)$.

We note also the following property of μ:

(iv) Let $E\in\exp(T)$, $\varepsilon>0$: If F is a $\mathscr{D}(\mu)$-set containing E such that $F\supseteq F'\supseteq E$, $F'\in\mathscr{D}(\mu)$ imply $|\mu(F')-\mu(F)|<\varepsilon$, then $A, B\supseteq E$, $A-B\subseteq F-E$ and $A, B\in\mathscr{D}(\mu)$ imply $|\mu(A-B)|<2\varepsilon$. In fact,

$$|\mu(A-B)| = |\mu(A\cap F)-\mu(A\cap B\cap F)|\leqslant|\mu(A\cap F)-\mu(F)|+ \\ +|\mu(F)-\mu(A\cap B\cap F)|<2\varepsilon.$$

By the upper regularity there is a $\mathscr{D}(\mu)$-sequence $\{B_n\}$ such that $B_n\supseteq E_n$ and $|\mu(F)-\lambda(E_n)|$, $|\mu(F)-\mu(B_n)|<\varepsilon/2^{n+2}$ whenever $B_n\supseteq F\supseteq E_n$ and $F\in\mathscr{D}(\mu)$. Setting $F_n = \bigcup\limits_{i=1}^{n} B_i$, $n=1,2,\ldots$, we have the increasing $\mathscr{D}(\mu)$-sequence $\{F_n\}$ such that $F_n\supseteq E_n$. It remains to show that $F_n\supseteq L\supseteq E_n$, $L\in\mathscr{D}(\mu)$ imply $|\mu(L)-\mu(E_n)|<\varepsilon$. We may suppose $n>1$. Since

$$|\mu(L)-\lambda(E_n)|\leqslant|\mu(L)-\mu(L\cap B_n)|+|\mu(L\cap B_n)-\lambda(E_n)|<|\mu(L-B_n)|+\frac{\varepsilon}{2^{n+2}},$$

it will suffice to show that $|\mu(L-B_n)|<\frac{1}{2}\varepsilon$. By (iii),

$$L-B_n = (L\cap F_n)-(L\cap B_n) = \bigcup_{k=1}^{n}(L\cap B_i)-(L\cap B_n)$$

$$= \bigcup_{k=1}^{n-1}\Big[\bigcup_{i=k}^{n}(L\cap B_i)-\bigcup_{i=k+1}^{n}(L\cap B_i)\Big].$$

The required conclusion will follow as in (a), using (iv), if we show that, for $1\leqslant k<n$,

$$x\in\bigcup_{i=k}^{n}(L\cap B_i)-\bigcup_{i=k+1}^{n}(L\cap B_i) \quad \text{implies} \quad x\in B_k-E_k.$$

We have $x\in L\cap B_k$. If we suppose that $x\in E_k$, then $x\in L\cap B_i$ for $k<i\leqslant n$, a contradiction.

We call a function ν *continuous* if, whenever (E_n), (F_n) are decreasing, increasing $\mathscr{D}(\nu)$-sequences, respectively, such that $\lim_n E_n\subseteq\lim_n F_n$, we have

$$\lim_n[\nu(E_n)-\nu(E_n\cap F_n)] = 0.$$

Specializing each sequence, in turn, to an appropriate constant sequence, we see that a *continuous function is* (σ,δ)-*additive*.

In the rest of this section, $\lambda\colon \mathscr{L}\to G$ is a strongly additive monotonely convergent continuous function whose domain is a lattice $\mathscr{D}(\lambda)=\mathscr{L}$.

2.2. LEMMA. *λ_σ is continuous.*

Proof. Let (E_n), (F_n) be decreasing, increasing $\mathscr{L}_\sigma$-sequences, respectively, such that $\lim_n E_n\subseteq\lim_n F_n$. It must be shown that $\lambda_\sigma(E_n)-\lambda_\sigma(E_n\cap F_n)\to 0$. Let $\varepsilon>0$. By Lemma 2.1 there is a decreasing $\mathscr{L}$-sequence (L_n) such that $L_n\subseteq E_n$ and $L_n\subseteq L\subseteq E_n$, $L\in\mathscr{L}$ imply $|\lambda(L)-\lambda_\sigma(E_n)|<\varepsilon$. Also, $L_n\subseteq H\subseteq E_n$, $H\in\mathscr{L}_\sigma$ imply $|\lambda_\sigma(H)-\lambda_\sigma(E_n)|<2\varepsilon$ (for we may choose $L\in\mathscr{L}$ such that $L_n\subseteq L\subseteq H$ and $|\lambda(L)-\lambda_\sigma(H)|<\varepsilon$). Now $E_n\cap F_n$, $L_n\cap F_n$ are $\mathscr{L}_\sigma$-sets contained in E_n such that $E_n\cap F_n-L_n\cap F_n\subseteq E_n-L_n$, so, since λ_σ is strongly additive, $|\lambda_\sigma(E_n\cap F_n)-\lambda_\sigma(L_n\cap F_n)|<4\varepsilon$, and therefore $|[\lambda_\sigma(E_n)-\lambda_\sigma(E_n\cap F_n)]-[\lambda(L_n)-\lambda_\sigma(L_n\cap F_n)]|<5\varepsilon$. Since $|.|$ and $\varepsilon>0$ are arbitrary, this reduces the proof to showing that $\lambda(L_n)-\lambda_\sigma(L_n\cap F_n)\to 0$.

