Generalized manifolds (ANR’s and AR’s) and null decompositions of manifolds

by

S. Singh (Altoona, Penn.)

Abstract. We prove the following theorem: Theorem. For each topological $n$-manifold $M^n$, $n \geq 5$, there exists an uncountable family $\mathcal{M}$ of $n$-dimensional absolute nbd. retracts such that each $X$ in $\mathcal{M}$ satisfies: (1) $X$ has the (proper) homotopy type of $M^n$, and (2) $X$ does not contain any strongly movable proper subset of dimension $\geq 2$; furthermore, if $M^n$ is a manifold without boundary, then each $X$ in $\mathcal{M}$ is a generalized $n$-manifold satisfying $X \times B^2$ is homeomorphic to $M^n \times B^2$ in addition to (1) and (2). Moreover, each $X$ in $\mathcal{M}$ contains movable subsets of dimension $\geq 2$ and the space $X$ is obtained from $M^n$ as a decomposition space associated with a null decomposition of $M^n$ into arcs and singletons. Some other related matters are discussed; for instance, it is shown that a strongly movable continuum has UV if for small loops (actually more is shown, see the statement at the end of the proof of Proposition (6.3.2)).

1. Introduction, notation and terminology

(1.1) Introduction. Suppose $G$ is a cell-like upper semicontinuous decomposition of a compact and connected $n$-manifold $M^n$ without boundary such that the decomposition space $M^n/G$ is finite dimensional. It is well-known that $M^n/G$ is a generalized $n$-manifold which is not, in general, an $n$-manifold. Recently, J. W. Cannon [13] has used a disjoint disk property (DDP) in his remarkable solution to the double suspension problem and later R. D. Edwards has proved the following far reaching extension of [13] and [36]: If $n \geq 5$ and $M^n/G$ has DDP, then the projection $p: M^n \to M^n/G$ can be approximated (arbitrarily close) by a homeomorphism. Thus, DDP appears to be the definitive (and minimal) condition whose presence guarantees that the generalized $n$-manifold $M^n/G$ is an $n$-manifold. On the other hand, the failure of DDP can be successfully used to produce generalized $n$-manifolds with rather exotic topological structure; the results of this note may be interpreted in this context. A recent deep theorem of F. Quinn [33] states that, in fact, every generalized $n$-manifold is a cell-like image of a topological $n$-manifold; therefore, the failure of DDP is the only obstacle for any generalized $n$-manifold to be an $n$-manifold, see Lacher [27] and J. W. Cannon [12] for historical and other details. The failure of DDP can cause enough damage that the generalized $n$-manifold, $n \geq 3$, does not contain any proper compact subset of dimension $\geq 2$ which looks like a polyhedron (or ANR). More specifically, we prove the following theorem:
THEOREM. For each topological $n$-manifold $M^n$, $n \geq 3$, there exists a family $\mathcal{M}$ of topologically distinct $n$-dimensional absolute nbd. retracts such that each $X \in \mathcal{M}$ satisfies (1) $X$ has the homotopy type of $M^n$, and (2) $X$ does not contain any strongly movable proper subset of dimension $\geq 2$; furthermore, if $M^n$ is a manifold without boundary, then each $X \in \mathcal{M}$ is a generalized $n$-manifold satisfying $X \times E^1$ is homeomorphic to $M^n \times E^1$ in addition to (1) and (2).

Each space $X$ in the theorem is constructed as a decomposition space associated with a certain upper semicontinuous decomposition $G$ of the manifold $M^n$ such that the set of all the nondegenerate elements of $G$ form a (countable) null collection of arcs; therefore, the decomposition $G$ is the minimal in the sense that it is a null collection and $G$ is the simplest since each of its nondegenerate elements is an arc which is the simplest cell-like continuum. Furthermore, it is shown that if $G$ is an arbitrary decomposition of an $n$-manifold $M^n$ whose nondegenerate elements form a null collection of arcs, then the associated decomposition space $X$ contains movable proper subsets of dimension $\geq 2$. Therefore, the results of this note are the best possible.

We have relied heavily on [37] for many technical details on linking, and we have also depended on Wright [46] whenever possible. We have tried to preserve the geometric flavor of its predecessors [5, 38, 39, 40, 41] and most notably the work of Bing and Borsuk [5]. The main ingredient is a Cantor set construction of Daverman and Edwards (we often refer to Daverman and Edwards by DE) which plays a crucial role in our construction, see Daverman [16] for an exposition. Another ingredient is the existence of certain wild arcs in $S^3$ whose complements have certain specific fundamental groups, see M. Brown [11] and Roslaniec [34]. We also use several results and techniques from the shape theory of K. Borsuk [8], the theory of retracts, K. Borsuk [7], and the theory of cell-like decompositions, Lacher [27].

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1.2 Notation and terminology. We denote by $B^n$ the closed ball of unit radius in the $n$-dimensional Euclidean space $\mathbb{E}^n$ and we denote by $S^{n-1}$ the boundary sphere of $B^n$. A set $X$ is uncountable if $X$ has the cardinality of the set of the real numbers. Any Cantor set constructed in $\mathbb{E}^1$, $S^1$, or $B^n$ by the Daverman and Edwards construction [16, 17] will be called a DE Cantor set (or a DE embedding of the Cantor set). We refer to Daverman and Edwards by DE whenever convenient. A generalized $n$-manifold $X$ is an ENR (Euclidean neighborhood retract) such that $H_n(X, X - \{x\}; Z) = H_n(\mathbb{E}^n, \mathbb{E}^n - \{0\}; Z)$ for any $x \in X$. One may consult J. W. Cannon [14] for an interesting discussion of generalized $n$-manifolds where many other references may also be found. By a proper subset $A$ of a space $X$ we mean $A \neq X$ and $A \neq \emptyset$. We shall use the terminology ANR, AR, FANR, and strongly movable only for compact metric space (cf. [7, 8]). We refer to Zastrow [7, 8], Cen non [12], Daverman [16], and Lacher [27] for specific results and also for references to the work of many others. A collection of subsets of a metric space $X$ is called a null collection if for $\epsilon > 0$ all the finitely many sets in the collection have diameter $> \epsilon$ (in $X$). All spaces are assumed to be metrizable. Two isomorphic groups are sometimes considered to be equal.

2. Sequences of groups

Throughout this section we shall be concerned with sequences of groups. By $\{G_n\}$ we mean the sequence $G_n, G_{n+1}, \ldots$

(2.1) Definition. A group $G$ has property $(*)$ if there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $G_{n_i + 1} \rightarrow G_{n_i}$ whenever $i \neq j$, where $G_n$ is the free product (cf. [29]) of $n_k$ copies of $G$ for $k = 1, 2, \ldots$. The sequence of groups $\{G_n\}$, as above, will be called a $\ast$-sequence for $G$.

