Some surgery formulae for maps into $\Omega^\infty S^\infty(RP^\infty)$

by

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Abstract. We derive cohomological formulae for the Kervaire surgery obstructions and the mod 2 index obstructions for maps given as compositions of the form $M \to \Omega^\infty S^\infty(RP^\infty) \xrightarrow{\lambda} \Omega^\infty S^\infty$. Here λ is the James map.

Introduction. If M is a smooth compact manifold of dimension $m = 2n$ and $f\colon M \to G/\mathrm{TOP}$ is any map, then one defines the Kervaire surgery obstruction $s_K(M, f) \in Z/2Z$ of the map f. Similarly, if M is a so-called "$Z/2Z$-manifold" of dimension $m = 4n$, then one defines the mod 2 index surgery obstruction $s_I(M, f) \in Z/2Z$ of the map f, [2].

Cohomological formulae for these obstructions proved to be very useful. They were used, for example, in the determination of the structure of PL and TOP bordism rings by G. Brumfiel, I. Madsen and R. J. Milgram, [2]. Those formulae turn out to be easier to handle if the map f is given as a composition of some map $f'\colon M \to SG$ and of the canonical projection $i\colon SG \to G/\mathrm{TOP}$.

On the other hand J. Jones and E. Rees, [4], have given a short proof of the Browder theorem on the non-existence of π-manifolds with the Kervaire invariant one in dimensions other than $m = 2^n - 2$. The proof is based on a factorization of the respective map $g\colon S^m \to SG$ as a composition of a certain $g'\colon S^m \to \Omega^\infty S^\infty(RP^\infty)$ and of the canonical map $\bar{\lambda}\colon \Omega^\infty S^\infty(RP^\infty) \to SG$.

The proof in question suggests that it may be useful to give some cohomological formulae for the surgery obstructions $s_K(M, f)$ (resp. $s_I(M, f)$) of maps $f\colon M \to G/\mathrm{TOP}$ having an explicit decomposition of the form

$$M \xrightarrow{f'} \Omega^\infty S^\infty(RP^\infty) \xrightarrow{i\circ\bar{\lambda}} G/\mathrm{TOP}.$$

The aim of this paper is to derive such formulae.

We should perhaps mention that this research was partly motivated by our attempt to find the image of the transformation $i_*\colon [RP^\infty, SG] \to [RP^\infty, G/\mathrm{TOP}]$. The cohomological formulae for the surgery obstructions obtained in this paper turned out to be too weak for this purpose. And, afterwards, W. H. Lin's proof of the completion conjecture for the group $Z/2Z$, [16], solved this problem anyway.

Nevertheless, we think that the cohomological formulae themselves may be of some interest.

The paper is organized as follows. In Section 1 we recall the Kahn–Priddy theorem. In Section 2 we express the Kervaire obstructions of maps $f\colon M^{2n} \to \Omega^\infty S^\infty(RP^\infty)$ in terms of appropriate quadratic forms and of the suspension of the Wu class. In Section 3 we express the index obstructions of maps $f\colon M^{4n} \to \Omega^\infty S^\infty(RP^\infty)$ in a similar form. Finally, Section 4 is technical and devoted exclusively to the proof of Lemma 2.12.

We adopt the following notation: for a prime p and a space X, $X_{(p)}$ is the localization of X at p, while $\Omega^\infty S^\infty(S^0)_i$ is the component of degree i maps.

1. The Kahn–Priddy theorem. In this section we recall the Kahn–Priddy theorem [5], [10].

If X is a space with a base-point we shall write $Q(X)$ for the space $\Omega^\infty S^\infty X = \varinjlim_n \Omega^n S^n X$, where $\Omega^n S^n X$ is the n-fold loop-space of the n-fold reduced suspension of X. If $f\colon X \to Y$ is a base-point preserving map, we write $Q(f)\colon Q(X) \to Q(Y)$ for the map $\Omega^\infty S^\infty(f)$.

Let $\lambda\colon RP^\infty \to SG = Q(S^0)_1$ be the composition

$$RP^\infty \xrightarrow{\lambda'} SO(\infty) \xrightarrow{j} SG,$$

where $\lambda'\colon RP^\infty \to SO(\infty)$ is the map which takes a line $L \subset R^N$ to the reflection through its normal hyperplane composed with the reflection $A\colon R^N \to R^N$, $A(x_1, x_2, \dots, x_N) = (-x_1, x_2, \dots, x_N)$.

Let RP_+^∞ be the infinite real projective space with a disjoint base-point added. We write

$$\lambda_+\colon RP_+^\infty \to Q(S^0)$$

for the extension of λ which takes the disjoint base-point of RP_+^∞ to the base-point of $Q(S^0)_0$.

Let $\mu_X\colon Q(Q(X)) \to Q(X)$ be the natural infinite loop space transformation, [7], p. 43. Consider the composition

$$\tilde{\lambda}\colon Q(RP_+^\infty) \xrightarrow{Q(\lambda_+)} Q(Q(S^0)) \xrightarrow{\mu_{S^0}} Q(S^0).$$

$\tilde{\lambda}$ is an infinite loop map since both $Q(\lambda_+)$ and μ_{S^0} are such. Furthermore one has $Q(RP_+^\infty) \simeq Q(RP^\infty) \times Q(S^0)$. Let $l\colon Q(RP^\infty) \to Q(RP_+^\infty)$ be the embedding $l(x) = (x, 1)$, where 1 is the base-point of $Q(S^0)_1$, and let

$$\bar{\lambda}\colon Q(RP^\infty) \to SG$$

be the composition $\bar{\lambda} = \tilde{\lambda} \circ l$.

THEOREM (Kahn–Priddy). *There is a map $\beta\colon SG \to Q(RP^\infty)$ such that the composition $\bar{\lambda} \circ \beta$ becomes a homotopy equivalence after being localized at the prime 2.*

The Atiyah–Hirzebruch spectral sequence arguments applied to the stable homotopy theory show that the groups $\pi_i(Q(RP^\infty))$, $i \geqslant 0$ are all finite and 2-primary. Hence $Q(RP^\infty)_{(2)} \simeq Q(RP^\infty)$ and we get

COROLLARY 1.1. *For any* CW-*complex X*

$$\bar{\lambda}_{(2)*}\colon [X, Q(RP^\infty)] \to [X, SG_{(2)}]$$

is a split epimorphism.

2. The Kervaire obstructions of $[M, Q(RP_+^\infty)]$. If $m \in Z$ let $Q(RP_+^\infty)_m$ denote the subspace $Q(RP^\infty) \times Q(S^0)_m \subset Q(RP^\infty) \times Q(S^0) \simeq Q(RP_+^\infty)$. Observe that $\tilde{\lambda}(Q(RP_+^\infty)_m) \subset Q(S^0)_m$.