Let $\varepsilon>0$. For $n=1,2,\ldots$ there is an increasing $\mathscr{L}$-sequence $(F_{n,m})_{m=1}^{\infty}$ converging to F_n, such that $F_{n,1}\subseteq L\subseteq F_n$, $L\in\mathscr{L}$ imply $|\lambda(L)-\lambda_\sigma(F_n)|<\varepsilon$. Let $K_m=\bigcup_{n=1}^{m}F_{n,m}$; then $K_m\subseteq F_m$, $K_m\in\mathscr{L}$ and $K_m\uparrow\lim_n F_n$. Also $K_m\subseteq K\subseteq F_m$, $H\in\mathscr{L}_\sigma$ imply $|\lambda_\sigma(H)-\lambda_\sigma(F_m)|<2\varepsilon$. Since $L_n\cap F_n$, $L_n\cap K_n$ are $\mathscr{L}_\sigma$-sets contained in F_n such that $L_n\cap F_n-L_n\cap K_n\subseteq F_n-K_n$, $|\lambda_\sigma(L_n\cap F_n)-\lambda(L_n\cap K_n)|<4\varepsilon$. This reduces the proof to showing that $\lambda(L_n)-\lambda(L_n\cap K_n)\to 0$. But this follows from the continuity of λ, because $\lim_n L_n\subseteq\lim_n K_n$.

2.3. LEMMA. *λ_δ is λ_σ-upper regular.*

Proof. Let $E\in\mathscr{L}_\delta$. By Lemma 2.5 of [2], $\mu(E)=\lim_{H\supseteq E,\, H\in\mathscr{L}_\sigma}\lambda_\sigma(H)$ exists. It must be shown that $\mu(E)=\lambda_\delta(E)$. Let $\varepsilon>0$. There exists $K\in\mathscr{L}_\sigma$ such that $E\subseteq K$ and $E\subseteq H\subseteq K$, $H\in\mathscr{L}_\sigma$ imply $|\lambda_\sigma(H)-\mu(E)|<\varepsilon$. Let (L_n) be a decreasing $\mathscr{L}$-sequence converging to E. Then

$$|\mu(E)-\lambda(L_n)|\leqslant|\mu(E)-\lambda_\sigma(K)|+|\lambda_\sigma(K)-\lambda(L_n)|$$

$$<\varepsilon+|\lambda_\sigma(K-L_n)|+|\lambda_\sigma(L_n-K)|$$

$$=\varepsilon+|\lambda_\sigma(K)-\lambda_\sigma(K\cap L_n)|+|\lambda_\sigma(L_n\cup K)-\lambda_\sigma(K)|$$

$$<\varepsilon+2\varepsilon+|\lambda_\sigma(L_n\cup K)-\lambda_\sigma(K)|.$$

Since λ_σ is δ-additive (Lemma 2.2), $|\lambda_\sigma(L_n\cup K)-\lambda_\sigma(K)|\to 0$ so $|\mu(E)-\lambda(L_n)|<4\varepsilon$ if n is great enough. Hence $\lambda(L_n)\to\mu(E)$. On the other hand, $\lambda(L_n)\to\lambda_\delta(E)$, so $\mu(E)=\lambda_\delta(E)$.

Lemma 2.3, with Theorem 1.1, yields the following result:

2.4. THEOREM. *A strongly additive monotonely convergent continuous function λ on a lattice $\mathscr{L}$ of subsets of a set T, taking values in G, extends uniquely to a strongly additive (σ,δ)-additive function on the (σ,δ)-lattice generated by $\mathscr{L}$.*

Let λ be an additive function on a ring $\mathscr{R}$ of subsets of T, taking values in G. If, for every disjoint sequence $\{R_n\}$, in $\mathscr{R}$, $\lambda(R_n)\to 0$, λ is said to be *s-bounded* [9, p. 654]. We note that if λ is s-bounded then, for every increasing sequence $\{R_n\}$ in $\mathscr{R}$, $\{\lambda(R_n)\}$ converges. In fact, if this were not so, there would exist an increasing sequence $\{R_n\}$ in $\mathscr{R}$, $\varepsilon>0$, a continuous quasi-norm $|.|$ on G and a strictly increasing sequence of positive integers $p_1<q_1<p_2<q_2\ldots$ such that $|\lambda(R_{q_n}-R_{p_n})|\geqslant\varepsilon$. This contradicts the s-boundedness, because $\{R_{q_n}-R_{p_n}\}$ is a disjoint sequence in $\mathscr{R}$.

The following corollary is the extension theorem of Sion [10, p. 92] as improved by Drewnowski [1, p. 441]:

2.5. COROLLARY. *An s-bounded σ-additive function λ on a ring $\mathscr{R}$ of subsets of T, taking values in G, extends uniquely to a σ-additive function on the σ-ring $\sigma(\mathscr{R})$ generated by $\mathscr{R}$.*

Proof. Since λ is σ-additive, it is strongly additive and continuous. Since λ is s-bounded, $\{\lambda(L_n)\}$ converges for every increasing $\mathscr{R}$-sequence. If $\{L_n\}$ is a decreasing $\mathscr{R}$-sequence, $\{\lambda(L_1-L_n)\}$ converges and $\lambda(L_1-L_n)=\lambda(L_1)-\lambda(L_n)$, so $\{\lambda(L_n)\}$ converges. Hence λ is monotonely convergent. The corollary follows from the theorem because the (σ,δ)-lattice generated by $\mathscr{R}$ is $\sigma(\mathscr{R})$.

3. Ring extension. The Carathéodory process of measure extensions was applied by Sion [10] to a group-valued function on a ring. The purpose of this section is to extend the application to a group-valued function on a lattice.

If $\mathscr{L}$ denotes a lattice (of subsets of T), $\mathscr{H}(\mathscr{L})=\{H\in\exp(T)\colon H\subseteq L$ for some $L\in\mathscr{L}\}$ is the smallest hereditary ring containing $\mathscr{L}$. If we wish to indicate that an at most countable union $\bigcup_n E_n$ is disjoint we write it $E_1+E_2+\dots$ or $\sum_n E_n$ and refer to it as a *sum*. For $E\in\exp(T)$, E^c denotes the complement of E in T.

