THEOREM 3.1. *Let X be a continuum in E^n having the property that for any neighborhood U of X the only loop that belongs to the normal closure in U of each neighborhood W of X is the trivial loop. Then X is nearly-1-movable if and only if X is* 1-UV.

Proof. Let X be nearly-1-movable. Let U be an open set containing X. Choose V so that each loop in V belongs to the normal closure in U of each open W. $X \subset W \subset V$. But only such loops are trivial loops. Thus X is 1-UV.

COROLLARY. *If X is as above, then X has property* 1-UV *if and only if E^n/X is locally simply connected.*

Proof. Clear.

As a corollary, we get the following theorem of D. R. McMillan [10].

THEOREM. *If X is compact connected strongly* 1*-acyclic, then X is* 1-UV *if and only if E^n/X is locally simply connected.*

Proof. Strongly acyclic continua satisfy the property in Theorem 3.1.

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Yosida-Fukamiya's theorem for f-rings

by

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Abstract. We introduce the concept of super-infinitely small element and prove that in a commutative f-ring with unity the J-radical coincides with the set of all super-infinitely small elements.

Preliminaries. We follow the notation and terminology of [1] and [5]. A lattice-ordered ring is an *f-ring* if $ax \wedge y = xa \wedge y = 0$ whenever $x \wedge y = 0$ and $a \geqslant 0$. If we put $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ + x^-$, then a lattice-ordered ring is a *d-ring* if $|xy| = |x| \cdot |y|$, $\forall x, y$. The term ideal must be understood in the ring-theoretic sense. An ideal I is an *l-ideal* if $|x| \leqslant |y|$, $y \in I \Rightarrow x \in I$. We denote by $\langle a \rangle$ the l-ideal generated by $a \in A$. Following [1], an element $a \in A$ such that $\langle a \rangle = A$ is called a *formal unity*. An l-ideal I is a *band* if, whenever a subset of I has a supremum in A, that supremum belongs to I. The *J-radical* $J(A)$ of an f-ring A is defined as the intersection of all maximal (two-sided) l-ideals, if there is any. Otherwise, $J(A) = A$ by definition. The ring A is *J-semisimple* if $J(A) = 0$. An element $x \in A$ is *infinitely small* with respect to the element $y \in A$ whenever $n|x| \leqslant |y|$ holds for $n = 1, 2, \ldots$ If we put $I_0(A) = \bigcup_{y \in A} I_0(y)$, where $I_0(y) = \{x \in A \mid x$ is infinitely small with respect to $y\}$, then A is Archimedean if and only if $I_0(A) = 0$. A lattice-ordered ring is *Dedekind complete* if every non-empty subset which is bounded from above has a supremum.

Introduction. In vector lattices with a strong unit the Yosida–Fukamiya's theorem [7] asserts that the *radical* — intersection of all maximal l-vector subspaces — is the set of all infinitely small elements. Here, for a commutative f-ring with unity, we obtain a result that is parallel to that of Yosida–Fukamiya. But in this context infinitely small elements are no more appropriate and it has been necessary to introduce a notion of "smallness" related to the product of the ring: that of *super-infinitely small element*. And the set of all super-infinitely small elements of A is proved to be $J(A)$.

Super-infinitely small elements and pseudoarchimedean rings.

DEFINITION 1. The element x of the lattice-ordered ring A is called *super-infinitely* small element with respect to $y \in A$ whenever $|a| \cdot |x| \leqslant |y|$ and $|x| \cdot |a| \leqslant |y|$ hold for every $a \in A$.

As we did with infinitely small elements, we write $I'_0(A) = \bigcup_{y\in A} I'_0(y)$, where $I'_0(y) = \{x \in A|\ x$ is super-infinitely small element with respect to $y\}$.

Analogously, we make the following natural definition:

Definition 2. The lattice-ordered ring A is said to be *Pseudoarchimedean* if and only if $I'_0(A) = 0$, that is, if and only if, given $x, y \in A$ such that $|a|\,|x| \leqslant |y|$ and $|x|\cdot|a| \leqslant |y|$ hold for every $a \in A$, we have $x = 0$.