(2.2) The $\ast$-set $\mathcal{S}$. Let $\{G_n\}$ be a $\ast$-sequence for $G$. We observe that every subsequence $\{G_{n_k}\}$ of $\{G_n\}$ is also a $\ast$-sequence for $G$. Two subsequences $\{G_{n_k}\}$ and $\{G_{n'_k}\}$ are defined to be distinct if and only if the sets $\{G_n: 1 \leq n < \infty\}$ and $\{G_n: 1 \leq n < \infty\}$ are distinct. It is easy to see that the set $\mathcal{S}$ consisting of all distinct subsequences of the $\ast$-sequence $\{G_n\}$ has the power of the continuum (an uncountable set). The set $\mathcal{S}$ will be called the $\ast$-set for $G$ corresponding to the $\ast$-sequence $\{G_n\}$.

(2.3) Main example. Roslaniec [34] has shown that the group $G$ having the presentation

$$\prod_{i=0}^{\infty} C_i \cup \bigcup_{i=0}^{\infty} \{C_i, C_i, C_{i+1}, C_{i+1}, C_{i+2}, C_{i+2}\}$$

has a $\ast$-sequence $\{G_n\}$, for a suitable sequence $\{n_i\}$ of positive integers, see [34] for more details. The $\ast$-set for $G$ corresponding to the sequence $\{G_n\}$ will serve as an important example for us. Recall that the group $G$ is the fundamental group of the complement of the celebrated Artin–Fox arc in the 3-sphere $S^3$ [21].

3. Arcs in $S^n$, $n \geq 3$

3.1 Preliminary results on arcs. We shall use the following results concerning arcs in $S^n$.

(3.1.1) THEOREM (M. Brown [11]). For every arc $a \subset S^n$, there is an arc $a^* \subset S^{n+1}$ such that $S^{n+1} - a^*$ has the homotopy type of $S^{n+1} - a^*$.\n
(3.1.2) THEOREM (Roslaniec [34]). There exists a null sequence $\{A_j\}$ of disjoint arcs in $S^n$ and a group $G$ with a $\ast$-sequence $\{G_n\}$ such that for each $j$, $1 \leq j < \infty$, $\pi_j(S^n - A_j) \cong G_j$, where $G_j$ is the free product of $j$-many copies of $G$.

(3.1.3) THEOREM (Brown–Roslaniec). For each $n, n \geq 3$, there exists a null sequence $\{A_j\}$ of arcs in $S^n$ and a group $G$ with a $\ast$-sequence $\{G_n\}$ such that for each $j$, $1 \leq j < \infty, \pi_j(S^n - A_j) \cong G_j$, where $G_j$ is the free product of $j$-many copies of $G$.

(3.1.4) Remark. Theorem 3.1.3 follows from Theorems 3.1.1 and 3.1.2. Roslaniec [34] has given a specific group $G$, see (2.3), and a specific $\ast$-sequence
(3.2) A method of attaching an arc to a given arc. By \( \langle x_1, x_2, \ldots, x_n \rangle \) we mean an arc in \( E^s, n \geq 3 \), such that the arc starts at \( x_1 \), goes through \( x_2, x_3, \ldots, x_{n-1} \) in this order, and ends at \( x_n \). Suppose \( M^s \) is a compact and connected P.L. n-manifold in \( E^s \) with boundary \( \partial M^s \). Suppose \( \langle ab \rangle \) is an arc contained in \( (M^s - \partial M^s) \). Choose a point \( c \) in \( \partial M^s \). It is easy to see that there exists a P.L. arc \( \langle bc \rangle \subset M^s \) such that \( \langle ab \rangle \cap \langle bc \rangle = \{ b \} \) and \( \langle bc \rangle \cap \partial M^s = \{ c \} \). Let \( d \) be a point in \( (E^s - M^s) \) such that \( d \) is not in the component of \( (E^s - M^s) \) whose closure contains \( c \). Choose two balls \( B_1 \) and \( B_2 \) centered at \( d \) such that \( B_1 \subset B_2 \) and \( B_1 \cap \partial M^s = \emptyset \). It is easy to see that there exists a P.L. arc \( \langle cd \rangle \subset (E^s - \partial M^s) \) such that \( \langle cd \rangle \cap \partial M^s = \{ c \} \) and \( \langle cd \rangle \cap \partial B_2 = \{ c \} \). The arc \( \langle ab \rangle \cup \langle bc \rangle \cup \langle cd \rangle \cup \langle dc \rangle \) is obtained by attaching \( \langle dc \rangle \) to \( \langle bc \rangle \) by the P.L. arc \( \langle bd \rangle \). Put \( U = E^s - (B_2 \cup \langle ab \rangle) \) and \( V = \text{Int}(B_1) - \langle cd \rangle \). By the Seifert and Van Kampen Theorem (cf. [29]), we have shown that \( \pi_1(E^s - \langle ab \rangle) \approx \pi_1(E^s - \langle ab \rangle) \approx \pi_1(E^s - \langle de \rangle) \approx \pi_1(E^s - \langle de \rangle) \approx \pi_1(U \cup V) \approx \pi_1(U) \approx \pi_1(V) \) since \( U \cup V \) is simply connected.

4. Preliminary results

(4.1) The DE embeddings of the Cantor set and arcs in \( E^s, n \geq 3 \). A closed \((n-2)\)-manifold \( M \) in \( E^s, n \geq 3 \), has a (topological) tubular neighborhood \( M_8 \) in \( E^s \) if \( M_8 \) is homeomorphic to \( M \times B^n \) under a homeomorphism which carries \( M \) onto \( M \times \{ 0 \} \). For notational convenience, we shall identify \( M_8 \) with \( M \times B^n \) and forget the homeomorphism. We shall also refer to \( M_8 \) as a tube with center \( M \) and normal disk \( B^n \). All manifolds considered in this note will be P.L. and orientable. All the closed manifolds in \( E^s \) will have a (topological) tubular neighborhoods. We begin with some results from [16,17].

(4.1.1) Theorem (DE [16, 17]). Suppose \( M, M_8, M_8, \) and \( E^s \) are as above in (4.1). Then for each \( \epsilon > 0 \), there exists a finite set \( \{ M_8 : 1 \leq i \leq n \} \) of disjoint n-manifolds such that: (a) For each \( i, 1 \leq i \leq n \), \( M_8 \subset M_8 \), the diameter of \( M_8 \) is less than \( \epsilon \), and \( M_8 \) is a tube with a center which is an \((n-2)\)-manifold; (b) a loop \( \gamma \) in \( (E^s - M_8) \) is nullhomotopic in \( (E^s - M) \) if and only if \( \gamma \) is nullhomotopic in \( (E^s - M) \).

Theorem (4.1.1) may be considered as a generalization of the classic Antoine's construction [1]. Indeed, the goal is to produce a wild Cantor set, analogous to the celebrated "Antoine's necklace" [1, 6, 32], by iterating this construction for each \( M_8, 1 \leq i \leq n \).