In his Lemma 2.2, Sion [10, p. 91] abstracts the essential step of the Carathéodory process. The following lemma is the version needed here:

3.1. LEMMA. *If $\mu\colon \mathscr{R}\to G$ is a function on a ring $\mathscr{R}$, such that $\mu(\emptyset)=0$, then $M(\mu)=\{M\in\mathscr{R}\colon \mu(R)=\mu(R\cap M)+\mu(R-M)$ for all $R\in\mathscr{R}\}$ is a subring of $\mathscr{R}$ and the restriction $\mu|M(\mu)$, of μ to $M(\mu)$, is additive.*

Proof. Let $M_1, M_2\in M(\mu)$ and let $R\in\mathscr{R}$. Then

$$\mu(R)=\mu(R\cap M_1\cap M_2)+\mu(R\cap M_1\cap M_2^c)+\mu(R\cap M_1^c\cap M_2)+\mu(R\cap M_1^c\cap M_2^c).$$

Replacing R by $R\cap(M_1\cup M_2)$,

$$\mu[R\cap(M_1\cup M_2)]=\mu(R\cap M_1\cap M_2)+\mu(R\cap M_1\cap M_2^c)+\mu(R\cap M_1^c\cap M_2).$$

Therefore $\mu(R)=\mu[R\cap(M_1\cup M_2)]+\mu[R-(M_1\cup M_2)]$, proving that $M_1\cup M_2\in M(\mu)$. Replacing R by $R-(M_1-M_2)=R\cap(M_1^c\cup M_2)$ in the first equation, we conclude similarly that $M_1-M_2\in M(\mu)$, proving that $M(\mu)$ is a ring. Supposing that $M_1\cap M_2=\emptyset$ and setting $R=M_1+M_2$ in the second equation, we have $\mu(M_1+M_2)=\mu(M_1)+\mu(M_2)$, proving the additivity of $\mu|M(\mu)$.

Let $\lambda\colon\mathscr{L}\to G$ be a function on a lattice $\mathscr{L}$. If, for every increasing (decreasing) $\mathscr{L}$-sequence (L_n), $(\lambda(L_n))$ converges we say that λ is *increasingly convergent* (*decreasingly convergent*). Weakening the first condition, we say that λ is *locally increasingly convergent* if, for every increasing $\mathscr{L}$-sequence (L_n) such that $L_n\subseteq L$ $(n=1,2,\dots)$ for some $L\in\mathscr{L}$, $(\lambda(L_n))$ converges. Thus, λ is monotonely convergent if and only if it is increasingly convergent and decreasingly convergent. We say that λ is *locally monotonely convergent* if it is decreasingly convergent and locally increasingly convergent. Weakening the δ-additivity condition, we say, following Halmos [3, p. 39], that λ is *continuous at* $\emptyset$, if, for every decreasing $\mathscr{L}$-sequence (L_n) converging to $\emptyset$, $\lambda(L_n)\to 0$.

3.2. LEMMA. *If $\lambda\colon\mathscr{L}\to G$ is an increasingly convergent (locally increasingly convergent) function on a lattice $\mathscr{L}$, then $\lim_{L\subseteq E,\,L\in\mathscr{L}}\lambda(L)$ exists for every $E\in\exp(T)$ (for every $E\in\mathscr{H}(\mathscr{L})$).*

Proof. Suppose that λ is locally increasingly convergent and that $\lim_{L\subseteq E,\,L\in\mathscr{L}}\lambda(L)$ does not exist for some $E\in\mathscr{H}(\mathscr{L})$. There exists a continuous quasi-norm $|.|$ on G and $\varepsilon>0$ such that, for every $\mathscr{L}$-set A contained in E, there is an $\mathscr{L}$-set B such that $A\subseteq B\subseteq E$ and $|\lambda(A)-\lambda(B)|\geqslant\varepsilon$. Hence we may construct inductively an increasing $\mathscr{L}$-sequence (L_n) such that $L_n\subseteq E$ and $|\lambda(L_n)-\lambda(L_{n+1})|\geqslant\varepsilon$ $(n=1,2,\dots)$. There is an $\mathscr{L}$-set K containing E, so that $L_n\subseteq K$ $(n=1,2,\dots)$. The hypothesis excludes the existence of such a sequence (L_n), so the second statement of the lemma is proved. The same argument, with K suppressed, proves the first statement.

Let $\lambda\colon\mathscr{L}\to G$ be a function on a lattice $\mathscr{L}$. By Lemma 3.2, if λ is increasingly convergent (locally increasingly convergent), $\lambda_*(E)=\lim_{L\subseteq E,\,L\in\mathscr{L}}\lambda(L)$ is defined for all $E\in\exp(T)$ (for all $E\in\mathscr{H}(\mathscr{L})$); we note that $\lambda_*|\mathscr{L}=\lambda$. Assuming λ to be strongly additive and locally increasingly convergent, we say that λ is *λ-inner tight* if $\lambda_*(L-K)=\lambda(L-K)$ for all $L,K\in\mathscr{L}$.