Let M^{2n} be a closed manifold. Suppose we are given a map

$$f\colon M^{2n} \to Q(RP_+^\infty)_1, \quad f = (f_1, f_2)$$

where $f_1\colon M^{2n} \to Q(RP^\infty)$, $f_2\colon M^{2n} \to Q(S^0)_1$. Let us consider the composition

$$g = i \circ \tilde{\lambda} \circ f\colon M^{2n} \to G/\mathrm{TOP}.$$

In this section we shall aim at giving a description of the Kervaire invariant $s_K(M, g) \in Z/2$ of the map g.

To this end we shall use the theory of E. H. Brown, Jr., as described in [2; Sec. 5]. Let us recall some of these results. $\{\cdot,\cdot\}$ denotes stable homotopy classes of maps and $K_n = K(Z/2, n)$. We use cohomology with $Z/2$-coefficients throughout this section. The following is Theorem 5.2 of [2].

THEOREM 2.1. (i) *There is an exact sequence*

$$0 \to Z/2 \xrightarrow{k_*} \{M, K_n\} \xrightarrow{j_*} H^n(M) \to 0.$$

(ii) *The suspension $s\colon H^n(M) \to \{M, K_n\}$ is quadratic; that is, if $x, y \in H^n(M)$, then $s(x+y) = s(x) + s(y) + k_*(x \cdot y) \in \{M, K_n\}$.*

Let $\varphi\colon M^{2n} \to SG$ be a map. There are associated a degree one normal map

$$\begin{array}{ccc} \nu_{M'} & \xrightarrow{\hat{\pi}} & \nu_M \\ \downarrow & & \downarrow \\ M' & \xrightarrow{\pi} & M \end{array}$$

where ν_M it the normal bundle of M, and a collapsing map $D\hat{\pi}\colon S^q \wedge (M_+) \to S^q \wedge (M'_+)$.

Let $\bar{\psi}\colon \{M, K_n\} \to Z/4$ be a linear homomorphism such that $\bar{\psi} k_*(1) = 2$. Then the functions $\bar{\psi}(D\hat{\pi})^* s\colon H^n(M') \to Z/4 \subset Q/2Z$ and $\bar{\psi}(D\hat{\pi})^* \pi^* s\colon H^n(M) \to Z/4 \subset Q/2Z$ are quadratic over the respective cup product pairings in the sense of [2; Def. 4.1] and, consequently, their Arf invariants $A(H^n(M'), \bar{\psi}(D\hat{\pi})^* s) \in Z/8$ and $A(H^n(M), \bar{\psi}(D\hat{\pi})^* \pi^* s) \in Z/8$ are defined (see [2; Thm. 4.3]).

The following result is Theorem 5.3 of [2] applied to the special case of a map $\varphi\colon M^{2n} \to SG$.

THEOREM 2.2 (Brown). *The Kervaire surgery obstruction* $s_K(M, i\circ\varphi) \in Z/2$ *of* $i\circ\varphi\colon M \to G/\mathrm{TOP}$ *is given by*

$$4s_K(M, i\circ\varphi) = A\big(H^n(M'), \bar{\psi}(D\hat{\pi})^* s\big) - A\big(H^n(M), \bar{\psi}(D\hat{\pi})^*\pi^* s\big) \in Z/8,$$

where $4\colon Z/2 \to Z/8$ *is the inclusion.*

Finally, the following is Thm. 5.4 of [2].

THEOREM 2.3 (Brown).

$$A\big(H^n(M), \bar{\psi}s\big) - A\big(H^n(M), \bar{\psi}(D\hat{\pi})^*\pi^* s\big) = 2\bar{\psi}s\big((V(M)\varphi^*(\sigma(V)))_n\big) \in Z/8,$$

where $2\colon Z/4 \to Z/8$ *is the inclusion,* $\sigma(V) \in H^*(SG)$ *is the suspension of the Wu class* $V \in H^*(BSG)$ *and* $V(M) \in H^*(M)$ *is the Wu class of* M.

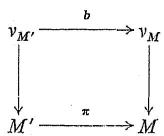
Let $\mathrm{Ad}(f)\colon S^N \wedge (M_+) \to S^N \wedge (RP^+_\infty)$, N-large, be the adjoint of the map $f\colon M \to Q(RP^\infty_+)_1$, $f = (f_1, f_2)$. We may assume that $\mathrm{Ad}(f)$ is transversal to $RP^\infty \subset S^N \wedge (RP^\infty_+)$. Then $M' = \mathrm{Ad}(f)^{-1}(RP^\infty) \subset S^N \wedge (M_+)$ is a manifold of dimension $2n$ and there are a collapsing map $D\hat{\pi}\colon S^N \wedge (M_+) \to S^N \wedge (M'_+)$ and a projection $\pi\colon M' \to M$. Furthermore, we have a decomposition of $\mathrm{Ad}(f)$

(2.4) $$S^N \wedge (M_+) \xrightarrow{D\hat{\pi}} S^n \wedge (M'_+) \xrightarrow{S^N\wedge \Theta_f} S^N \wedge (RP^\infty_+),$$

where $\Theta_f\colon M' \to RP^\infty$ is the restriction of $\mathrm{Ad}(f)$. The composition

$$S^N \wedge (M_+) \xrightarrow{D\hat{\pi}} S^N \wedge (M'_+) \xrightarrow{S^N\wedge \pi_+} S^N \wedge (M_+)$$

is homotopic to the map $\hat{\gamma}\colon S^N \wedge (M_+) \to S^N \wedge (M_+)$, $\hat{\gamma}(x, m) = (\mathrm{Ad}(f_2)(x, m), m)$ for $x \in S^N$, $m \in M$. Observe that the normal map corresponding to f_2 is of the form



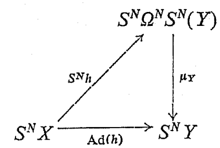
and its collapsing map is $D\hat{\pi}$.

Suppose that $\bar{\psi}\colon \{M, K_n\} \to Z/4$ is a linear homomorphism such that $\bar{\psi}k_*(1) = 2$.