Some elementary properties follow without proof:

Theorem 1. *In a lattice-ordered ring A the following holds.*

(a) *$I'_0(A)$ is an l-ideal.*

(b) *If A has a (ring) unity, then $I'_0(A) \subset I_0(A)$.*

Note that the inclusion given in part (b) of Theorem 1 makes it reasonable, for the unitary case, to call the elements of $I'_0(A)$ super-infinitely small elements.

Examples

1. *A Pseudoarchimedean ring that is not Archimedean.* Consider the non-pseudo-compact topological space X, $X \neq \emptyset$, and let $C(X)$ be the f-ring af all real continuous functions on X, under pointwise ordering and operations. Let M be a hyper-real maximal ideal [3] in $C(X)$ and let $A = C(X)/M$ be the canonically ordered quotient ring. Then we have $I'_0(A) = 0$ and $I_0(A) \neq 0$. This example shows also that the inclusion $I'_0(A) \subset I_0(A)$ (in a ring whith unity) may be strict.

2. *Totally ordered ring that is neither Archimedean nor Pseudoarchimedean.* Let $\boldsymbol{R}[x]$ be the ring of polynomials in an indeterminate x with real coefficients, endowed with the usual operations and a total ordering defined as follows: if $P(x) = a_n x^n + \ldots + a_{n-k} x^{n-k} (a_{n-k} \neq 0)$, then $P > 0$ if and only if $a_{n-k} > 0$. Then $A = (\boldsymbol{R}[x], +, \cdot, \leqslant)$ is a totally ordered non-Archimedean ring, and it is not Pseudo-archimedean since $x \in I'_0(1)$.

3. *An Archimedean ring that is not Pseudoarchimedean.* Consider $A = \boldsymbol{R}\times\boldsymbol{R}$. Addition and ordered are defined coordinatewise, and the rule for multiplication is given by $(a, b)(c, d) = (0, ac)$. Then we get a lattice-ordered ring such that $I_0(A) = 0$ and $I'_0(A) = \{(0, y)|\ y \in \boldsymbol{R}\}$.

We derive now some sufficient conditions for Pseudoarchimedeanity:

Theorem 2. *Given the lattice-ordered ring with unity A, then the following implications hold:*

(a) *If A is Archimedean, it is also Pseudoarchimedean.*

(b) *If A is commutative and every $x > 0$ is a formal unity, then A is Pseudo-archimedean.*

(c) *If A is commutative and M is a maximal l-ideal, then the quotient ring A/M is Pseudoarchimedean.*

Proof. (a) It follows from Theorem 1 (b). (b) Evident. (c) It is a consequence of part (b).

Theorem 3. *Let A be a Dedekind complete f-ring containing an element $\underline{d}$ that is not divisor of zero. Then A is Pseudoarchimedean.*

Proof. Assume that $x \in I'_0(A)$. Then the set $\{|a|\cdot|x||\ a \in A\}$ is bounded from above and, therefore, the least upper bound $z = \bigvee_{a\in A} |a|\cdot|x|$ exists in A by hypothesis. There exists also $t = \bigvee_{a\in A} (|a|+|d|)|x|$ in A and $t \leqslant z$. Now, from $z + |d|\cdot|x| = t \leqslant z$ we obtain $dx = 0$. Hence $x = 0$.

Let us now present one more example. Let H be a complex Hilbert space, with inner product $(x|y)$. The set of all bounded linear Hermitian operators in H will be denoted by $\mathscr{H}$. Under the usual algebraic operations, $\mathscr{H}$ is a real vector space. Moreover, $\mathscr{H}$ is an ordered vector space by defining that $A \geqslant 0$ holds whenever $(Ax|x) \geqslant 0$ holds for all $x \in H$. Let $\mathscr{D}$ a subset of $\mathscr{H}$ such that all elements of $\mathscr{D}$ commute mutually and let $\mathscr{C}''(\mathscr{D})$ be its second commutant [5]. With the same ordering of $\mathscr{H}$, $\mathscr{C}''(\mathscr{D})$ is an f-ring.