(4.1.2) Theorem (DE [16, 17]). Suppose \( M \) and \( M_8 \) as in Theorem (4.1.1). Then there exists a Cantor set \( C \subset \text{Int}(M_8) \) such that any loop \( \gamma \) in \( (E^s - M_8) \) is nullhomotopic in \( (E^s - C) \) if and only if \( \gamma \) is nullhomotopic in \( (E^s - M) \).

(4.2) Definitions and terminology. A set \( \{ X_i : 1 \leq i \leq n \} \) consisting of disjoint subsets of a metric space \( X \) will be called a \( \delta \)-chain provided each \( X_i \) has diameter \( < \delta \) and each of the sets \( \{ X_1 \cup X_2, X_2 \cup X_3, \ldots, X_{n-1} \cup X_n \} \) and \( \{ X_1 \cup X_2, X_2 \cup X_3 \} \) has diameter less than \( 2 \delta \). The finite set \( \{ M_8 : 1 \leq i \leq n \} \) given in Theorem (4.1.1) will be called an \( \epsilon \)-chain of n-manifolds substituting for \( M_8 \) at the first stage of DE construction provided (1) this set is an \( \epsilon \)-chain, and (2) each loop in \( M_8 \) is nullhomotopic in \( (E^s - M_8) \) and each loop in \( M_8^{(i+1)} \) is nullhomotopic in \( (E^s - M_8^{(i+1)}) \) where \( 1 \leq i \leq n \) and \( i+1 \) if \( i = n \) (consider \( i \) modulo \( n \)). The construction of an \( \epsilon \)-chain substituting for \( M_8 \) can be done, for instance, as an added feature of DE construction; this follows from "the standard replacement technique" of DE. The concept of an \( \epsilon \)-chain substituting for \( M_8 \) is simply a way of ordering the manifolds and that there is very little geometry between the consecutive two manifolds other than the assertion (2) in the definition. We omit details and we refer the reader to Wright [46; p. 123-124] for a relevant discussion. Furthermore, we require that in the first stage of DE construction we use only an \( \epsilon \)-chain substituting for the manifold \( M_8 \), for a suitable \( \epsilon > 0 \), whenever we use DE construction; while, the manifolds used at any successive stage will be called a chain and which may or may not be an \( \epsilon \)-chain substituting for the manifolds under consideration.

(4.3) The parallel DE Cantor sets. Suppose the first stage of the DE construction is finished, i.e., suppose an \( \epsilon \)-chain of n-manifolds \( \{ M_8 : 1 \leq i \leq n \} \) substituting for \( M_8 \) is given. Since each \( M_8 \) is a tube we identify it with \( N_8 \times B^n \), see (4.1). Now, choose two disjoint tubes \( T_8 = N_8 \times B_8 \) and \( T_8 = N_8 \times B_8 \) satisfying: (1) the tubes miss the center of \( N_8 \times B^n \); (2) there exist \( b \) and \( b' \) inside \( B^n \) such that \( T_8 = \{ b \} \) and \( T_8 = \{ b' \} \) are the respective centers for \( T_8 \) and \( T_8 \). \( T_8 \) and \( T_8 \) are two subdisks of \( B^n \) with respective centers \( b \) and \( b' \). We stipulate that the chain inside each \( M_8 \), required in the second stage of the DE construction, is contained in \( T_8 \). This requirement will apply only to the second stage. Now, iterate the usual DE construction to construct the third stage, ..., \( n \)th stage, ..., and a Cantor set \( C \subset M_8 \). Clearly, \( C \) is the disjoint union of the Cantor sets \( C_{2i} = C \cap M_8^{(i+1)} \). Note that for each \( i, C_{2i} \) is contained in \( T_8 \). Observe that there exists an isometry of \( M_8 \) taking \( T_8 \) onto \( T_8 \) such that the disk \( B_8 \) goes onto \( B_8 \) inside \( B^n \) and the boundary of \( M_8 \) remains fixed throughout the isometry; we shall refer to the isometry of this type as a vertical isotopy. Therefore, the Cantor set \( C \) goes onto a Cantor set \( C' \) where \( C' \) is the disjoint union of the Cantor sets \( C_{2i+1} = C \cap M_8^{(i+1)} \) and \( C_8 \) is the image of \( C_8 \) at the end of the isotopy. We say \( C_{2i} \) and \( C_{2i+1} = C \) are two parallel Cantor sets in \( M_8 \) in \( M_8 \).

(4.4) The dyadic arcs. For each two consecutive elements \( M_8 \) and \( M_8^{(i+1)} \) we
choose a simple closed curve $S$ such that (1) $S$ lies in the complement of $M_{d+1}$, (2) $S$ and the boundary $\Sigma$ of a normal disk of $M_{d+1}$ bound an annulus, and (3) $\Sigma$ meets $M_d$ in a loop. Let $A_{xy} = \{xy\}$ and $A'_{xy} = \{x'y'\}$ where $x$ goes to $x'$ and $y$ goes to $y'$ under the vertical isometry which carries $A_{xy}$ onto $A'_{xy}$. Our notation (ab) means an oriented arc running from $a$ to $b$ and then to $c$. By extending the arc if necessary we assume $J = \{xy\}$. Choose $x$ and $y$ in $S$ such that $S = \{x'y'\} \cup \{x'y'\}$ where $J = \{xy\}$. Let $L_{xy} = \{A_{xy} \cup A'_{xy}\} \cup \{x'y'\}$. The arc $L_{xy}$ will be called a dyadic arc substituting for $M_d$. Observe that $L_{xy} \cup M_d$ contains a loop which links $M_{d+1}$, or $L_{xy} \cup L_{xy'}$ contains a loop which links $M_{d+1}$. It is clear that $L_{xy} = \{1 \leq i < \ell \leq n_1\}$ is an $x$-chain of arcs which will be called the $2x$-chain of dyadic arcs (or dyadic chain) substituting for $M_d$.

5. A family of decompositions

(5.1) A family of manifolds in $E^n$, $n \geq 3$. Let $S^d(\theta)$ denote the family of all the PL $(n-2)$-manifolds in $E^n$ with rational vertices and with tubular neighborhoods, see (4.1). It follows from the results in [37] and from the connected sum construction that for each continuum $A \subset E^n$ of dimension $\geq 2$ and for each open set $U \subset E^n$ with $(U \cap A) = \emptyset$, there exists $M \in \Theta(\theta)$ such that $A$ and $M$ have the same fundamental group, each for $\theta$ small enough. The family $\Theta(\theta)$ contains a simple closed curve which is linked with $M$ (see [37] for more details) and $(\cap U)$ is nonempty. Let $M_1, M_2, M_3, \ldots$, be an enumeration of the family $\Theta(\theta)$ such that each manifold in $\Theta(\theta)$ appears an infinite number of times. It is clear that the above discussions remain valid when $\Theta(\theta)$ is replaced by $S^n$.