LEMMA 2.5. *The Kervaire obstruction of the map* $g = i\circ\tilde{\lambda}\circ f\colon M \to G/\mathrm{TOP}$ *is given by*

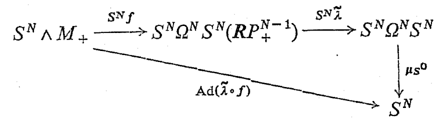
$$4s_K(M, g) = 4s_K(M, f_2) - 2\bar{\psi}s\big((V(M)f_2^*(\sigma(V)))_n\big) + $$
$$+ 2\bar{\psi}s\big((V(M)(\tilde{\lambda}\circ f)^*(\sigma(V)))_n\big) - 2\bar{\psi}(D\hat{\pi})^* s\big((V(M')\Theta_f^*\lambda^*(\sigma(V)))_n\big)$$
$$\in Z/8.$$

Proof. Suppose that there is a map $h\colon X \to \Omega^N S^N(Y)$, where X, Y are based topological spaces. Then the diagram

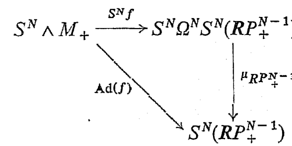


commutes. Here $\mu_Y\colon S^N\Omega^N S^N(Y) \to S^N Y$ is given by $\mu_Y(t, f) = f(t)$, $t \in S^N$, $f \in \Omega^N S^N Y$.

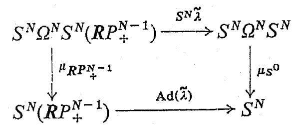
In particular, we have commutative diagrams (N large)



and



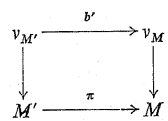
Moreover, since $\tilde{\lambda}\colon Q(RP^\infty_+) \to Q(S^0)$ is an infinite-loop map, we have a commutative diagram



Consequently,

(2.6) $$\mathrm{Ad}(\tilde{\lambda}\circ f) = \mathrm{Ad}(\tilde{\lambda})\circ \mathrm{Ad}(f).$$

Since λ factors through $SO(\infty)$, it follows from (2.4) and (2.6) that the normal map corresponding to $g = i\circ\tilde{\lambda}\circ f$ is of the form



for a certain b' and the collapsing map of g is

$$S^N \wedge M_+ \xrightarrow{D\hat{\pi}} S^N \wedge M'_+ \xrightarrow{\theta_f} S^N \wedge M'_+,$$

where $\theta_f(t, y) = \big((\lambda \circ \mathcal{O}_f(y))(t), y\big)$ for $t \in S^N$, $y \in M'$.

It now follows from Thm. 2.2 that

$$(2.7)\qquad 4s_K(M, g) = A\big(H^n(M'), \overline{\psi}(D\hat{\pi})^*\theta_f^* s\big) - A\big(H^n(M), \overline{\psi}(D\hat{\pi})^*\theta_f^*\pi^* s\big).$$

We are now going to transform both expressions on the right hand side of (2.7).

From the second diagram on page 104 of [2] it follows that $\overline{\overline{\psi}} = \overline{\psi}(D\hat{\pi})^*$: $\{M'_+, K_n\} \to Z/4$ is a linear homomorphism with $\overline{\overline{\psi}} k_*(1) = 2$.

Now consider the map $\lambda \circ \mathcal{O}_f$: $M' \to SG$. Its corresponding normal map is of the form

$$\begin{array}{ccc} \nu_{M'} & \xrightarrow{\ c\ } & \nu_{M'} \\ \downarrow & & \downarrow \\ M' & \xrightarrow{\ \mathrm{id}\ } & M' \end{array}$$

where c is the bundle map induced by $\theta_{f'}$ and θ_f itself is the collapsing map of $\lambda \circ \mathcal{O}_f$. Hence, according to Thm. 2.3,

$$(2.8)\qquad A\big(H^n(M'), \overline{\overline{\psi}}s\big) - A\big(H^n(M'), \overline{\overline{\psi}}\theta_f^*\mathrm{id}^* s\big) = 2\overline{\overline{\psi}}s\Big(\big(V(M')(\lambda \circ \mathcal{O}_f)^*(\sigma(V))\big)_n\Big).$$

Since g factors through SG, we may also apply Thm. 2.3 to its Arf invariant and then

$$(2.9)\qquad A\big(H^n(M), \overline{\psi}s\big) - A\big(H^n(M), \overline{\psi}(D\hat{\pi})^*\theta_f^*\pi^* s\big) = 2\overline{\psi}s\Big(\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big).$$

Finally, since $i \circ f_2$ also factors through SG,

$$(2.10)\qquad A\big(H^n(M), \overline{\psi}s\big) - A\big(H^n(M), \overline{\psi}(D\hat{\pi})^*\pi^* s\big) = 2\overline{\psi}s\Big(\big(V(M)\cdot f_2^*(\sigma(V))\big)_n\Big).$$

It follows from (2.7), (2.8) and (2.10) that

$$\begin{aligned} 4s_K(M, g) &= A\big(H^n(M'), \overline{\psi}(D\hat{\pi})^* s\big) - A\big(H^n(M), \overline{\psi}s\big) + \\ &\quad + 2\overline{\psi}s\Big(\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big) - 2\overline{\psi}(D\hat{\pi})^* s\Big(\big(V(M')(\lambda \circ \mathcal{O}_f)^*(\sigma(V))\big)_n\Big) \\ &= 4s_K(M, f_2) + 2\overline{\psi}s\Big(\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big) - \\ &\quad - 2\overline{\psi}(D\hat{\pi})^* s\Big(\big(V(M')(\lambda \circ \mathcal{O}_f)^*(\sigma(V))\big)_n\Big) - 2\overline{\psi}s\Big(\big(V(M)f_2^*(\sigma(V))\big)_n\Big). \ \blacksquare \end{aligned}$$

Corollary 2.11. *If the map f_2 is null-homotopic then the Kervaire obstruction of $g = i \circ \tilde{\lambda} \circ f$: $M \to G/\mathrm{TOP}$ is given by*

$$4s_K(M, g) = 2\overline{\psi}s\Big(\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big) - 2\overline{\psi}(D\hat{\pi})^* s\Big(\big(V(M')\mathcal{O}_f^*\lambda^*(\sigma(V))\big)_n\Big).$$

Let α^*: $H^*(M') \to H^*(M)$ be the splitting of π^*: $H^*(M) \to H^*(M')$ induced by the normal map corresponding to f_2, i.e.

$$\alpha^* = \Sigma^{-N} \circ (D\hat{\pi})^* \circ \Sigma^N,$$

where Σ is the cohomology suspension, and let $K^*(M') = \mathrm{Ker}(\alpha^*) \subset H^*(M')$.

Lemma 2.12. *If f_2 is null-homotopic then*

$$\pi^*(\tilde{\lambda}f)^*(\sigma(V)) - (\lambda \circ \mathcal{O}_f)^*(\sigma(V)) \in K^*(M').$$

We postpone the proof of Lemma 2.12 till Section 4.

