$H^*(G/B, \underline{L})$ for G of semi-simple rank 2

by

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Abstract. Let G be a semi-simple linear algebraic group of rank 2, B a Borel subgroup, and $\underline{L}$ a line bundle on the flag variety G/B. The structure as G-modules of $H^*(G/P, \pi_*(\underline{L}))$ and $H^*(P/B, \underline{L}_{P/B})$ are obtained by explicit calculations, where P is a maximal parabolic subgroup and π: $G/B \to G/P$ the canonical map. These are used to study the G-module structure of $H^1(G/B, \underline{L})$.

The purpose of this paper is to calculate the structure as G-modules of the submodules of the cohomology space H^1 of line bundles on the generalized flag variety G/B in char. p, when G is a semi-simple algebraic group of semi-simple rank 2.

In 1958 Bott showed that if the ground field was $\mathbf{C}$, then $H_q(G/B, \underline{L})$ vanished for all values of q with one possible exception. The non-vanishing cohomology space is irreducible as a G-module and is isomorphic to $H^0(\underline{L}')$ for a prescribed $\underline{L}'$ ([6]). Demazure proved this for any algebraically closed field of char. 0 ([7]). Mumford first showed that the vanishing theorem was false in char. 2; it was subsequently shown to be false for any positive characteristic ([8]). It is quite easy to see that the representation-theoretic part of Bott's theorem fails in char. p also. (Bott's theorem holds for G of arbitrary semi-simple rank.)

Some vanishing theorems in char. p are known ([1], [2], [3], [8], [9], [11]) but no result valid for all G and $\underline{L}$ is yet available. Representation-theoretic results are even scarcer and are essentially complete only for G of type A_1.

The following notation will be used throughout the paper. k will denote an algebraically closed field of char. $p>0$ unless otherwise specified. G will be a semi-simple algebraic group defined over k. Unless otherwise stated G will be assumed to be of semi-simple rank 2. B will be a Borel subgroup of G; P will be some parabolic subgroup containing B. $\underline{L}$ will denote a line bundle on G/B or G/P. T will denote a maximal torus contained in B; weights will be taken with respect to T. W will denote the Weyl group of G and w will be an element of W. α_i $(i = 1, 2)$ are the simple roots of G. There are many references to these notions (e.g. [3], [4], [5]).

The first step is to describe the various flag varieties explicitly. As noted in [3], G may be replaced by any specific semi-simple algebraic group of the same type for the purposes of these calculations.

Case I. G is of type A_2 (which is the same as type D_2). In this case G may be taken to be SL(3). SL(3)/B is well-known to be the space of flags in A^3: $\{(V_1, V_2) | V_1 \subseteq V_2 \subseteq A^3, \dim V_i = i, V_i$ being a vector subspace of A^3 considered as a k-vector space$\}$ ([5]).

If P_1 and P_2 denote the parabolic subgroups containing B then the canonical maps $SL/B \to SL/P_i$ are simply the maps $(V_1, V_2) \to V_i$. $SL/P_i \cong \boldsymbol{P}^2$ for $i = 1, 2$.

SL(3)/B, like all other flag varieties, has a system of coordinates induced from the Grassmann coordinates ([5], [10]).

Case II. G is of type C_2 (which is the same as type B_2). In this case G may be taken to be Sp(4). It is easily seen that Sp(4)/B is the following flag variety: $\{(V_1, V_2, V_3) | V_1 \subseteq V_2 \subseteq V_3 \subseteq A^4$, V_i as above, V_3 being orthogonal to V_1 with respect to the alternating bilinear form left invariant by Sp(4)$\}$. Since V_3 is completely determined by V_1, it may be omitted.

The identification of this variety with Sp(4)/B proceeds by noting both are of dimension 4. (The dimension of Sp(4)/B may be calculated from the Weyl group; it is the maximum length of any element of W ([3]).) Since the flag variety is clearly projective, the stabilizer of any points is connected, and since Sp(4) acts transitively on it, the result follows from [5].