Theorem 4. (a) *$\mathscr{C}''(\mathscr{D})$ is Pseudoarchimedean.*

(b) *If the set $\{\|BA\||\ B \in \mathscr{C}''(\{A\})\}$ is bounded, then $A = 0$.*

Proof. (a) $\mathscr{C}''(\mathscr{D})$ is Dedekind complete [5], hence Archimedean. It suffices now to apply Theorem 2 (a).

(b) Suppose that $\|BA\| \leqslant k$ holds for every $B \in \mathscr{C}''(\{A\})$. Therefore, being the norm of $\mathscr{C}''(\mathscr{D})$ compatible with the lattice structure, we have $\||B|\cdot|A|\| \leqslant k$, $\forall B \in \mathscr{C}''(\{A\})$. On account of $|B|\cdot|A| \geqslant 0$ we have now that

$$\||B|\cdot|A|\| = \sup\{((|B|\cdot|A|)x|x)/\|x\|^2|\ x \neq 0\}$$

and so $((|B|\cdot|A|)x|x) \geqslant ((kI)x|x)$ holds for every $x \in H$, that is, $A \in I'_0(\mathscr{C}''(\{A\}))$. Hence $A = 0$, by part (a).

Natural questions arise about the incidence of the Pseudoarchimedean condition on the structure of the ring. Some results related with this problem follow.

Lemma 1. *If* $\mathrm{Ann}(A)$ *is the annihilator of the lattice-ordered ring A, then* $\mathrm{Ann}(A) \subset I'_0(A)$.

Proof. Obvious.

Theorem 5. *Every commutative Pseudoarchimedean d-ring is an f-ring.*

Proof. Suppose that $x \wedge y = 0$ and $a \geqslant 0$; we have to prove that $ax \wedge y = 0$. Indeed, the element $ax \wedge y$ is a left-annihilator of the ring [1] and, therefore, by the commutativity assumption, $ax \wedge y \in \mathrm{Ann}(A)$. By Lemma 1, $ax \wedge y \in I'_0(A)$ and consequently $ax \wedge y = 0$, as it had to be shown.

Remark. Dropping the Pseudoarchimedean hypothesis and substituting it by Archimedeanity, Theorem 5 is no more valid, as may be shown by means of Example 3. Also, if we do not assume that the ring is a d-ring, Theorem 5 is not true: by way of example, take $A = \boldsymbol{R}\times\boldsymbol{R}$, with coordinatewise operations and order it by: $(x, y) \geqslant 0$ if and only if $x \geqslant y \geqslant 0$. Then $I'_0(A) = 0$, but A is not an f-ring.

THEOREM 6. *If A is an Archimedean, Pseudoarchimedean f-ring, then it has no non-zero nilpotent elements. Hence A is a subdirect product of totally ordered rings without non-zero divisors of zero.*

Proof. In an Archimedean f-ring every nilpotent element is an annihilator of the ring [1]. Hence $N(A)\subset \mathrm{Ann}(A)$. Combining this with Lemma 1 and Pseudoarchimedeanity we obtain $N(A) = 0$. The other statement follows now from a well-known characterization of f-rings without nilpotency [4].

COROLLARY 1. *Let A be a Dedekind complete f-ring with some non-zerodivisor. Then $N(A) = 0$.*

Proof. It is sufficient to apply Theorems 3 and 6.

Remark. Theorem 6 does not necessarily hold for an Archimedean, Pseudoarchimedean ring that is not an f-ring: consider the $M_2(R)$ of 2×2-matrices with real coefficients, with the usual operations and the ordering defined pointwise. Then $I_0'(A) = I_0(A) = 0$, but $N(A) \neq 0$.

A Lattice characterization of the J-radical.

THEOREM 7. *If A is an f-ring with formal unit $\underline{u}$, then $J(A)\subset I_0'(u)$.*

Proof. Assume that $x\notin I_0'(u)$. Then there exist an $\underline{a}\in A$ such that $|a|\cdot|x|\nleqslant u$ or $|x|\cdot|a|\nleqslant u$. Suppose that we are in the first case (if we were in the second one, the argument would be similar). There are two possibilities:

(i) $|a|\cdot|x|\geqslant u$. Then x is also a formal unity and, therefore, $x\notin M$ for every maximal l-ideal M. So $x\notin J(A)$ and the theorem is proved.