Theorem 2.13. *If f_2 is null-homotopic then*

$$2s_K(M, g) = \big(\overline{\psi}(D\hat{\pi})^* s\big)\Big(\big[V(M')\big((\tilde{\lambda} \circ f \circ \pi)^*(\sigma(V)) + (\lambda \circ \mathcal{O}_f)^*(\sigma(V))\big)\big]_n\Big) \in Z/4.$$

Proof. Let us substitute a for $\big(V(M')(\tilde{\lambda} \circ f \circ \pi)^*(\sigma(V))\big)_n$ and b for $\big(V(M')(\lambda \circ \mathcal{O}_f)^*(\sigma(V))\big)_n$. Since $\overline{\psi}(D\hat{\pi})^* s$ is a quadratic function over the cup product pairing in $H^n(M)$,

$$\overline{\psi}(D\hat{\pi})^* s(a) - \overline{\psi}(D\hat{\pi})^* s(b) = \overline{\psi}(D\hat{\pi})^* s(a-b) + 2(b, a-b)$$

(recall: 2: $Q/Z \to Q/2Z$). Observe that $a - b \in K^n(M')$. This follows from (2.12) and the fact that $K^*(M')$ may be characterized as the subspace of $H^*(M')$ orthogonal to $\mathrm{Im}\,\pi^* \subset H^*(M')$.

Now, $V(M') = \pi^*(V(M))$ and $a \in \mathrm{Im}\,\pi^*$. Hence $(a, a-b) = 0$ and $(b, a-b) = -(a-b, a-b) + (a, a-b) = -(a-b, a-b) = 0$ since the cup pairing is even on $K^*(M')$, see [1; sec. III. 3].

Hence we get

$$(2.14)\qquad \overline{\psi}(D\hat{\pi})^* s(a+b) = \overline{\psi}(D\hat{\pi})^* s(a) - \overline{\psi}(D\hat{\pi})^* s(b).$$

Since f_2 is null-homotopic, the map $(S^N\pi) \circ (D\hat{\pi})$: $S^N \wedge M_+ \to S^N \wedge M_+$ is homotopic to the identity. Consequently

$$a = \Big(\big(\pi^*(V(M))\big)\big(\pi^*(\tilde{\lambda} \circ f)^*(\sigma(V))\big)\Big)_n = \pi^*\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n$$

and

$$\begin{aligned} \overline{\psi}(D\hat{\pi})^* s(a) &= \overline{\psi}(D\hat{\pi})^* s\Big(\pi^*\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big) \\ &= \overline{\psi}(D\hat{\pi})^*\pi^* s\Big(\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big) \\ &= \overline{\psi}s\Big(\big(V(M)(\tilde{\lambda} \circ f)^*(\sigma(V))\big)_n\Big). \end{aligned}$$

The conclusion of Thm. 2.13 follows now from (2.14) and (2.11). $\blacksquare$

Let $x_f = \big[V(M')\big((\tilde{\lambda} \circ f \circ \pi)^*(\sigma(V)) + (\lambda \circ \mathcal{O}_f)^*(\sigma(V))\big)\big]_n \in H^n(M')$ and let $s(x_f)$: $S^N \wedge M'_+ \to S^N \wedge K_n$ be the suspension of x_f. Let $Sq^{n+1}_{s(x_f)D\hat{\pi}}(\Sigma^N(\iota_n)) \in H^{2n+N}(S^N \wedge M_+)$ be the functional square of the map $s(x_f) \circ D\hat{\pi}$: $S^N \wedge M_+ \to S^N \wedge K_n$.

COROLLARY 2.15. *If f_2 is null-homotopic then*

$$s_K(M, g) = \left(\Sigma^{-N} Sq^{n+1}_{s(x_f)D\hat{\pi}}(\Sigma^N(\iota_n))\right)[M] \in Z/2 .$$

Proof. Since $x_f \in K^n(M')$, the conclusion of Cor. 2.15 follows from [2; p. 105] and Thm. 2.13. ■

3. The index obstructions of $[M, Q(RP^\infty)]$. In this section we aim at giving a description of the index obstruction $s_I(M, g)$ of a map $g: M \to G/\mathrm{TOP}$ in the case where g is a composition $g = i \circ \bar{\lambda} \circ f$ and $f: M \to Q(RP^\infty)$. We assume that M^m is a $Z/2$-manifold and $\dim M = m = 4n$. Our main reference is [2; Chap. 2 and 6]. Till the end of the section we use the notation $K_t = K(Q/Z, t)$ unless otherwise stated.

Let $z_1(M) \in H^1(M; Z)$ be an integral class such that its mod 2 reduction is the first Stiefel–Whitney class $w_1(M)$ of M. Suppose that $z_1(M)$ is represented by a map $z_1: M \to S^1$ which has $1 \in S^1$ as its regular value. Let $N = z_1^{-1}(1) \subset M$. Then N is an oriented $(4n-1)$-manifold. Let

$$T^{2n-1}(N) = H^{2n-1}(N, Q/Z)/\mathrm{Image}(H^{2n-1}(N, Q) \to H^{2n-1}(N, Q/Z)) .$$

There is a nonsingular bilinear pairing

$$L: T^{2n-1}(N) \otimes T^{2n-1}(N) \to Q/Z$$

induced by the pairing

$$L': H^{2n-1}(N, Q/Z) \otimes H^{2n-1}(N, Q/Z) \to Q/Z$$

defined by $L'(x \otimes y) = \langle x \cup \beta y, [N] \rangle$. Here $\cup$ is the cup product associated to the pairing $Q/Z \otimes Z \to Q/Z$ and $\beta: H^{2n-1}(N, Q/Z) \to H^{2n}(N, Z)$ is the Bockstein of the coefficient system $0 \to Z \to Q \to Q/Z \to 0$.

Let me now recall [2; Thm. 6.2]:

THEOREM 3.1. (i) *If N^{4n-1} is an oriented $(4n-1)$-manifold then there is an exact sequence*

$$0 \to Q/Z \xrightarrow{i_*} \{N, K_{2n-1}\} \xrightarrow{j_*} H^{2n-1}(N, Q/Z) \to 0 .$$

(ii) *The suspension s: $H^{2n-1}(N; Q/Z) \to \{N, K_{2n-1}\}$ is quadratic; that is if $x, y \in H^{2n-1}(N; Q/Z)$, then*

$$s(x+y) = s(x) + s(y) + i_*(L'(x, y)) .$$

Let $\bar{\varphi}$: $\{N, K_{2n-1}\} \to Q/2Z$ be a linear function such that

(3.2)
- (i) $\bar{\varphi}(\mathrm{Image}\{N, K(Q, 2n-1)\}) = 0$;
- (ii) the composition $\bar{\varphi} i_*$: $Q/Z \to \{N, K_{2n-1}\} \to Q/2Z$ is the isomorphism, multiplication by 2;
- (iii) if f: $S^q \wedge N \to S^q \wedge K_{2n-1}$ represents an element of $\{N, K_{2n-1}\}$, then $\varrho\bar{\varphi}(f) = \langle f^*(\Sigma^q(\iota \cup \beta\iota)), [S^q \wedge N] \rangle \in Q/Z$, where ϱ: $Q/2Z \to Q/Z$ is the projection.