3.3. THEOREM. *Let $\lambda\to G$ be a function on a lattice $\mathscr{L}$. If λ is strongly additive, continuous at $\emptyset$, locally increasingly convergent and λ-inner tight then its Pettis extension is σ-additive. If, further, λ is increasingly convergent, it extends uniquely to a σ-additive function on $\sigma(\mathscr{L})$.*

Proof. It will be shown that $\mathscr{L}\subseteq M(\lambda_*|\mathscr{R}(\mathscr{L}))$. Let $L\in\mathscr{L}$, $R\in\mathscr{R}(\mathscr{L})$. There exists an $\mathscr{L}$-set K contained in R such that $K\subseteq H\subseteq R$, $H\in\mathscr{L}$ imply $|\lambda(H)-\lambda_*(R)|<\varepsilon$. There exist $\mathscr{L}$-sets K_1, K_2 bearing the same relation to $R-L$, $R\cap L$, respectively. We may suppose that $K_1+K_2\subseteq K\subseteq R$. Since $\lambda(K-L)=\lambda_*(K-L)$, there exists an $\mathscr{L}$-set A such that $K_1\subseteq A\subseteq K-L\subseteq R-L$ and $|\lambda(A)-\lambda(K-L)|<\varepsilon$. Then, by the definition of K_1, $|\lambda(A)-\lambda_*(R-L)|<\varepsilon$, therefore

(1) $$|\lambda(K-L)-\lambda_*(R-L)|<2\varepsilon.$$

Since $K_2\subseteq K\cap L\subseteq R\cap L$ and $\lambda(K\cap L)=\lambda_*(K\cap L)$, there exists an $\mathscr{L}$-set B such that $K_2\subseteq B\subseteq K\cap L\subseteq R\cap L$ and $|\lambda(B)-\lambda(K\cap L)|<\varepsilon$. Then, by the definition of K_2, $|\lambda(B)-\lambda_*(R\cap L)|<\varepsilon$, therefore

(2) $$|\lambda(K\cap L)-\lambda_*(R\cap L)|<2\varepsilon.$$

Since λ is strongly additive, (1) and (2) imply

$$|\lambda_*(R)-\lambda_*(R-L)-\lambda_*(R\cap L)|=\big|(\lambda_*(R)-\lambda(K))-\\-(\lambda_*(R-L)-\lambda(K-L))-(\lambda_*(R\cap L)-\lambda(K\cap L))\big|<5\varepsilon.$$

Since $|.|$ and ε are arbitrary, this proves what was claimed.

It follows from this and Lemma 3.1 that $M(\lambda_*|\mathscr{R}(\mathscr{L}))=\mathscr{R}(\mathscr{L})$ and that $\lambda_*|\mathscr{R}(\mathscr{L})$ is additive. Therefore $\lambda_*|\mathscr{R}(\mathscr{L})$ is the Pettis extension of $\lambda\colon\mathscr{L}\to G$. Let (R_n) be a decreasing $\mathscr{R}(\mathscr{L})$-sequence converging to $\emptyset$. Since λ_* is λ-lower regular, Lemma 2.1 asserts the existence of a decreasing $\mathscr{L}$-sequence (L_n) such that $L_n\subseteq R_n$ and $|\lambda(L_n)-\lambda_*(R_n)|<\varepsilon$ $(n=1,2,\dots)$. Since λ is continuous at $\emptyset$, $\lambda(L_n)\to 0$. Hence, $|.|$ and ε being arbitrary, $\lambda_*(R_n)\to 0$. This proves that $\lambda_*|\mathscr{R}(\mathscr{L})$ is σ-additive.

Suppose, further, that λ is increasingly convergent. Let (R_n) be an increasing $\mathscr{R}(\mathscr{L})$-sequence. There exists an $\mathscr{L}$-sequence (L_n) such that $L_n\subseteq R_n$ and $L_n\subseteq L\subseteq R_n$, $L\in\mathscr{L}$ imply $|\lambda(L)-\lambda_*(R_n)|<\varepsilon$ $(n=1,2,\dots)$. Setting $K_n=\bigcup_{i=1}^{n}L_i$ $(n=1,2,\dots)$ we obtain the increasing $\mathscr{L}$-sequence (K_n) such that $|\lambda(K_n)-\lambda_*(R_n)|<\varepsilon$ $(n=1,2,\dots)$. Then $(\lambda(K_n))$ is Cauchy. Since $|.|$, ε are arbitrary, it follows that $(\lambda_*(R_n))$ is Cauchy. Thus the σ-additive function $\lambda_*|\mathscr{R}(\mathscr{L})$ is increasingly convergent and so extends uniquely, by Corollary 2.5, to a σ-additive function on $\sigma(\mathscr{L})$.

3.4. Remark. Theorem 3.3 is contained in Lipecki's result [7, p. 110], proved by a different method. It is included here because it is one of four extension theorems coming out of the same Carathéodory argument.

3.5. Lemma. *If* $\lambda\colon \mathscr{L} \to G$ *is a decreasingly convergent function on a lattice* $\mathscr{L}$, $\lim_{L\supseteq E, L\in\mathscr{L}} \lambda(L)$ *exists for every* $E \in \mathscr{H}(\mathscr{L})$.

Proof. Dual of the proof of the first statement of Lemma 3.2.

Let $\lambda\colon \mathscr{L} \to G$ be a function on a lattice $\mathscr{L}$. By Lemma 3.5, if λ is decreasingly convergent, $\lambda^*(E) = \lim_{L\supseteq E, L\in\mathscr{L}} \lambda(L)$ is defined for all $E \in \mathscr{H}(\mathscr{L})$. We note that $\lambda^*|\mathscr{L} = \lambda$. Assuming λ to be strongly additive and decreasingly convergent, we say that λ is λ-*outer tight* if $\lambda^*(L-K) = \lambda(L-K)$ for all $L, K \in \mathscr{L}$.

3.6. Theorem. *Let* $\lambda\colon \mathscr{L} \to G$ *be a function on a lattice* $\mathscr{L}$. *If* λ *is strongly additive,* σ*-additive, decreasingly convergent and* λ*-outer tight then its Pettis extension is* σ*-additive. If, further,* λ *is increasingly convergent, it extends uniquely to a* σ*-additive function on* $\sigma(\mathscr{L})$.