(ii) $|a|\cdot|x|\nleqslant u$ and $|a|\,|x|\ngeqslant u$. Equivalently, $(|a|\cdot|x|-u)^+>0$ and $(|a|\cdot|x|-u)^->0$. Consider now the nonzero ideal $I = \langle(|a|\cdot|x|-u)^-\rangle$; being A an f-ring, we have $\langle(|a|\cdot|x|-u)^+\rangle \cap \langle(|a|\cdot|x|-u)^-\rangle = 0$, hence $I\neq A$ on account of $(|a|\cdot|x|-u)^+ \neq 0$. Hence there exist a maximal l-ideal M containing I. Now, considering the canonical mapping onto the quotient ring A/M, $x\mapsto \bar{x}$, we have $\bar{0} = \overline{(|a|\cdot|x|-u)^-}\geqslant \bar{u}-|\bar{a}|\cdot|\bar{x}|$. Hence $x\notin M$.

COROLLARY 2. *Every Pseudoarchimedean f-ring with formal unity is J-semisimple.*

It is an immediate consequence of Theorem 7.

Note that this corollary is a generalization of the following result of Johnson [4]: Every Archimedean f-ring with unity is J-semisimple.

A theorem of Birkhoff-Pierce [2] asserts that an Archimedean f-ring with unity has no nonzero nilpotent elements. The preceding theorem allows us to strengthen this result:

COROLLARY 3. *If A is a commutative Pseudoarchimedean f-ring with unity, then it contains no nonzero nilpotent elements.*

Proof. It follows from the inclusions $N(A)\subset J(A)\subset I_0'(A)$.

The main theorem is the following one:

THEOREM 8. *If A is a commutative f-ring with unity 1, then $J(A) = I_0'(A) = I_0'(1)$.*

Proof. Theorem 7 gives us the inclusion $J(A)\subset I_0'(1)$. For the proof that $I_0'(A)\subset J(A)$ holds, assume that M is a maximal l-ideal. Then $I_0'(A/M) = 0$ by Theorem 2 (b). Hence we obtain $I_0'(A)\subset J(A)$ by noting that $(I_0'(A))/M\subset I_0'(A/M)$.

In view of this theorem it is now easy to see that there are sufficiently many Pseudoarchimedean f-rings:

COROLLARY 4. *For every commutative f-ring with unity, $A/I_0'(A)$ is Pseudoarchimedean.*

Proof. By Theorem 8, $I_0'(A/I_0'(A)) = J(A/J(A)) = 0$, on account of $A/J(A)$ being J-semisimple [4].

We also obtain as a consequence of Theorem 8 a result that is already known [1] (part (c)):

COROLLARY 5. *For a commutative f-ring A with unity the following statements are equivalent:*

(a) *A is J-semisimple.*

(b) *A is Pseudoarchimedean.*

(c) *For the arbitrary elements $a, b\in A$ there exist an $x\in A$ such that $|a|\cdot|x|\nleqslant|b|$.*

Remark. Theorem 8 makes it easy to obtain an f-ring for which the J-radical and the algebraic Jacobson radical do not coincide: consider again the ring of Example 2; by Theorem 8 we have that $J(A) = I_0'(A) = I_0'(1) = (x)$, where (x) is the ideal generated by x. But in this case, if $R(A)$ is the Jacobson radical, $R(A) = N(A) = 0$.

COROLLARY 6. *If $\emptyset \neq \mathscr{D}\subset\mathscr{H}$ and $\mathscr{D}$ is a set of operators that commute mutually, then $\mathscr{C}''(\mathscr{D})$ is J-semisimple. Hence $C''(\mathscr{D})$ is a subdirect product of totally ordered rings without nonzero divisors of zero and without nontrivial l-ideals.*

Proof. It follows from Corollary 5 and Theorem 4(a).

We recall that an f-ring is said to satisfy the *descending chain condition* for l-ideals if every properly descending chain $I_1\supset I_2\supset I_3\supset \ldots$ of l-ideals is finite.