(see [2; p. 111]).

Let h: $M^{4n} \to G/\mathrm{TOP}$ be a map. Then the index obstruction $s_I(M, h) \in Z/2$ is defined, see [2; p. 91]. If the map h factors through h': $M^{4n} \to SG$ then the index obstruction $s_I(M, h)$ can be computed as follows. Let

$$\begin{array}{ccc} \nu_{M'} & \xrightarrow{b} & \nu_M \\ \downarrow & & \downarrow \\ M' & \xrightarrow{\pi} & M \end{array}$$

be a normal invariant corresponding to h', let $D\hat{\pi}$: $S^q \wedge M_+ \to S^q \wedge M'_+$ be its collapsing map and let $N' = \pi^{-1}(N) \subset M'$ (we assume that π is transversal to N). Denote $M_0 = M$ — (open normal bundle of N), $M'_0 = M'$ — (open normal bundle of N'). Thus $\partial M_0 = 2N$, $\partial M'_0 = 2N'$ and we have a map of pairs π: $(M'_0, \partial M'_0) \to$ $\to (M_0, \partial M_0)$.

THEOREM 3.3. *The index obstruction of the map h': $M^{4n} \to SG$ is given by*

$$s_I(M, h') = \mathrm{index}(M'_0) - \mathrm{index}(M_0) + 2\big(A(T^{2n-1}(N'), \bar{\varphi}(D\hat{\pi})^* s) -$$
$$- A(T^{2n-1}(N), \bar{\varphi}s)\big) + 4\bar{\varphi}s\big((V(N)(h'|_N)^*(\sigma(V)))_{2n-1}\big) \in 8Z/16Z = Z/2 .$$

See [2; (6.7)].

Let me finally recall [2; Thm. 6.6]. Suppose that γ: $N \to SG$ is a map, π: $N' \to N$ is a corresponding degree one normal map and $D\hat{\pi}$: $S^q \wedge N \to S^q \wedge N'$ is the collapsing map.

THEOREM 3.4.

$$A(T^{2n-1}(N), \bar{\varphi}s) - A(T^{2n-1}(N), \bar{\varphi}(D\hat{\pi})^* \pi^* s) = 2\bar{\varphi}s\big((V(N)\gamma^*(\sigma(V)))_{2n-1}\big) \in Z/8 ,$$

where the natural inclusions $Z/2 \subset Q/Z$ and $Z/4 \subset Q/2Z$ give a commutative diagram

$$\begin{array}{ccc} H^{2n-1}(N; Z/2) & \longrightarrow & H^{2n-1}(N; Q/Z) \\ \downarrow{\scriptstyle \bar{\varphi}s} & & \downarrow{\scriptstyle \bar{\varphi}s} \\ Z/4 & \subset & Q/2Z \end{array}$$

2: $Z/4 \to Z/8$ is the inclusion and $\sigma(V) \in H^*(SG; Z/2)$ is the suspension of the Wu class $V \in H^*(BSG; Z/2)$.

Let f: $M \to Q(RP^\infty_+)_1$ be a map, $f = (f_1, f_2)$. As in Section 2, $\mathrm{Ad}(f)$ can be factorized as

$$\mathrm{Ad}(f): S^q \wedge M_+ \xrightarrow{D\hat{\pi}} S^q \wedge M'_+ \xrightarrow{S^q \wedge \theta_f} S^q \wedge RP^\infty_+$$

and then $D\hat{\pi}$ is the collapsing map corresponding to f_2, while $\theta_f \circ D\hat{\pi}$ is the collapsing map corresponding to $\bar{\lambda} \circ f$.

Consider the map $\lambda\circ\mathcal{O}_f|_{N'}\colon N'\to SG$. Its corresponding normal map is of the form

$$\begin{array}{ccc} \nu_{N'} & \xrightarrow{c} & \nu_{N'} \\ \downarrow & & \downarrow \\ N' & \xrightarrow{\text{id}} & N' \end{array}$$

where c is the bundle map induced by $\theta_f|_{N'}$, and $\theta_f|_{N'}$ itself is the collapsing map of $\lambda\circ\mathcal{O}_f|_{N'}$. The function $\bar{\bar{\varphi}} = \bar{\varphi}(D\hat{\pi}|_N)^*\colon \{N', K_{2n-1}\}\to Q/2Z$ satisfies (3.2). Hence, according to Thm. 3.4,

(3.5) $$A(T^{2n-1}(N'), \bar{\varphi}(\theta_f|_{N'})^*s)$$
$$= A(T^{2n-1}(N'), \bar{\bar{\varphi}}s) - 2\bar{\bar{\varphi}}s\big((V(N')(\lambda\circ\mathcal{O}_f|_{N'})^*(\sigma(V)))_{2n-1}\big)\in Z/8\,.$$

LEMMA 3.6. *The index surgery obstruction of the map $g = i\circ\tilde{\lambda}\circ f\colon M\to G/\mathrm{TOP}$ is given by*

$$\begin{aligned} s_I(M, g) = s_I(M, f_2) &+ 4\bar{\varphi}s\big((V(N)(\tilde{\lambda}\circ f|_N)^*(\sigma(V)))_{2n-1}\big)-\\ &-4\bar{\varphi}s\big((V(N)f_2^*(\sigma(V)))_{2n-1}\big)-\\ &-4\bar{\varphi}(D\hat{\pi})^*s\big((V(N')(\lambda\circ\mathcal{O}_f|_{N'})^*(\sigma(V)))_{2n-1}\big)\\ &\in 8Z/16Z = Z/2\,. \end{aligned}$$