Proof. It will be shown that $\mathscr{L} \subseteq M(\lambda^*|\mathscr{R}(\mathscr{L}))$. Let $L \in \mathscr{L}$, $R \in \mathscr{R}(\mathscr{L})$. There exists an $\mathscr{L}$-set K containing R such that $K \supseteq H \supseteq R$, $H \in \mathscr{L}$ imply $|\lambda(H)-\lambda^*(R)| < \varepsilon$. There exist $\mathscr{L}$-sets K_1, K_2 bearing the same relation to $R-L$, $R \cap L$, respectively. We may suppose that $K \supseteq K_1 \supseteq R-L$ and $K \cap L \supseteq K_2 \supseteq R \cap L$. Then $R \cap L \supseteq R \cap K_2 \supseteq R \cap (R \cap L) = R \cap L$, so $R \cap L = R \cap K_2$. Since $\lambda(K_1-K_2) = \lambda^*(K_1-K_2)$, there exists an $\mathscr{L}$-set A such that $K_1 \supseteq A \supseteq K_1-K_2 = (K_1 \cup K_2)-K_2 \supseteq R-K_2 = R-(R\cap K_2) = R-(R \cap L) = R-L$ and $|\lambda(A)-\lambda(K_1-K_2)| < \varepsilon$. Then, by the definition of K_1, $|\lambda(A)-\lambda^*(R-L)| < \varepsilon$, therefore $|\lambda(K_1-K_2)-\lambda^*(R-L)| < 2\varepsilon$. Since $K \supseteq K_1 \cup K_2 \supseteq R$,

$$|\lambda^*(R)-\lambda^*(R-L)-\lambda^*(R\cap L)| =$$
$$= |(\lambda^*(R)-\lambda(K_1\cup K_2))-(\lambda^*(R-L)-\lambda(K_1-K_2))-(\lambda^*(R\cap L)-\lambda(K_2))| < 4\varepsilon .$$

Since $|.|$ and ε are arbitrary, this proves what was claimed.

It follows from this and Lemma 3.1 that $\lambda^*|\mathscr{R}(\mathscr{L})$ is the Pettis extension of $\lambda\colon \mathscr{L} \to G$. Let (R_n) be a decreasing $\mathscr{R}(\mathscr{L})$-sequence converging to $\emptyset$. Let L be an $\mathscr{L}$-set containing R_1, so that $(L-R_n) \uparrow L$. Since λ^* is λ-upper regular, Lemma 2.1 asserts the existence of an increasing $\mathscr{L}$-sequence (L_n) such that $L_n \supseteq L-R_n$ and $L_n \supseteq H \supseteq L-R_n$, $H \in \mathscr{L}$ imply $|\lambda(H)-\lambda^*(L-R_n)| < \varepsilon$ $(n = 1, 2, \ldots)$. Setting $K_n = L_n \cap L$ $(n = 1, 2, \ldots)$ we have $K_n \uparrow L$ and $|\lambda(K_n)-\lambda^*(L-R_n)| < \varepsilon$ $(n = 1, 2, \ldots)$. Since λ is σ-additive, $|\lambda(L)-\lambda^*(L-R_n)| < 2\varepsilon$ for n large enough. Since $L, R_n \in \mathscr{R}(\mathscr{L})$, $\lambda(L)-\lambda^*(L-R_n) = \lambda(L)-(\lambda^*(L)-\lambda^*(R_n)) = \lambda^*(R_n)$, so $|\lambda^*(R_n)| < 2\varepsilon$ for n large enough. Since $|.|$, ε are arbitrary this proves that $\lambda^*(R_n) \to 0$, from which it follows that $\lambda^*|\mathscr{R}(\mathscr{L})$ is σ-additive.

Suppose, further, that λ is increasingly convergent. Let (R_n) be an increasing $\mathscr{R}(\mathscr{L})$-sequence. By Lemma 2.1, there exists an increasing $\mathscr{L}$-sequence (L_n) such that $|\lambda(L_n)-\lambda^*(R_n)| < \varepsilon$ $(n = 1, 2, \ldots)$. Since $|.|$, ε are arbitrary, it follows that $(\lambda^*(R_n))$ is Cauchy, proving that $\lambda^*|\mathscr{R}(\mathscr{L})$ is increasingly convergent. By Corollary 2.5, $\lambda^*|\mathscr{R}(\mathscr{L})$ extends uniquely to a σ-additive function on $\sigma(\mathscr{L})$.

3.7. Lemma. *Let* $\lambda\colon \mathscr{L} \to G$ *be a strongly additive locally monotonely convergent* δ*-additive function on a lattice* $\mathscr{L}$. *Then* $\lambda_\delta = \lambda^*|\mathscr{L}_\delta$ *is strongly additive, locally monotonely convergent and* δ*-additive.*

Proof. It will be shown first that if (L_n) is a decreasing $\mathscr{L}$-sequence converging to a set K then $\lambda(L_n) \to \lambda_\delta(K)$. There exists an $\mathscr{L}$-set L containing K such that $L \supseteq A \supseteq K$, $A \in \mathscr{L}$ imply $|\lambda(A)-\lambda_\delta(K)| < \varepsilon$. We have

$$|\lambda(L_n)-\lambda_\delta(K)| \leqslant |\lambda(L_n)-\lambda(L)|+|\lambda(L)-\lambda_\delta(K)| = |\lambda(L_n-L)-\lambda(L-L_n)|+|\lambda(L)-\lambda_\delta(K)|$$
$$\leqslant |\lambda(L_n\cup L)-\lambda(L)|+|\lambda(L)-\lambda(L\cap L_n)|+|\lambda(L)-\lambda_\delta(K)|$$
$$< |\lambda(L_n\cup L)-\lambda(L)|+2\varepsilon+\varepsilon .$$

Since λ is δ-additive, $|\lambda(L_n\cup L)-\lambda(L)| \to 0$, so $|\lambda(L_n)-\lambda_\delta(K)| < 4\varepsilon$ for large enough n. This, with $|.|$, ε arbitrary, suffices.