(5.2) Some choices. Fix an integer $n > 3$. Choose a group $G$ with a $\ast$-sequence $\{G_\ell\}$ such that for each $\ell$, $1 \leq \ell < \infty$, there exists an arc $A_{\ell} = S^d$. The set $\{G_\ell\} \subset G$ corresponding to $G_\ell$, see (2.1.1). Choose an enumeration $M_1, M_2, M_3, \ldots$, of the family $\Theta(\theta)$ satisfying the assertions of (5.1) and fix this enumeration. The group $G$ and the $\ast$-sequence $\{G_\ell\}$ will remain fixed throughout; however, we do not make fixed choices for the arcs $A_{\ell}$. For each element $\ell \in \theta$, where $\theta$ is the $\ast$-set given above, we shall construct a decomposition space $S^d$ of $S^n$ such that $\lambda \neq \mu$ implies $S^d$ is not homeomorphic to $S^d$ (details will follow).

(5.3) The construction. Suppose $\lambda = \{G_\ell\}$ is given. Choose a tube $T_1$ containing $M_1$ such that $T_1$ is contained inside $M_{d+1}$. Let $T_{i+1}$ be a $\ast$-equivalence of $d$, see (4.1.3). Let $\{L_{xy} = 1 \leq i < \ell \leq n_1\}$ denote the 2-chain of dyadic arcs substituting for $T_1$, see (4.4). Put $H_{11} = \pi_1(S^n - L_{11})$. There are two cases and we consider them separately.

Case I. For each $i$, the group $H_{ii}$ is not isomorphic to any group in the $\ast$-sequence $\{G_\ell\}$.

Case II. There exists a group $H_{ii}$ such that $H_{ii} = G_\ell$ for some $i$.

Suppose Case I is true. We construct an arc $A_{\ell} = S_\ell$ such that $\pi_1(S_\ell - A_{\ell}) = G_\ell$ and $A_{\ell}$ is disjoint from the arcs in $\{L_{11} = 1 \leq i < \ell \leq n_1\}$, and the diameter of $A_{\ell}$ is less than 1. This finishes our construction for Case I.

Suppose Case II is true, i.e., $H_{ii}$ is isomorphic to $G_\ell$, as above. Choose an index $m_1 > n_1$. Now $G_\ell$ equals to the free product of $G_{m_1}$ with $G_\ell$ where $G_{m_1}$ equals to the free product of $(m_1 - n_1)$-many copies of $G$. By (3.2) we modify our arc $L_{11}$ to obtain a new arc $L_{11}$ sufficiently near to $L_{11}$ such that $\pi_1(S^n - L_{11}) = G_\ell$. As a consequence, in either case we may assume without loss of generality that the chain of $\{L_{11} = 1 \leq i < \ell \leq n_1\}$ satisfies the property:

(P) For each $i$, the group $H_{ii} = \pi_1(S^n - L_{11})$ is either isomorphic to some $G_\ell$ or it is not isomorphic to any $G_\ell$.

In addition, we construct an arc $A_{\ell} = S_\ell$ of diameter less than 1 such that $\pi_1(S_\ell - A_{\ell}) = G_\ell$ and $A_{\ell}$ is disjoint from the arcs in $\{L_{11} = 1 \leq i < \ell \leq n_1\}$.

Suppose the construction has been performed inside the tubes $T_{11}, T_{21}, \ldots, T_{i1}$.

We next describe this construction inside $T_{i1}$ where $T_{i1}$ is a tube contained in $N(M_{d+1}) = \{x \in E^n : d(x, M) < 1\}$ for some $m \in M_{d+1}$. Choose a 1-chain $\{M_{d+1} = 1 \leq i < \ell \leq n_1\}$ substituting for $T_{i1}$ and let $\{L_{i1} = 1 \leq i < \ell \leq n_1\}$ denote the 2-chain of dyadic arcs substituting for $M_{d+1}$ satisfying (1) these arcs are disjoint from the finitely many arcs employed in the chains of arcs substituting for $T_{11}, \ldots$, and $T_{i1}$, (2) we require that the chain $\{L_{i1} = 1 \leq i < \ell \leq n_1\}$ satisfies the property (P); furthermore, we construct an arc $A_{\ell} = S_\ell$ of diameter less than 1 disjoint from all the arcs previously employed such that $\pi_1(S_\ell - A_{\ell}) = G_\ell$. The requirement (1) in the previous sentence can be fulfilled by requiring the DE Cantor sets to be disjoint which is a feature of the DE construction and then generalizing the arcs modulo the Cantor sets, see Wright [46] for a related discussion. This finishes our inductive step in the construction. Let $S_\ell$ denote the decomposition space $S^d$ such that the set $S_\ell$ of all the nondegenerate elements of $S_\ell$ is the union of all the chains of arcs, $\{L_{i1} = 1 \leq i < \ell \leq n_1\}$ where $1 \leq i < \ell \leq n_1$, with the set of arcs $\{A_{\ell} = 1 \leq i < \ell \leq n_\ell\}$. Clearly, for each $\phi > 0$ all but finitely many arcs in $S_\ell$ have diameters greater than $\phi$, i.e., the nondegenerate elements of the decomposition $S_\ell$ form a null collection. This finishes our construction of the decomposition space $S^d$. In order to avoid proliferation of symbols we denote the associated decomposition space again by $S^d$.

6. Decompositions of $S^d$, $n \geq 3$

Suppose $n \geq 3$ is arbitrary but fixed for the following discussions. We assume several results from the context of cell-like mappings and we refer the reader to Lacher’s excellent survey article [27] for the discussion of these results and other related matters. We start with the following well-known proposition (cf. [27]).

(6.1) Proposition. For each $\ell$, in $S^d$ the decomposition space $S^d$ is a generalized n-manifold (definition later) such that the projection $\pi_1 : S^d \to S^d$ is a (simple) homotopy equivalence.

A finite dimensional ANR (ENR) $X$ is a generalized n-manifold if $H_\ell(X, X - \{x\}; Z)$ is isomorphic to $H_\ell(E, E - \{0\}; Z)$ for each $x$ in $X$ where $Z$ is
the group of the integers (under addition). A generalized $n$-manifold is locally orientable, i.e., the orientation sheaf (generated by the presheaf $U \rightarrow H_n(\mathcal{O}, X \setminus U; Z)$ where the homology can be taken in the sense of Borel-Moore or Čech) is locally constant, see Cannon [14] for an interesting discussion of these matters where the reference to the work of Bredon and others may also be found.

(6.2) The Groups $\pi_i(S^n_+ - \{x\})$. Suppose $\lambda = \{G_{n, i}\}$ is an (fixed) element of $\mathcal{S}_n$. We observe that the group $\pi_1(S^n_+ - \{x\}) = G_{n, 1}$ or $H_1$, respectively, when $p_{1}^{-1}(x)$ consists of exactly one point, $p_{1}^{-1}(x) = A_{n, 1}$ or $p_{1}^{-1}(x) = E_{n, 1}$. We reindex and arrange the groups $H_1$'s into a sequence which is denoted by $(H_1)^\alpha$. Hence, every nonzero group $\pi_i(S^n_+ - \{x\})$ appears in the sequence $(H_1)^\alpha$, or in $(G_{n, i})$. Let $\lambda = \{(H_1)^\alpha, \{G_{n, i}\}\}$ denote the (ordered) pair of these sequences. The pair $\lambda$ has the following property:

$(P_\alpha)$ Each group $H_1^{\alpha}$ is either isomorphic to some group in the sequence $(G_{n, 1})$, or is not isomorphic to any group in the sequence $(G_{n, 1})$. This follows from the property $(F)$ of (5.2).