Proof. According to Thm. 3.3

$$\begin{aligned} s_I(M, g) = \operatorname{index}(M_0') &- \operatorname{index}(M_0)+\\ &+2\big(A(T^{2n-1}(N'), \bar{\varphi}(D\hat{\pi})^*\theta_f^*s) - A(T^{2n-1}(N), \bar{\varphi}s)\big)+\\ &+4\bar{\varphi}s\big((V(N)(\tilde{\lambda}\circ f|_N)^*(\sigma(V)))_{2n-1}\big) \end{aligned}$$

and

$$\begin{aligned} s_I(M, f_2) &= \operatorname{index}(M_0') - \operatorname{index}(M_0)+\\ &+2\big(A(T^{2n-1}(N'), \bar{\varphi}(D\hat{\pi})^*s) - A(T^{2n-1}(N), \bar{\varphi}s)\big)+4\bar{\varphi}s\big((V(N)f_2^*(\sigma(V)))_{2n-1}\big)\,. \end{aligned}$$

Hence

$$\begin{aligned} s_I(M, g) &= s_I(M, f_2) + 2\big(A(T^{2n-1}(N'), \bar{\varphi}(D\hat{\pi})^*\theta_f^*s) - A(T^{2n-1}(N'), \bar{\varphi}(D\hat{\pi})^*s)\big)+\\ &\quad+4\bar{\varphi}s\big((V(N)(\tilde{\lambda}\circ f|_N)^*(\sigma(V)))_{2n-1}\big) - 4\bar{\varphi}s\big((V(N)f_2^*(\sigma(V)))_{2n-1}\big)\,. \end{aligned}$$

The conclusion of Lemma 3.6 now follows if we apply (3.5). ■

COROLLARY 3.7. *If f_2 is null-homotopic then*

$$\begin{aligned} s_I(M, i\circ\tilde{\lambda}\circ f) = 4\big(&\bar{\varphi}s\big((V(N)(\tilde{\lambda}\circ f|_N)^*(\sigma(V)))_{2n-1}\big)-\\ &-\bar{\varphi}(D\hat{\pi})^*s\big((V(N')(\lambda\circ\mathcal{O}_f|_{N'})^*(\sigma(V)))_{2n-1}\big)\big)\in 8Z/16Z\,, \end{aligned}$$

where $4\colon Z/4Z\to 4Z/16Z$ is the isomorphism.

THEOREM 3.8. *If f_2 is null-homotopic then*

$$\begin{aligned} s_I(M, i\circ\tilde{\lambda}\circ f) &= 4\bar{\varphi}(D\hat{\pi})^*s\big((V(N')[(\tilde{\lambda}\circ f\circ\pi|_{N'})^*(\sigma(V))+\\ &\qquad+(\lambda\circ\mathcal{O}_f|_{N'})^*(\sigma(V))])_{2n-1}\big)\\ &\in 8Z/16Z = Z/2\,. \end{aligned}$$

Proof. Let us substitute a for $(V(N')(\tilde{\lambda}\circ f\circ\pi|_{N'})^*(\sigma(V)))_{2n-1}$ and b for $(V(N')(\lambda\circ\mathcal{O}_f|_{N'})^*(\sigma(V)))_{2n-1}$. Then $a\in\pi^*(H^*(N))\subset H^*(N')$ and

$$\bar{\varphi}s\big((V(N)(\tilde{\lambda}\circ f|_N)^*(\sigma(V)))_{2n-1}\big) = \bar{\varphi}(D\hat{\pi})^*s(a)$$

since $(D\hat{\pi})^*\pi^* = \text{id}$. Denote $\varphi = \bar{\varphi}(D\hat{\pi})^*s$. It follows from (3.7) that, in particular, $\varphi(a)-\varphi(b)\in Z/2\subset Z/4\subset Q/2Z$.

According to Lemma 2.12 (see Section 4), $a+b\in K^*(N')$. Thus $L(a+b, a) = \langle(a+b)\cup\beta(a), [N']\rangle = \langle(a+b)Sq^1(a), [N']\rangle = 0$ since $Sq^1(a)\in\pi^*(H^*(N))\subset\subset H^*(N')$.

Furthermore, since a and b are elements of order 2,

$$\varphi(b) = \varphi(a+b)+\varphi(a)+2L(a+b, a) = \varphi(a+b)+\varphi(a)\,.$$

Consequently, $\varphi(a)-\varphi(b) = \varphi(b)-\varphi(a) = \varphi(a+b)$ and

$$s_I(M, i\circ\tilde{\lambda}\circ f) = 4(\varphi(a)-\varphi(b)) = 4\varphi(a+b)\,.$$

Thus (3.8) is proved. ■

4. The Wu class. In this section we prove Lemma 2.12. Homology and cohomology with $Z/2$-coefficients is used and $K_n = K(Z/2, n)$ throughout the section.

For a given cohomology class $x\in H^t(X)$ we may consider a cohomology class $Q(x)\in H^t(Q(X))$. If x is represented by a map $h\colon X\to K_t$ then $Q(x)$ is the cohomology class represented by the composition $Q(X)\xrightarrow{Q(h)}Q(K_t)\xrightarrow{\gamma_t}K_t$, where the map γ_t comes from the infinite-loop space structure of K_t.

Let $\sigma(V) = \sum_{j=1}\sigma(v_j)\in H^*(SG)$ be the suspension of the Wu class $V = \sum_{j=0}v_j\in H^*(BSG)$ and let $w_j\in H^j(BSG)$ be the jth Stiefel–Whitney class. Then $\sigma(v_{2^r}) = \sigma(w_{2^r})$ and $\sigma(v_j) = 0$ for $j\neq 2^r$, see [2; Remark 5.6] or [4; Sec. 3, Lemma]. $\sigma(V)$ may be extended trivially to a cohomology class in $H^*(Q(S^0))$. We shall write $\sigma(V)$ for this extended class as well.

There are two cohomology classes, $Q((\lambda_+)^*(\sigma(V)))$ and $\tilde{\lambda}^*(\sigma(V))$, in $H^*(Q(RP_+^\infty))$. They are not the same. We shall now proceed to describe how they both evaluate on the homology of $Q(RP_+^\infty)$.

Let us first recall what $H_*(Q(RP_+^\infty))$ looks like. Our main reference for this matter is [3; Part I]. For a finite sequence of integers $I = (s_1, \ldots, s_k)$ such that $s_j\geqslant 0$ we define

$$d(I) = \sum_{j=1}^{k}s_j\,,\qquad l(I) = k\qquad\text{and}\qquad e(I) = s_1-\sum_{j=2}^{k}s_j\,.$$

I is said to be *admissible* if $2s_j \geq s_{j-1}$ for $2 \leq j \leq k$. The empty sequence $I_\emptyset$ is admissible and satisfies $d(I_\emptyset) = l(I_\emptyset) = 0$, $e(I_\emptyset) = \infty$.