To show that λ_δ is strongly additive, let $A, B \in \mathscr{L}_\delta$ and let (A_n), (B_n) be decreasing $\mathscr{L}$-sequences converging to A, B respectively. Applying the first paragraph and the strong additivity of λ,

$$\lambda_\delta(A)+\lambda_\delta(B)-\lambda_\delta(A\cup B)-\lambda_\delta(A\cap B)$$
$$= \lim_n [\lambda(A_n)+\lambda(B_n)-\lambda(A_n\cup B_n)-\lambda(A_n\cap B_n)] = 0 .$$

To show that λ_δ is δ-additive (and, in particular, that λ_δ is decreasingly convergent), let (K_n) be a decreasing $\mathscr{L}_\delta$-sequence converging to a set K. For each $n = 1, 2, \ldots$, there is a decreasing $\mathscr{L}$-sequence $(L_{nm})_{m=1}^{\infty}$ converging to K_n. Write $A_m = \bigcap_{n=1}^{m} L_{nm}$ $(m = 1, 2, \ldots)$, so that (A_m) is a decreasing $\mathscr{L}$-sequence converging to K such that $K_m \subseteq A_m \subseteq L_{mm}$ $(m = 1, 2, \ldots)$. By the first paragraph, $\lambda(A_m) \to \lambda_\delta(K)$. We may suppose the $(L_{nm})_{m=1}^{\infty}$ $(n = 1, 2, \ldots)$ chosen so that $L_{n1} \supseteq A \supseteq K_n$, $A \in \mathscr{L}$ imply $|\lambda(A)-\lambda_\delta(K_n)| < \varepsilon$. Then $|\lambda(A_m)-\lambda_\delta(K_m)| < \varepsilon$ $(m = 1, 2, \ldots)$. Since $|.|$, ε are arbitrary, this shows that $\lambda_\delta(K_m) \to \lambda_\delta(K)$.

It remains to show that λ_δ is locally increasingly convergent. Let (K_n) be an increasing $\mathscr{L}_\delta$-sequence such that, for some $K \in \mathscr{L}_\delta$, $K_n \subseteq K$ $(n = 1, 2, \ldots)$. Then $K \subseteq L$ for some $L \in \mathscr{L}$. Since λ_δ is λ-upper regular, Lemma 2.1 asserts the existence of an increasing $\mathscr{L}$-sequence (L_n) such that $L_n \supseteq K_n$ and $L_n \supseteq A \supseteq K_n$, $A \in \mathscr{L}$ imply $|\lambda(A)-\lambda_\delta(K_n)| < \varepsilon$ $(n = 1, 2, \ldots)$. Writing $A_n = L_n \cap L$ $(n = 1, 2, \ldots)$, (A_n) is an increasing $\mathscr{L}$-sequence such that $A_n \subseteq L$ and $|\lambda(A_n)-\lambda_\delta(K_n)| < \varepsilon$ $(n = 1, 2, \ldots)$. Since $(\lambda(A_n))$ converges and $|.|$, ε are arbitrary, $(\lambda_\delta(K_n))$ converges.

Let $\lambda\colon \mathscr{L} \to G$ be a strongly additive locally monotonely convergent δ-additive function on a lattice $\mathscr{L}$. Since λ_δ is locally increasingly convergent (Lemma 3.7), Lemma 3.2 asserts that λ_δ extends to

$$(\lambda_\delta)_*(E) = \lim_{K\subseteq E, K\in\mathscr{L}_\delta} \lambda_\delta(K), \quad E \in \mathscr{H}(\mathscr{L}_\delta) = \mathscr{H}(\mathscr{L}) .$$

We say that λ is λ_δ-*inner tight* if

$$(\lambda_\delta)_*(L-K) = \lambda(L-K) \quad \text{for all } L, K \in \mathscr{L}.$$

3.8. LEMMA. *If $\lambda\colon \mathscr{L} \to G$ is a strongly additive locally monotonely convergent δ-additive function on a lattice $\mathscr{L}$, then $(\lambda_\delta)_*\colon \mathscr{H}(\mathscr{L}) \to G$ is δ-additive.*

Proof. Let (H_n) be a decreasing $\mathscr{H}(\mathscr{L})$-sequence converging to a set H. Let K be an $\mathscr{L}_\delta$-set contained in H such that $K \subseteq A \subseteq H$, $A \in \mathscr{L}_\delta$ imply $|\lambda_\delta(A)-(\lambda_\delta)_*(H)| < \varepsilon$. Since $(\lambda_\delta)_*$ is λ_δ-lower regular, Lemma 2.1 asserts the existence of a decreasing $\mathscr{L}_\delta$-sequence (K_n) such that $K_n \subseteq H_n$, $|\lambda_\delta(K_n)-(\lambda_\delta)_*(H_n)| < \varepsilon$ $(n = 1, 2, \ldots)$ and $K \subseteq B \subseteq H$, where $B = \lim_n K_n$. We have

$$|(\lambda_\delta)_*(H_n)-(\lambda_\delta)_*(H)| \leqslant |(\lambda_\delta)_*(H_n)-\lambda_\delta(K_n)| + |\lambda_\delta(K_n)-\lambda_\delta(B)| + |\lambda_\delta(B)-(\lambda_\delta)_*(H)|$$
$$< \varepsilon + |\lambda_\delta(K_n)-\lambda_\delta(B)| + \varepsilon.$$

Since λ_δ is δ-additive (Lemma 3.7), $|\lambda_\delta(K_n)-\lambda_\delta(B)| \leqslant \varepsilon$ for large enough n and then $|(\lambda_\delta)_*(H_n)-(\lambda_\delta)_*(H)| < 3\varepsilon$. Since $|.|$, ε are arbitrary, this proves that $(\lambda_\delta)_*(H_n) \to (\lambda_\delta)_*(H)$.