COROLLARY 7. *If A is a commutative f-ring with unity that satisfies the descending chain condition and contains no nonzero nilpotent elements, then A is Pseudoarchimedean.*

Proof. Being A an f-ring with unity and satisfying the descending chain condition we have [4]: $J(A) = N(A)$. By Theorem 8, $I_0'(A) = N(A)$ and so the conclusion follows.

As an application of Theorem 8 we shall investigate some conditions under with $J(A)$ is a band.

A commutative lattice-ordered ring A is called *p-distributive* [6] if the product by positive elements preserves the suprema of subsets of A.

THEOREM 9. *If A is a p-distributive, commutative f-ring with unity, then $J(A)$ is a band.*

Proof. By Theorem 8 it suffices to prove that $I_0'(A)$ is a band. To this end, suppose that $\{x_i|i\}\subset I_0'(1)$ and $x = \bigvee_i x_i$ exists in A; we must prove that $x \in I_0'(1)$. It follows easily by convexity that $x^- \in I_0'(1)$. On the other hand, $|a|x^+ = \bigvee_i |a|x_i^+$ holds for every $a \in A$ by p-distributivity. But $|a|x_i^+ \leqslant 1$, $\forall i$, $\forall a \in A$, and, therefore, $x^+ \in I_0'(1)$.

Let us recall that an ordered ring with unity 1 is of *bounded inversion* if every element greater than 1 is a unit. If A is a commutative f-ring with unity and S is the set of non-zero-divisors, $(S_0^{-1}A, +, \cdot, \leqslant)$ will denote the *total* ring of fractions, ordered by the cone $(S_0^{-1}A)^+ = \{a/s|\ as \geqslant 0\}$. Then we have

COROLLARY 8. *Given a commutative f-ring A with unity, each of the following conditions is sufficient for $J(A)$ to be a band.*

(a) *The mapping $A \mapsto S_0^{-1}A$, $x \mapsto x/1$ preserves all suprema of subsets of A.*
(b) *A is of bounded inversion.*
(c) *Every non-unit is a zerodivisor.*

Proof. Each of these conditions implies p-distributivity [6].

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Intersections of separators and essential submanifolds of I^N

by

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Abstract. A compactum X in $I^N = I^m \times I^n$ is *essential* in the first m directions if and only if the projection of X to I^m is a stable map. Similarly define Y to be essential in the last n directions. We discuss conditions under which X and Y must have nonempty intersection. If $m \in \{1, 2\}$ then $X \cap Y \neq \emptyset$, while for $m, n > 2$ examples of disjoint essential compacta are constructed. We give applications, including an apparently new characterization of dimension in terms of mappings into R^n, and a generalization of the Cantor manifold concept.

1. Introduction. The boundary S^{N-1} of I^N can be written as the non-singular join of two distinct canonical lower dimensional spheres S^{m-1} and S^{n-1} for each choice of m, n with $m+n = N$. These spheres bound convex balls D^m and D^n whose intersection is nonempty. The balls are examples of compacta which are essentially embedded in the sense that D^k does not retract to S^{k-1}, $k \in m, n$. Suppose we replace D^m and D^n by different essentially embedded compacta X and Y; then is it possible that $X \cap Y = \emptyset$? Indeed this is possible as we shall show in Section 4, while in Section 3 we shall show it is impossible whenever $m \in \{1, 2\}$. A final result in Section 3 is an apparently new characterization of dimension in terms of mappings into R^n. In Section 5, we will generalize the Cantor manifold concept.

It is not known whether all infinite dimensional compacta have infinite cohomological dimension. A solution to this longstanding problem would be equivalent to a solution of the CE-map dimension raising problem and related problems [E1]. In 3.1 of [W] it was shown that any compactum which can be written as the intersection of separators of co-infinitely many faces of the Hilbert cube has infinite cohomological dimension. In Section 4 we shall show that, at least in finite dimensional cubes, there are essentially embedded manifolds which cannot be written as intersections of separators in the non-essential directions. This situation is related to the one described above, concerning non-intersecting essentially embedded compacta. It may shed light on the question of which compacta in the Hilbert cube can be written as the intersection of co-infinitely many separators of faces of the Hilbert cube. We note that quite recently Roman Pol [P] proved the existence of a compact metric space X which is neither countable dimensional nor strongly

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