Let $e_i \in H^i(RP^\infty)$, $i \geq 0$, be the nontrivial homology classes and let $P(RP_+^\infty) = P[Q^I(e_i) | \, i \geq 0,\ I$ — admissible, $e(I) > i$, $\deg Q^I(e_i) > 0]$ be the graded polynomial ring over $Z/2$ generated by $Q^I(e_i)$'s. Here $\deg Q^I(e_i) = i + d(I)$. Let us consider the group ring $Z/2[Z]$ as being trivially graded. Then

(4.1) $$H_*(Q(RP_+^\infty)) \cong P(RP_+^\infty) \otimes_{Z/2} Z/2[Z],$$

see [3; Thm. 4.2 and p. 42]. $Q^I(\cdot)$'s correspond to the Dyer–Lashof operations in $Q(RP_+^\infty)$. Let $b_0 \in Z/2[Z]$ be the unit and let $[1]$ be the element of $Z/2[Z]$ given by $1 \in Z$. The embedding $j\colon RP_+^\infty \hookrightarrow Q(RP_+^\infty)$ induces $j_*\colon H_*(RP_+^\infty) \to H_*(Q(RP_+^\infty))$ and $j_*(e_i) = Q^{I_\emptyset}(e_i) \otimes b_0$ for $i > 0$, $j_*(e_0) = 1 \otimes [1]$.

For $a \in P(RP_+^\infty)$, $b \in Z/2[Z]$ we say that $a \otimes b$ is decomposable if and only if a is such. If $a \otimes b \in P(RP_+^\infty) \otimes Z/2[Z]$ is decomposable then

(4.2) $$\langle Q(x), a \otimes b \rangle = 0$$

for any cohomology class $x \in H^t(RP_+^\infty)$. Indeed, if a map $h\colon RP_+^\infty \to K_t$ represents x then

$$\langle Q(x), a \otimes b \rangle = \langle Q(h)^* \gamma_t^*(\iota_t), a \otimes b \rangle = \langle \iota_t, \gamma_{t*} Q(h)_*(a \otimes b) \rangle,$$

where $\iota_t \in H^t(K_t)$ is the characteristic class. Since both γ_t and $Q(h)$ were infinite-loop maps and $a \otimes b$ was decomposable, $\gamma_{t*} Q(h)_*(a \otimes b)$ is also decomposable. Then (4.2) follows since ι_t is primitive for dimensional reasons.

Suppose now that I is a nonempty sequence and $e(I) > i$. Then $Q^I(e_i) \otimes b_0 = Q^I(e_i \otimes b_0)$, where $Q^I(\cdot)$ on the right hand side means the Dyer–Lashof operation in $H_*(Q(RP_+^\infty))$. Since K_t is connected,

$$\gamma_{t*} Q(h)_*(Q^I(e_i) \otimes b) = \varepsilon(b) \gamma_{t*} Q(h)_*(Q^I(e_i) \otimes b_0)$$

for any $b \in Z/2[Z]$, where $\varepsilon\colon Z/2[Z] \to Z/2$ is the augmentation. $\gamma_t Q(h)$ is an infinite-loop map, hence $\gamma_{t*} Q(h)_*$ commutes with the Dyer–Lashof operations. Furthermore, the Dyer–Lashof operations in $H_*(K_t)$ are all trivial, see [3; Lemma 6.1, p. 60]. Hence

$$\gamma_{t*} Q(h)_*(Q^I(e_i) \otimes b) = \varepsilon(b) \gamma_{t*} Q(h)_*(Q^I(e_i \otimes b_0))$$
$$= \varepsilon(b)\, Q^I(\gamma_{t*} Q(h)_*(e_i \otimes b_0)) = 0.$$

Therefore, for any $x \in H^t(RP_+^\infty)$

(4.3) $$\langle Q(x), Q^I(e_i) \otimes b \rangle = 0$$

if I is nonempty and $e(I) > i$.

Finally, since the composition

$$RP_+^\infty \hookrightarrow Q(RP_+^\infty) \xrightarrow{Q(h)} Q(K_t) \xrightarrow{\gamma_t} K_t \text{ is equal to } h,$$

(4.4) $$\langle Q(x), e_i \otimes b \rangle = \varepsilon(b) \cdot \langle x, e_i \rangle.$$

Let us also recall that

(4.5) $$\langle \sigma(w_{i+1}), \lambda_* e_i \rangle = 1,$$

see [2; Lemma 3.5].

Formulae (4.2)–(4.5) describe completely the action of $Q((\lambda_+)^*(\sigma(V)))$ on $H_*(Q(RP_+^\infty))$. We shall now investigate the action of $\tilde{\lambda}^*(\sigma(V))$. The class $\sigma(V)$ is primitive for the loop sum product in SG, [2; Lemma 3.5], and $\tilde{\lambda}$ is an H-space map. Thus, if $a \otimes b \in H_*(Q(RP_+^\infty))$ is decomposable then

(4.6) $$\langle \tilde{\lambda}^*(\sigma(V)), a \otimes b \rangle = 0.$$

Let us recall ([3; p. 42]) that $H_*(Q(S^0)) \cong P(S^0) \otimes Z/2[Z]$, where $P(S^0) = P[Q^I[1] | \, I$-admissible, $e(I) > 0$, $d(I) > 0]$. Since $\tilde{\lambda}$ is an infinite-loop map and $\tilde{\lambda}(RP^\infty) \subset SG$, the induced algebra homomorphism $\tilde{\lambda}_*\colon H_*(Q(RP_+^\infty)) \to H_*(Q(S^0))$ is of the form

$$\tilde{\lambda}_*(Q^I(e_i) \otimes b_0) = Q^I(\lambda_*(e_i))$$

and

$$\tilde{\lambda}_*(1 \otimes b) = 1 \otimes b.$$

Furthermore $\lambda(e_i) = Q^i[1] * [-1]$, where $*$ means the algebra multiplication in $H_*(Q(S^0))$, see [2; (8.13)]. Hence

$$\langle \tilde{\lambda}^*(\sigma(V)), Q^I(e_i) \otimes b \rangle = \langle \sigma(V), \tilde{\lambda}_*(Q^I(e_i) \otimes b) \rangle = \langle \sigma(V), \tilde{\lambda}(Q^I(e_i)) * b \rangle$$
$$= \langle \sigma(V), Q^I(Q^i[1] * [-1]) * b \rangle$$
$$= \langle \sigma(V), Q^I(Q^i[1] * [-1]) * [1 - 2^{l(I)}] * [2^{l(I)} - 1] * b \rangle$$
$$= \varepsilon_0(2^{l(I)} - 1] * b) \langle \sigma(V), Q^I(Q^i[1] * [-1]) * [1 - 2^{l(I)}] \rangle,$$

where $\varepsilon_0\colon Z/2[Z] \to Z/2$, $\varepsilon_0[m] = 0$ for $m \in Z$, $m \neq 0$, and $\varepsilon_0[0] = 1$.