Suppose $\lambda = \{(H_1)^\alpha, \{G_{n, i}\}\}$ and $\beta = \{(H_1)^\beta, \{G_{n, j}\}\}$ are given where $\lambda = \{G_{n, i}\}$ and $\mu = \{G_{n, j}\}$ are two distinct elements of the $\mathcal{S}_n$ set $\mathcal{S}_n$, see Section 2. Since $\lambda \neq \beta$, we let without loss of generality $G_{n, a}$ be a group which is not isomorphic to any group in the sequence $(G_{n, i})$. In this setting, we have proved the following Lemma (6.3).

(6.3) Lemma. The distinct pairs $\lambda$ and $\beta$, given above, have the property that the group $G_{n, a}$ does not appear in either of the component sequences $(H_1)^\alpha$ or $(G_{n, i})$ or $\beta$.

(6.4) Lemma. If $\lambda$ and $\mu$ are two distinct elements of $\mathcal{S}_n$ then the decomposition spaces $S^n_+$ and $S^n_+$ are topologically distinct.

Proof. Let $\lambda$, $\mu$, and $\beta$ be given as above satisfying the assertions of Lemma (6.3). We do not lose any generality by making this assumption. This means that the group $G_{n, a}$ is not isomorphic to $\pi_i(S^n_+ - \{x\})$ for any $x$ in $S^n_+$. This finishes the proof of Lemma (6.4).

For each $n \geq 3$ and $\mathcal{S}_n$ we denote by $S^n_+$ the set $S^n_+$ of the decomposition spaces $S^n_+$ containing the decomposition spaces. The following theorem summarizes our results thus far:

(6.5) Theorem. For each $n \geq 3$, the set $\mathcal{S}_n$ consists of uncountably many topologically distinct generalized $n$-manifolds such that each $S^n_+$ has the (simple/finite) homotopy type of $S^n_+$ and this (simple/finite) homotopy equivalence is induced by the projection $p_1: S^n_+ \rightarrow S^n_+$.

Since the nondegenerate elements for our decomposition are a null collection of arcs, the following proposition is a consequence of some results of Meyer [30]:

(6.6) Proposition. For each $n \geq 3$, any space $S^n_+$ in $\mathcal{S}_n$ is an $(n+1)$-manifold. More precisely, $S^n_+ \times S^n_+$ is homeomorphic to $S^n \times S^n$.

(6.7) Remark. It is well-known that every generalized $n$-manifold is an $n$-manifold when $n \geq 2$; moreover, every (compact) generalized $n$-manifold with the homotopy type of $S^n_+$ induced by the homotopy type of $S^n_+$ where $n = 1$ or 2 (cf. [45]). This may be contrasted with Theorem (6.5) and some other results of this note. The following observation may also be useful in the sequel: An open subset of a generalized $n$-manifold is itself a generalized $n$-manifold.

(6.8) The subsets of $S^n_+$ and the "backing-up technique". We fix a decomposition space $S^n_+$ throughout the following discussion. We illustrate the backing-up technique in the following simpler but important setting.

(6.8.1) Setting. Given a closed proper subset $A$ of $S^n_+$ satisfying $(1)$ $A$ has $U_1$, $(2)$ $A$ has dimension $\geq 2$, and $(3)$ $A$ contains a simple closed curve $C$ such that $C = p_1^{-1}(C)$ is a simple closed curve $C = p_1^{-1}(C)$ where $p_1: S^n_+ \rightarrow S^n_+$ is the projection.

(6.8.2) Goal. Our immediate goal is to use the "backing-up argument" to reach a contradiction and thereby prove that $S^n_+$ does not contain any proper closed subset of dimension $\geq 2$ with $U_1$.

It follows from a theorem of Shar [25] that $A$ and $A'$ have the same shape. Since $U_1$ is a shape invariant (cf. [8]), it follows that $A'$ has $U_1$. Let $W_1$ be an open subset of $S^n_+$ such that the complement $(S^n_+ - W_1)$ is an ndb. of $A$. Now apply $U_1$ of $A'$ to find a nest $V_1 = V_1 \supset \ldots$ of open saturated ndbs. of $A'$ in $S^n_+$ such that $(1)$ $V_1$ does not intersect $W_1$ (choose $V_2$ inside $(S^n_+ - W_1)$, $(2)$ $A' \cap V_1 = \emptyset$, and $(3)$ each loop inside $V_1$ is nullhomotopic inside $V_1$. There exists a manifold $M$ belonging to the family $\mathcal{S}_n$ such that $C'$ links $M$ and $M$ intersects $W_1$. This means that there exists a tube $T = M \times D^2$ with center $M_1 = M$ such that a normal disk $S^1 \times D^2$ is contained in $W_1$, see Sections 4 and 5. We put $M_2 = M_1$ so that we may use the notation of Section 4. Let $(M_4; 1 \leq i \leq n)$ and $(L_4; 1 \leq i \leq n)$ denote, respectively, the chain of manifolds and $2k_i$-chain of dyadic arcs subordinating for $M_4$ (where $2k_i - 1$, see Section 4. The simple closed curve $C'$ bounds a PL singular disk $D^2 \rightarrow V_4$ since $C'$ is contained in $V_{4+1}$. Since $C'$ and the center $M_2$ of $M_4$ are linked, it follows from 16 that there exists a punctured disk $p > D^2$ and a map $f': (p, D^2) \rightarrow (M_4, M_3)$ for some $i$, $1 \leq i \leq n$, such that $f'$ is $I$-essential in the sense of 16, and $f'(p, D^2) = V_5$. It follows from 16 that each arc $A_{n, i}$ and $A_{n, i}$ intersects $f'(p, D^2)$; furthermore, it follows that the dyadic arc $L_{2, i}$ is contained in $V_5$ since $V_5$ is saturated. We have set this up so that we may use the notation and terminology of (4.3) and (4.4). In this setting we state and prove the following:

(6.8.3) "The Backing-up Lemma". The hypothesis $L_{2, i}$ contained in $V_{4, 1}$, as above, implies $L_{2, i} \in V_{4, 1}$ is contained in $V_{4, 1}$.