If P_1 and P_2 are as above, the canonical maps remain the same. $\mathrm{Sp}(4)/P_1 \cong \boldsymbol{P}^3$. To identify Sp(4)/$P_2$, note that V_2 can be a term in some flag iff it is self-orthogonal under the bilinear form. Hence Sp(4)/P_2 is a subvariety of Grass (2,4). If the bilinear form is taken to be $X_1Y_3 + X_2Y_4 - X_3Y_1 - X_4Y_2$, the self-orthogonality condition is $p_{13} + p_{14} = 0$ in Grassmann coordinates.

Case III. G is of type G_2. In this case G may be replaced by the automorphisms of the Cayley numbers (or octonoions) over k denoted by Aut($\mathbf{0}$). (Throughout this paper, whenever G is of type G_2 the additional hypothesis that $\mathrm{char}(k) \neq 2$ will be imposed.) From [3], G/P_1 is a quadric of codimension 1 in P^6.

Consider G/P_2. From [13], G can be imbedded in SO(7), hence in SL(7). P_2 must contained in some maximal parabolic subgroup P_{SL} of SL(7). Hence $G/P_2 \subseteq \mathrm{SL}(7)/P_{\mathrm{SL}}$, which is a Grassmannian.

Returning to the general case, the maps of the type $G/B \to G/P$ will be the principal tool used. The structure of $H^*(G/P, \underline{L})$ as a G-module and $H^*(P/B, \underline{L})$ as a P-module must first be calculated. (The fibres of $G/B \to G/P$ are non-canonically isomorphic to P/B.) Since $P/B \cong \boldsymbol{P}^1$ for the parabolic subgroups of interest here, the latter structure is already known, but since it can be readily deduced from the general theory given here I will do so.

Assume X is a projective G-homogeneous space (or P-homogeneous space) such that i) $\mathrm{Pic}\, X \cong \boldsymbol{Z}$ and ii) X can be imbedded G-equivariantly in a Grassmannian (with some appropriate G-structure) such that X is a complete intersection considered as a subvariety of the Grassmannian. Suppose $\underline{L}$ is a line bundle on X. The degree of $\underline{L}$ is the integer associated to its isomorphism class.

Lemma 1. *If the degree of $\underline{L}$ is non-negative, then $H^q(X, \underline{L}) = (0)$ for $q > 0$.*

$H^0(X, \underline{L})$ is generated by the residues of monomials in appropriate Grassmann coordinates.

Proof. The first statement is really a special form of Kempf's theorem ([3], [11]). Since X is a projective G (or P)-homogeneous space, it is of the form G/P_a (or P/P_a) where P_a is a parabolic subgroup ([5]). Let $f\colon G/B \to G/P_a$ be the canonical map (or $P/B \to P/P_a$). By Kempf's theorem $H^q(G/B, f^*(\underline{L})) = (0)$ for $q > 0$. Since $f_*(f^*(\underline{L})) = \underline{L}$ and $R^q f_*(f^*(\underline{L})) = (0)$ for $q > 0$ (since $f^*(\underline{L})$ has degree 0 on the fibers of f), $H^q(X, \underline{L}) = H^q(G/B, f^*(\underline{L})) = (0)$ by the Leray Spectral Sequence, for $q > 0$.

To show the second statement describe X by a sequence $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_m = X$, where X_0 is a Grassmannian and X_i is a locally principal Cartier divisor on X_{i-1}. It may be assumed that $m < \dim X_0$, since the lemma is obvious for a finite set of points. For any line bundle $\underline{M}$ on X_0, $H^q(X_0, \underline{M}) = (0)$ for $0 < q < \dim X_0$ (this follows from the first part of the lemma and Serre duality). By taking the long exact sequence in cohomology of

$$0 \to \underline{L}(-X_1) \to \underline{L} \to \underline{L}_{X_1} \to 0 \tag{1}$$

it follows that $H^q(X_1, \underline{M}) = (0)$ for $0 < q < \dim X_1$ and that $H^0(X_0, \underline{L}) \to H^0(X_1, \underline{L})$ is surjective. By induction $H^0(X_0, \underline{L}) \to H^0(X, \underline{L})$ is surjective. Since $H^0(X_0, \underline{L})$ is generated by monomials ([15]) the lemma follows immediately.