If I is nonempty and $d(I) + i > 0$ then

$Q^I(Q^i[1] * [-1]) \equiv Q^I(Q^i[1]) * [-2^{l(I)}]$ modulo decomposable elements in $H_*(Q(S^0))$.

Thus, since $\sigma(V)$ is primitive,

$$\langle \sigma(V), Q^I(Q^i[1] * [-1]) * [1 - 2^{l(I)}] \rangle = \langle \sigma(V), Q^I(Q^i[1]) * [1 - 2^{l(I)+1}] \rangle.$$

It was proved in [15; (6.3)], see also [3; p. 127–128], that

$$\langle \sigma(V), Q^I(Q^i[1]) * [1 - 2^{l(I)+1}] \rangle = 0$$

except for the case when I is empty or $i = 0$.

Hence, if I is nonempty and $i > 0$

(4.7) $$\langle \tilde{\lambda}^*(\sigma(V)), Q^I(e_i) \otimes b \rangle = 0.$$

Finally, for $i \geq 0$

(4.8) $$\langle \tilde{\lambda}^*(\sigma(V)), e_i \otimes b \rangle = \langle \sigma(V), \tilde{\lambda}_*(e_i \otimes b) \rangle = \langle \sigma(V), \tilde{\lambda}_*(e_i) * b \rangle$$
$$= \langle \sigma(V), Q^i[1] * [-1] * b \rangle$$
$$= \varepsilon_0(b) \langle \sigma(V), Q^i[1] * [-1] \rangle = \varepsilon_0(b) \langle \sigma(V), \lambda_*(e_i) \rangle.$$

Observe that $\varepsilon_0(b_0) = \varepsilon(b_0)$.

It follows from the formulae (4.2)–(4.8) that

$$\langle Q((\lambda_+)^*(\sigma(V))), y\rangle = \langle\tilde{\lambda}^*(\sigma(V)), y\rangle$$

if $y \in H_*(Q(RP_+^\infty))$ and either y is decomposable or $y = Q^I(e_i)\otimes b$ with $i>0$ and I nonempty, or $y = e_i\otimes b_0$ with $i\geqslant 0$.

Let $d_0: RP_+^\infty \to S^0$ be the nontrivial map. Consider the induced homomorphism

$$Q(d_0)_*: H_*(Q(RP_+^\infty)) \to H_*(Q(S^0)).$$

Then $Q(d_0)_*$ commutes with the Dyer–Lashof operations, $Q(d_0)_*(e_i) = 0$ for $i>0$ and $Q(d_0)_*(e_0) = [1]$. Let us write $D = \operatorname{Ker}(Q(d_0)_*)$ or the kernel of $Q(d_0)_*$ and let $D_1 = D \cap H_*(Q(RP_+^\infty)_1)$. Then for any $z \in D_1$

$$\langle Q((\lambda_+)^*(\sigma(V))), z\rangle = \langle\tilde{\lambda}^*(\sigma(V)), z\rangle. \tag{4.9}$$

Proof of Lemma 2.12. We follow the notation of Section 2. Lemma 2.12 will be proved if we show that

$$(D\hat{\pi})^*\Sigma^N((\tilde{\lambda}\circ f\circ\pi)^*(\sigma(V)) - (\lambda_+\circ\mathcal{O}_f)^*(\sigma(V))) = 0, \tag{4.10}$$

where Σ is the suspension in cohomology.

Since f_2 is null-homotopic,

$$\begin{aligned}(D\hat{\pi})^*\Sigma^N\pi^*(\tilde{\lambda}\circ f)^*(\sigma(V)) &= (\pi\circ D\hat{\pi})^*\Sigma^N f^*(\tilde{\lambda}^*(\sigma(V)))\\ &= \Sigma^N f^*(\tilde{\lambda}^*(\sigma(V))).\end{aligned} \tag{4.11}$$

Furthermore $(S^N\wedge\mathcal{O}_f)\circ D\hat{\pi} = \operatorname{Ad}(f)$, (see (2.4)), and $\operatorname{Ad}(f) = \mu_{RP_+^{N-1}}\circ(S^N f)$, (see the proof of Lemma 2.5), thus

$$\begin{aligned}(D\hat{\pi})^*\Sigma^N\mathcal{O}_f^*\lambda_+^*(\sigma(V)) &= (S^N\wedge\mathcal{O}_f\circ D\hat{\pi})^*\Sigma^N\lambda_+^*(\sigma(V))\\ &= \operatorname{Ad}(f)^*\Sigma^N\lambda_+^*(\sigma(V)) = (S^N f)^*\mu_{RP_+^{N-1}}\Sigma^N\lambda_+^*(\sigma(V)).\end{aligned}$$

Now, for any $x\in H^t(Y)$, Y being a space, one has

$$\mu_Y^*\Sigma^N x = \Sigma^N Q_N(x) \in H^{t+N}(S^N\Omega^N S^N(Y)).$$

Here $Q_N(x)$ is the cohomology class represented by the composition

$$\Omega^N S^N Y \xrightarrow{\Omega^N S^N(h)} \Omega^N S^N K_t \xrightarrow{\gamma_t} K_t \quad\text{and}\quad h: Y\to K_t$$

is the map representing x.

Hence

$$(D\hat{\pi})^*\Sigma^N\mathcal{O}_f^*\lambda_+^*(\sigma(V)) = (S^N f)^*\Sigma^N Q(\lambda_+^*(\sigma(V))) = \Sigma^N f^*(Q(\lambda_+^*(\sigma(V)))). \tag{4.12}$$

Since f_2 is null-homotopic, the composition $Q(d_0)\circ f$ is null-homotopic as well (indeed, $Q(d_0)\circ f$ is always homotopic to f_2). Consequently, $f_*(\tilde{H}_*(M))\subset D_1$. It now follows from (4.9) that

$$f^*(Q(\lambda_+^*(\sigma(V)))) = f^*\tilde{\lambda}^*(\sigma(V))$$

and, consequently, from (4.11) and (4.12)

$$(D\hat{\pi})^*\Sigma^N\pi^*(\tilde{\lambda}\circ f)^*(\sigma(V)) = (D\hat{\pi})^*\Sigma^N\mathcal{O}_f^*\lambda_+^*(\sigma(V)).$$

Thus (4.10) is proved and so is Lemma 2.12. ■

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