Proof of (6.8.3). Choose a point $a \in [A_{n, i} \cap f'(p, D^2)]$ and a point $d' \in [A_{n, i} \cap f'(p, D^2)]$. Our notation $(xy)\text{etc.}$ denotes an arc or a simple closed curve starting at $x$ and traversing through $y$, $z$ and ending at $w$ in the order these letters appear in $(xy)\text{etc.}$. Let $(xy)\text{etc.}$ be an arc inside $f'(p, D^2)$. We choose two subarcs $(xy)\text{etc.}$ and $(xy)\text{etc.}$ of arcs $A_{n, i}$ and $A_{n, i}$ respectively. Since the loop (or the simple closed 5 = Fundamenta Mathematicae 2005
curve $f(x,y,z,d)$ (see (4.1.5)) is contained in $M_d$, it is nullhomotopic in $(E^n - M_{d,h1})$ by (4.1.3). Therefore, the arcs $f(x,y,z)$ and $f(x,y,z')$ are homotopic inside $(E^n - M_{d,h1})$ with endpoints fixed. Now the loop $f(x,y,z')d$ links $M_{d,h1}$ since $S = f(x,y,z')d$ links $M_{d,h1}$; furthermore, since this loop $f(x,y,z')d$ is contained inside $V_n$, it is nullhomotopic in $V_n$. This means that we may apply our earlier arguments, see the argument following the statement (6.8.2), to deduce that $L_{d,h1}$ is contained in $V_n$. This finishes our proof of (6.8.3).

By iteratively applying (6.8.3), we have that $\bigcup_{i=1}^{n} L_{i} = L_{n}$ is contained inside $V_1$.

Since $W_0$ contains a normal disk, see above, it follows from [16] that this normal disk intersects $\bigcup_{i=1}^{n} L_{i}$, i.e., $W_0$ intersects $V_1$. We reach a contradiction to our assumption $(V_1 \cup W_0) = \emptyset$. This finishes our proof and our Goal (6.8.2) is reached.

We now start with the following setting and our Goal is the same as (6.8.2).

(6.8.4) Setting. Given a closed proper subset $A$ of $S^2$ such that $A$ has dimension $\geq 2$ and $A$ have UV

We recall some facts from [37] concerning linking. It is shown in [37] that there exists a manifold $M$ belonging to the family $\mathcal{G}_{k-2}$ such that each nbd. of $A$ contains a simple closed curve which links $M$. This allows us to apply the back-up argument given above and we conclude that no subset of $S^2$ satisfies (6.8.4). The following theorem can be regarded as a preliminary version of our main theorem which is stated here to summarize our discussions given above and to make the transition to our main result easier:

(6.8.5) Theorem (Preliminary Version). For each integer $n \geq 3$ the family $\mathcal{G}^n$ consisting of uncountably many topologically distinct generalized n-manifolds satisfies the following: Each $S^2$ belongs to $\mathcal{G}^n$ satisfies (1) $S^2 \times S^2$ is homeomorphic to $S^n \times S^n$, (2) $S^2$ has the homotopy type of $S^n$, and (3) $S^2$ does not contain any closed proper subset $A$ of dimension $\geq 2$ such that $A$ has UV

(6.8.6) Remark. We observe that the class of compacta with UV contains all compacta with trivial shape (FAR's), and hence, all the compact metric absolute retracts. This implies that $S^2$, as above in Theorem (6.8.5), does not contain any cell-like set of dimension $\geq 2$ and hence any cell of dimension $\geq 2$. Furthermore, one may replace (3) in Theorem (6.8.5) by the following: (3') If $A$ is any compactum with UV such that $A$ has dimension $\geq 2$, then $A$ cannot be embedded in $S^2$ as a proper subset of $S^2$.

(6.9) A property UV of $n$ and strong movability. We shall extend our results of (6.8.5); for instance, we shall show that $S^2$ does not contain any proper subset $A$ of dimension $\geq 2$ which is an ANR. We begin with some preliminary notions.

(6.9.1) Definition. A continuum $A$ contained in an ANR $X$ has the property UV for small loops (Abbrivate: UV($A$)) if there exists a positive number $\delta$ satisfying: For each nbd. $U$ of $A$ in $X$ there exists a nbd. $V \subset U$ of $A$ such that each $\delta$-loop (i.e., a loop of diameter $<\delta$) in $V$ is nullhomotopic in $U$.

(6.9.2) Proposition. Let $A$ be a continuum in an ANR $X$ such that $A$ is shape dominated by a compact polyhedron $P$. Then $A$ has the property UV($A$) in $X$.

Proof. Since $P$ shape dominates $A$, we let $f: A \to P$ and $g: P \to X$ denote two fundamental sequences in the sense of Borsuk [8] such that $gf: A \to X$ is equivalent to the identity fundamental sequence $f: A \to A$. Since $P$ is an ANR, the shape map $f: A \to P$ is represented by a map which we denote by $f: A \to P$; and furthermore, our definition of $f: A \to P$ and $g: P \to X$ considers $P$ embedded in the ANR $P$ itself rather than taking an embedding of $P$ in some larger AR-space which is usually done in defining fundamental sequences, this does not affect anything. We now choose a nest $V_1 \supset V_2 \supset V_3 \supset \ldots$ of nbd's of $A$ in $X$ such that (1) $A = \bigcap_{i=1}^{\infty} V_i$, and (2) for each $i$ the composite map $gf: V_{i+1} \to V_i$ is homotopic to the inclusion $V_{i+1} \to V_i$, where $f: V_{i+1} \to P$ and $g: P \to X$, respectively, are the maps in the definitions of $f: A \to P$ and $g: P \to X$. Let $\eta$ be a positive number such that each $\eta$-subset of $P$ (i.e., a subset of diameter $<\eta$) is contractible to a point in $P$. Choose a number $\delta > 0$ such that the image of each $\delta$-subset of $V_i$ is a $\eta$-subset of $P$. It is now easy to see that this suffices to prove the result and our proof is finished. We observe that this proof also shows that each $\eta$-subset of $A$ contracts to a point inside each $V_i$, $1 \leq i < \infty$.

(6.9.3) Remark. A continuum $A$ which is shape dominated by a compact polyhedron is referred to as "fan RK" or "strongly movable" (cf. [8]). The class $\mathcal{G}$ of continua each of which is shape dominated by a compact polyhedron is strictly larger than the class of continua whose members are shape equivalent to compact polyhedra (cf. [8]). It follows that UV($A$) is enjoyed by members of $\mathcal{G}$. This suffices for our applications and we do not pursue whether UV($A$) is a shape invariant. Easy examples show that a movable continuum may not have UV($A$); as an example, it is easy to see that the Hawaiian ear ring in $S^2$ (this is an infinite wedge of circles whose diameters converge to zero) is movable but does not have UV($A$).