3.9. LEMMA. *If $\lambda\colon \mathscr{L} \to G$ is a strongly additive locally monotonely convergent δ-additive λ_δ-inner tight function on a lattice $\mathscr{L}$, then $\lambda_\delta(K-L) = (\lambda_\delta)_*(K-L)$ whenever $K \in \mathscr{L}_\delta$ and $L \in \mathscr{L}$.*

Proof. Since λ_δ is strongly additive and δ-additive (Lemma 3.7), (L_n) being a decreasing $\mathscr{L}$-sequence converging to K, we have

$$\lambda_\delta(K-L) = \lambda_\delta(K)-\lambda_\delta(K \cap L) = \lim_n [\lambda(L_n)-\lambda(L_n \cap L)] = \lim_n \lambda(L_n - L).$$

Since λ is λ_δ-inner tight and $(\lambda_\delta)_*$ is δ-additive (Lemma 3.8), $\lim_n \lambda(L_n-L) = \lim_n (\lambda_\delta)_*(L_n-L) = (\lambda_\delta)_*(K-L)$.

3.10. THEOREM. *Let $\lambda\colon \mathscr{L} \to G$ be a function on a lattice $\mathscr{L}$. If λ is strongly additive, δ-additive, locally monotonely convergent and λ_δ-inner tight then its Pettis extension is σ-additive. If, further, λ is increasingly convergent, it extends uniquely to a σ-additive function on $\sigma(\mathscr{L})$.*

Proof. We note first that, if λ is increasingly convergent, so is λ_δ (Lemma 2.1). Then, taking account of Lemma 3.7, it suffices to retrace the proof of Theorem 3.3 with λ_δ in the approximating role of λ (and therefore $(\lambda_\delta)_*$ in the role of λ_*), applying λ_δ-inner tightness in the form of Lemma 3.9.

3.11. LEMMA. *Let $\lambda\colon \mathscr{L} \to G$ be a strongly additive locally monotonely convergent σ-additive function on a lattice $\mathscr{L}$. Then $\lambda_\sigma = \lambda_* | \mathscr{L}_\sigma$ is strongly additive, monotonely convergent and σ-additive.*

Proof. The dual of the proof of Lemma 3.7 shows that λ_σ is strongly additive, locally monotonely convergent and σ-additive. Then, the locally increasingly convergent function λ_σ, whose domain $\mathscr{L}_\sigma$ is a σ-lattice, is increasingly convergent.

Let $\lambda\colon \mathscr{L} \to G$ be a strongly additive locally monotonely convergent σ-additive function on a lattice $\mathscr{L}$. Since λ_σ is decreasingly convergent (Lemma 3.11), Lemma 3.5 asserts that λ_σ extends to

$$(\lambda_\sigma)^*(E) = \lim_{K \supseteq E,\, K \in \mathscr{L}_\sigma} \lambda_\sigma(K), \quad E \in \mathscr{H}(\mathscr{L}_\sigma).$$

We say that λ is λ_σ-*outer tight* if

$$(\lambda_\sigma)^*(L-K) = \lambda(L-K) \quad \text{for all } L, K \in \mathscr{L}.$$

3.12. LEMMA. *If $\lambda\colon \mathscr{L} \to G$ is a strongly additive locally monotonely convergent σ-additive function on a lattice $\mathscr{L}$, then $(\lambda_\sigma)^*\colon \mathscr{H}(\mathscr{L}_\sigma) \to G$ is σ-additive.*

Proof. Dual of the proof of Lemma 3.8, with Lemma 3.11 in the role of Lemma 3.7.

3.13. THEOREM. *If $\lambda\colon \mathscr{L} \to G$ is a strongly additive locally monotonely convergent σ-additive λ_σ-outer tight function on a lattice $\mathscr{L}$, then $\lambda_\sigma(K-L) = (\lambda_\sigma)^*(K-L)$ whenever $K \in \mathscr{L}_\sigma$ and $L \in \mathscr{L}$.*

Proof. Dual of the proof of Lemma 3.9, with Lemmas 3.11, 3.12 in the roles of Lemmas 3.7, 3.8, respectively.

We will need a generalization of Theorem 3.13. As intermediate step, we prove the following additive property of $(\lambda_\sigma)^*$:

3.14. LEMMA. *Let $\lambda\colon \mathscr{L} \to G$ be a strongly additive locally monotonely convergent σ-additive λ_σ-outer tight function on a lattice $\mathscr{L}$. If $L \subseteq K \subseteq E$, where $K, E \in \mathscr{L}_\sigma$ and $L \in \mathscr{L}$, then*

$$(\lambda_\sigma)^*[L+(E-K)] = \lambda(L)+(\lambda_\sigma)^*(E-K).$$

Proof. Let H be an $\mathscr{L}_\sigma$-set containing $E-K$ such that $H \supseteq A \supseteq E-K$, $A \in \mathscr{L}_\sigma$ imply $|\lambda_\sigma(A)-(\lambda_\sigma)^*(E-K)| < \varepsilon$. We may suppose that $E-K \subseteq H \subseteq E$. Let P be an $\mathscr{L}_\sigma$-set containing $L+(E-K)$ such that $P \supseteq A \supseteq L+(E-K)$, $A \in \mathscr{L}_\sigma$ imply $|\lambda_\sigma(A)-(\lambda_\sigma)^*[L+(E-K)]| < \varepsilon$. Writing $H_1 = P \cap H$, we have

$$L+(E-K) \subseteq P \cap (L \cup H) = L \cup H_1 \subseteq P,$$
$$E-K \subseteq H_1 \subseteq H.$$