(6.10) Strongly movable subset of $S^2$. Suppose $A$ is a proper closed and connected subset of $S^2$ such that $A$ is strongly movable and dimension of $A$ is $\geq 2$. The subset $A' = f_1^{-1}(A)$ of $S^2$ has the shape of $A$ and furthermore $A'$ is a proper continuum contained in $S^2$ of dimension $\geq 2$. Since strong movability is a shape invariant (cf. [8]), it follows that $A'$ is strongly movable. Suppose $W_0$ is an open subset of $(S^2 - A')$. By Proposition (6.9.2), we find a nest $V_1 \supset V_2 \supset V_3 \supset \ldots$ of saturated open nbd's of $A$ in $S^2$ and a number $\delta > 0$ such that (1) $A' = \bigcap_{i=1}^{\infty} V_i$, (2) for each $i$ each $\delta$-loop in $V_{i+1}$ is nullhomotopic inside $V_i$, and (3) $V_i$ is contained in $(S^2 - W_0)$. Choose a $\eta$-subset $B$ of $A'$ such that the dimension of $B$ is $\geq 2$. The fact that $B$ exists is elementary (cf. [25]). Our arguments given in [37] can be applied without change to find a manifold $M$ in $\mathcal{G}_{k-2}$ such that $B$-links $M$ in the sense of [37] and $M$ intersects $W_0$. It follows that each $V_i$ contains a simple closed curve $C_i$ of diameter $<\eta$ such that $C_i$ and $M$ are linked. Choose a tube $M_x \supset \eta$.
such that a normal disk of $M_n$ is contained in $W_0$ and the first stage chain \(\{M_n; 1 \leq i \leq n\}\) is a \(\delta\)-chain of $n$-manifolds substituting for $M_n$. The "backing-up technique" applies and we conclude that $S_2^n$ does not contain any strongly movable proper closed subset of dimension $\geq 2$. The following theorem summarizes our results:

(6.10.1) Theorem. For each integer $n \geq 3$ the family $\mathscr{D}^n$ consisting of uncountably many topologically distinct generalized $n$-manifolds satisfies the following: Each $S_1^n$ belonging to $\mathscr{D}^n$ satisfies (1) $S_1^n \times S^1$ is homeomorphic to $S^n \times S^1$, (2) $S_2^n$ has the (simple/finite) homotopy type of $S^n$, and (3) $S_3^n$ does not contain any closed proper subset of dimension $\geq 2$ which is strongly movable (see the remark below concerning (3)).

(6.10.2) Remark. The class $\mathscr{G}$ defined in (6.9.3) is precisely the class of strongly movable continua (cf. [3]). Every ANR-space (AR-space) belongs to $\mathscr{G}$, and hence, $\mathscr{G}$ contains all compact polyhedra; furthermore, $\mathscr{G}$ contains all cell-like sets (compacts of trivial shape), and more generally, $\mathscr{G}$ contains all continua of the shape of a compact polyhedron. Theorem (6.8.5) follows immediately from Theorem (6.10.1). The assertion (3) in Theorem (6.10.1) can be replaced by the following: (7) if $A$ is any strongly movable continuum (compactum) of dimension $\geq 2$ then there is no embedding $\varphi: A \to S_2^n$ such that $\varphi(A)$ is a proper subset of $S_2^n$ (it is obvious that we cannot rule out embeddings of $A$ onto $S_2^n$).

7. Decompositions of $B^n$ and $n$-manifolds, $n \geq 3$

Let $S^{n-1}$ denote the boundary of $B^n$. We consider $S^n$ as the one point compactification of the interior $B^\circ$ of $B^n$. Suppose $n \geq 4$. For each $A$, we consider the decompositions $S_2^n$ and $S_3^{n-1}$ (recall that we frequently identify decomposition and the decomposition space) of $S^n$ and $S^{n-1}$, respectively. Let $B_1^n$ denote the decomposition of $B^n$ which is induced from the decomposition $S_1^n$. We denote by $B_1^n$ the decomposition of $B^n$ obtained from the union of decompositions $B_1^n$ and $S_3^{n-1}$. It is clear that $B_1^n$ is a null collection, and hence, an upper semicontinuous decomposition. The associated decomposition space which we again denote by $B_1^n$ contains an $(n-1)$-dimensional ANR $S_2^{n-1}$. This is not desirable. This can be easily corrected by a method of "attaching chords."

(7.1) A method of attaching chords. Throughout the following discussions we let $n \geq 4$. Let $B^n$, $B^n_1$, and $S^{n-1}$ as above. A chord in $B^n$ is a PL arc $(pq)$ in $B^n$ such that the end points $p$ and $q$ belong to $S^{n-1}$. Choose distinct points $p_1, q_1, p_2, q_2, p_3, q_3, \ldots$ such that the sequence $\{d(p_i, q_i)\}$ of distances converges to zero and choose a sequence $\{k(p_i, q_i)\}$ of disjoint chords such that the sequence of diameters of these chords converges to zero. We shall refer to the sequence $\{k(p_i, q_i)\}$ as a sequence of chords attached to $S^{n-1}$ (in $B^n$). It is clear that one can construct a sequence of chords attached to $S^{n-1}$. For $B^n$ a rather specific sequence of chords is desired which is discussed in several other places, see [5, 38]. We observe that the fundamental group of the complement of a chord in $B^n$ is trivial (remember $n \geq 4$).

(7.2) Modifying the decomposition $B_2^n$. We construct a decomposition $B_2^n$ of $B^n$ such that the set of all the nondegenerate elements of $B_2^n$ is the union of the set of all the nondegenerate elements of $B_1^n$ and a set consisting of sequence of chords attached to $S^{n-1}$. There is no difficulty in choosing these chords so that all the nondegenerate arcs of $B_2^n$ are disjoint; for instance, this can be accomplished by inductively constructing chords along with other arcs in the interior $B^n$ which will yield chords and the induced decomposition of $B^n$ from $S_2^n$ simultaneously. Clearly, the decomposition $B_2^n$ is a null collection consisting of arcs and singletons and hence $B_2^n$ is an upper semicontinuous decomposition of $B^n$.

(7.3) Properties of the decomposition space $B_2^n$. Let $q_1$: $B^n \to B_1^n$ denote the projection. We observe that the image $q_1(S^{n-1})$ is not an ANR. This can be easily seen by observing that the fundamental group of $q_1(S^{n-1})$ is not finitely generated since each of the sets $\{p_1, q_1\}, \{p_2, q_2\}, \ldots$ goes to a point under $q_1$. This is elementary and we omit details. Suppose $A$ is a strongly movable proper subset of $B_2^n$ where dimension of $A$ is $\geq 2$. The case when $q_1^{-1}(A)$ is contained in $S^{n-1}$ or $B^n$ is treated by applying the backing-up technique in $S^{n-1}$ or $B^n$, respectively. The case when $q_1^{-1}(A)$ meets $S^{n-1}$ and $B^n$ needs some consideration. Since the dimension of $A$ is $\geq 2$, it follows that either $A_1 = q_1^{-1}(A) \cap S^{n-1}$ or $A_2 = q_1^{-1}(A) \cap B^n$ has dimension $\geq 2$ [25]. Suppose $A_1$ has dimension $\geq 2$. Apply the backing-up technique inside $S^{n-1}$ as follows. Choose a manifold $M \subset S^{n-1}$ such that $A_1 \subset M$. Choose a nest $V_1 \subset V_2 \subset \ldots$ of nbd of $q_1^{-1}(A)$ inside $B^n$ satisfying the properties given in (6.10) and $W_0 \subset S^{n-1}$ where $W_0 \subset V_1 \subset V_2 \subset M \neq \emptyset$ and $W_0$ is an open subset of $S^{n-1}$. The backing-up technique applies and we are done. The case when $A_2$ has dimension $\geq 2$ is handled exactly the same way. We shall now state the following:

(7.3.1) Theorem. For each integer $n \geq 3$, there is an uncountable family $\mathcal{P}$ of topologically distinct $n$-dimensional AR's such that each $B_2^n$ in $\mathcal{P}$ satisfies: (1) $B_2^n \times I$ is not homeomorphic to $B^n \times I$, (2) $B_2^n \times I$ is homeomorphic to $B^n \times I^2$, and (3) $B_2^n$ does not contain any strongly movable proper subset of dimension $\geq 2$.