By the definitions of H and P,

(1) $$|\{(\lambda_\sigma)^*[L+(E-K)]-\lambda(L)-(\lambda_\sigma)^*(E-K)\}-\{\lambda_\sigma(L \cup H_1)-\lambda(L)-\lambda_\sigma(H_1)\}| < 2\varepsilon.$$

Since λ_σ is strongly additive (Lemma 3.11), $\lambda_\sigma(L \cup H_1)-\lambda(L)-\lambda_\sigma(H_1) = -\lambda_\sigma(L \cap H_1)$, so (1) may be written

(2) $$|(\lambda_\sigma)^*[L+(E-K)]-\lambda(L)-(\lambda_\sigma)^*(E-K)+\lambda_\sigma(L \cap H_1)| < 2\varepsilon.$$

Since $\lambda_\sigma(H_1-L) = (\lambda_\sigma)^*(H_1-L)$ (Lemma 3.13), there is an $\mathscr{L}_\sigma$-set Q such that $H_1-L \subseteq Q \subseteq H_1$ and

(3) $$|\lambda_\sigma(Q)-\lambda_\sigma(H_1-L)| < \varepsilon.$$

Since $E-K \subseteq H_1$ and $L \cap (E-K) = \emptyset$, $E-K \subseteq H_1 - L \subseteq Q \subseteq H_1 \subseteq H$, so, by the definition of H,

(4) $$|\lambda_\sigma(Q) - \lambda_\sigma(H_1)| < 2\varepsilon.$$

By (3) and (4),

(5) $$|\lambda_\sigma(H_1 \cap L)| = |\lambda_\sigma[H_1 - (H_1 - L)]| = |\lambda_\sigma(H_1) - \lambda_\sigma(H_1 - L)| < 3\varepsilon.$$

By (2) and (5),

$$|(\lambda_\sigma)^*[L+(E-K)] - \lambda(L) - (\lambda_\sigma)^*(E-K)| < 5\varepsilon.$$

This, with $|.|$, ε arbitrary, proves the lemma.

3.15. Lemma. *If λ: $\mathscr{L} \to G$ is a strongly additive locally monotonely convergent σ-additive λ_σ-outer tight function on a lattice $\mathscr{L}$, then $\lambda_\sigma(E-K) = (\lambda_\sigma)^*(E-K)$ for all $E, K \in \mathscr{L}_\sigma$.*

Proof. We may suppose that $K \subseteq E$. Let (L_n) be an increasing $\mathscr{L}$-sequence converging to K, so that $[L_n + (E-K)] \uparrow E$. Since $(\lambda_\sigma)^*$ is σ-additive (Lemma 3.12),

(1) $$(\lambda_\sigma)^*[L_n + (E-K)] \to (\lambda_\sigma)^*(E) = \lambda_\sigma(E).$$

Applying Lemma 3.14, we may write (1) in the form

(2) $$\lambda(L_n) + (\lambda_\sigma)^*(E-K) \to \lambda_\sigma(E).$$

Because λ_σ is σ-additive (Lemma 3.11), (2) implies

(3) $$\lambda_\sigma(K) + (\lambda_\sigma)^*(E-K) = \lambda_\sigma(E).$$

Because λ_σ is strongly additive (Lemma 3.11), (3) may be written

$$(\lambda_\sigma)^*(E-K) = \lambda_\sigma(E) - \lambda_\sigma(K) = \lambda_\sigma(E-K).$$

3.16. Theorem. *Let λ: $\mathscr{L} \to G$ be a function on a lattice $\mathscr{L}$. If λ is strongly additive, σ-additive, locally monotonely convergent and λ_σ-outer tight, then λ extends uniquely to a σ-additive function on $\sigma(\mathscr{L})$.*

Proof. Retracing the first paragraph of the proof of Theorem 3.6 (taking account of Lemma 3.11) with λ_σ in the approximating role of λ (therefore $(\lambda_\sigma)^*$ in the role of λ^*), applying λ-outer tightness in the form of Lemma 3.15, we show that $(\lambda_\sigma)^*|\mathscr{R}(\mathscr{L})$ is the Pettis extension of λ: $\mathscr{L} \to G$.

By Lemma 3.12, $(\lambda_\sigma)^*|\mathscr{R}(\mathscr{L})$ is σ-additive. Since λ_σ is increasingly convergent (Lemma 3.11), it follows from Lemma 2.1 that $(\lambda_\sigma)^*$ is increasingly convergent. Hence, by Corollary 2.5, $(\lambda_\sigma)^*|\mathscr{R}(\mathscr{L})$ extends uniquely to a σ-additive function on $\sigma(\mathscr{L})$.

Remark. Let λ: $\mathscr{R} \to G$ be a σ-additive function on a ring $\mathscr{R}$. Then, in particular, λ is strongly additive and (σ, δ)-additive. If λ is increasingly convergent it is monotonely convergent (as shown in the proof of Corollary 2.5), so that its extensions λ_*, λ^* $(\lambda_\sigma)_*$ and $(\lambda_\sigma)^*$ are defined. Supposing λ to be increasingly convergent, $R_1, R_2 \in \mathscr{R}$ implies $R_1 - R_2 \in \mathscr{R}$, and therefore λ is λ-inner tight, λ-outer-tight, λ_σ-inner tight and λ_σ-outer tight. Consequently, each of Theorems 3.3, 3,6, 3.10 and 3.16 generalizes Corollary 2.5 — though the proof of each applies Corollary 2.5.

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