We have limited our discussions to $n \geq 4$. The case $n = 3$ follows from discussions in [38]. The techniques of [41] are different from this note and the families of AR's are also different. Our assertions (1) and (2) in Theorem (7.3.1) follows from [19]. We now state some easy extensions of our results.

(7.4) Decompositions of manifolds. We may extend our results to obtain the following:

(7.4.1) Theorem. For each topological $n$-manifold $M^n$ with $n \geq 3$, there exists an uncountable family $\mathcal{M}$ of topologically distinct $n$-dimensional ANR's such that each $M^n$ in $\mathcal{M}$ does not contain any strongly movable proper subset of dimension $\geq 2$ and $M^n$ has the (proper/simple/finite) homotopy type of $M^n$. Furthermore, if $M^n$ is a manifold without boundary, then each $M^n$ in $\mathcal{M}$ is a generalized $n$-manifold satisfying $M^n \times B^1 \cong M^n \times B^1 \cong M^n \times S^1 \cong M^n \times S^1$. \(\square\)
A proof can be easily constructed based on our arguments for $S^n$ and $B^r$. We shall indicate this in the following discussion. Choose a starfinite covering of the manifold consisting of coordinate neighborhoods and apply the construction inductively in these coordinate neighborhoods. The details are elementary but lengthy and we omit them. These details can be carried out in several different ways for a manifold with a triangulation or one may use a handle-body decomposition (cf. [20]).

(7.5) Rigid generalized $n$-manifolds. A topological space $X$ is rigid if the only homeomorphism of $X$ onto itself is the identity homeomorphism. The following result follows by combining arguments of this note with the technique of [34] (see also [41]).

(7.5.1) Theorem. The family $\mathcal{F}$, $\mathcal{F} = S^n$, $D^n$, or $M^n$, can be constructed such that each element of $\mathcal{F}$ is rigid in addition to the properties stated in Theorems (6.10.1), (7.3.1), or (7.4.1), respectively.

This shows that the group of homeomorphisms for a generalized $n$-manifold can be very small. This may be contrasted with some well-known results concerning the group of homeomorphisms for a manifold.

(7.6) UV$(d)$ revisited. Suppose $G$ is a cell-like upper semicontinuous decomposition of a metric space $X$ and suppose $A$ is a closed subset of the decomposition space $X/G$ with $UV^1(d)$ in $X/G$. The usual lifting arguments (cf. [27]) show that the set $p^{-1}(A)$ has $UV^1(d)$ in $X$, where $p: X \rightarrow X/G$ is the projection map. We observe that we have actually proved the following more general result:

(7.6.1) Theorem. The family $\mathcal{F}$, $\mathcal{F} = S^n$, $D^n$, or $M^n$, satisfies, in addition to the properties stated in Theorems (6.10.1), (7.3.1), or (7.4.1), that each $X$ belonging to $\mathcal{F}$ does not contain any proper (closed) subset of dimension $\geq 2$ with $UV^1(d)$ in $X$. The same applies to the family given in Theorem (7.5.1).

(7.7) A question of John Walsh (communicated to us by R. J. Daverman). Does there exist a cell-like map from a generalized $n$-manifold onto an $n$-manifold?

The answer to this question, in general, is negative. Consider any cell-like map $f: X \rightarrow Y$ from a generalized $n$-manifold $X$ belonging to the family $\mathcal{F}$, $\mathcal{F} = S^n$ or $M^n$, onto an $n$-manifold $Y$. Let $A$ be a strongly movable proper subset $N^n$ such that $dim(A) \geq 2$. It is easy to see that $p^{-1}(A)$ is strongly movable (in fact, $p^{-1}(A)$ is shape equivalent to $A$ [33]) and $p^{-1}(A)$ is a proper subset of $X$ of dimension $\geq 2$. This is a contradiction. We have actually proved the following:

(7.7.1) Theorem (Stability under cell-like mappings). Given $X$ as above. If $f: X \rightarrow Y$ is a surjective cell-like mapping, then $Y$ does not contain any proper subset $A$ of dimension $\geq 2$ with $UV^1(d)$ in $Y$.

(7.7) Movable subsets. The following theorem shows that our decomposition spaces contain enough movable subsets:

(7.8.1) Theorem. If $G$ is an upper semicontinuous decomposition of an $n$-manifold $M^n$ with $n \geq 2$ such that the nondegenerate elements of $G$ form a null collection of arcs, then the decomposition space $M^n/G$ contains a movable proper subset of dimension $\geq 0 \leq k \leq n$.

Proof. Choose a closed $k$-cell $D$ inside $M^n$. Let $D = p^{-1}[p(D)]$ denote the saturation of $D$ where $p: M^n \rightarrow M^n/G$ denotes the projection. It is easy to see that $D$ is a continuum of dimension $k$. It remains to show that $D$ is movable. This can be easily seen by shrinking $D = \bar{D}$ and observing that $D = \bar{D}$ is a 1-dimensional planar continuum (cf. [38]). This suffices to prove the theorem.

(7.8.2) Corollary. Suppose $X$ belongs to the family $\mathcal{F}$, $\mathcal{F} = S^n$, $D^n$, or $M^n$, and suppose $k$ is an integer satisfying $0 \leq k \leq n$. Then, $X$ contains a movable proper subset of dimension $k$.

(7.8.3) Concluding remark. Each compactum $A$ inside $X$, $X$ as in Corollary (7.8.2), can be approximated by a locally connected compactum in the following rather strong sense: For each $\varepsilon > 0$ there exists a map $\phi: A \rightarrow A_\varepsilon$ onto a locally connected subset $A_\varepsilon$ of $X$ such that $d(\phi, \phi(\varepsilon)) < \varepsilon$ for each $\varepsilon$ in $A$ [41]. Observe that $A_\varepsilon$ is pointed 1-movable. Theorem (7.8.1) is also a consequence of our more general results which are too technical to discuss here.

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THE PENNSYLVANIA STATE UNIVERSITY
Altoona, Pennsylvania 16602

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