Lemma 2. *Every G-submodule of $H^0(X, \underline{L})$ is generated by the residues of monomials.*

Proof. This lemma is well known to be true if $\mathrm{char}(k) = 0$ as well. Recall the argument: $H^0(X, L)$ is irreducible. For $H^0(X, \underline{L}) \cong H^0(G/B, f^*(\underline{L}))$ as in the proof of Lemma 1 and the right-hand side is irreducible by a theorem of Weyl ([14]).

Now assume that $\mathrm{char}(k) = p$. If X_0 is as in the proof of Lemma 1, then $H^0(X_0, \underline{L})$ has a monomial highest weight vector v which is in every submodule (v is the highest weight vector of the unique irreducible submodule).

Define $kx_{\boldsymbol{Z}/p}\boldsymbol{Z}/p^e$ to be the universal object making the following diagram commute:

$$\begin{array}{ccc} k & \longleftarrow & kx_{\boldsymbol{Z}/p}\boldsymbol{Z}/p^e \\ {\scriptstyle a \to a\cdot 1_k}\uparrow & & \uparrow \\ \boldsymbol{Z}/p & \xleftarrow{\text{quotient}} & \boldsymbol{Z}/p^e \end{array}$$

Define $kx_{\boldsymbol{Z}/p}\boldsymbol{Z}$ analogously.

Extend G to $kx_{\boldsymbol{Z}/p}\boldsymbol{Z}/p^e$ by base extension (resp. to $kx_{\boldsymbol{Z}/p}\boldsymbol{Z}$) and denote it by $G(\boldsymbol{Z}/p^e)$ or $G(\boldsymbol{Z})$. v may be pulled back to a highest weight vector of $G(\boldsymbol{Z}/p^e)$ or $G(\boldsymbol{Z})$ which is still a residue of a monomial. This vector will also be written as v. The translates of v are residues of polynomials in Grassmann coordinates of the same rank with coefficients involving multinomial coefficients and polynomials in the entries of matrices representing G. These translates over $G(\boldsymbol{Z}/p^e)$ may be obtained by reducing the coefficients of the translates over $G(Z)$ modulo p^e. Call the result y^e.

Since k is algebraically closed y^e is generated by monomials. Since v must be in every submodule and since the preceding paragraph shows they depend only on the power of p dividing the coefficients, all submodules are of the form y^e (and hence form a totally ordered set). This proves Lemma 2.

Now assume m is a positive integer. Let $\Delta(p^m)$ denote any submodule of $H^0(X, \underline{L})$ which is a Frobenius mth power of a submodule of $H^0(X, \underline{L}')$, $0 \leqslant \deg \underline{L}' < p$. Let $\Delta_1(1), \ldots, \Delta_b(p-1)$ be the various submodules of $H^0(X, \underline{L})$ with degree $\underline{L} < p$. Let $\Delta(p^{m_1}) \times \ldots \times \Delta(p^{m_s})$ denote the Cartesian product. ($\Delta(p^0)$ will mean any of $\Delta_1(1), \ldots, \Delta_b(p-1)$.) Since there is a map $\Pi_1^s H^0((X, \underline{L}_i)) \to H^0(X, \otimes_1^s \underline{L}_i)$, $\Delta(p^{m_1}) \times \ldots \times \Delta(p^{m_s})$ may be identified with a submodule of $H^0(X, \otimes_1^s \underline{L}_i)$ for which the same notation will be used.

Lemma 3. i) *$\pi_i \Delta_i(p^{m_i})$ is a G-module.*

ii) *If $l =$ degree $\underline{L}$, then all submodules of $H^0(X, \underline{L})$ are of the form $\pi_i \Delta_i(p^{m_i})$ where $\sum \deg \Delta_i(p^{m_i}) = l$.*

Proof. i) is clearly true, since G acts linearly. The only non-trivial point in ii) is to show that every submodule is of the given form.

As shown in the proof of Lemma 2, any submodule is described completely by its degree l and by a reduction modulo p^e for some $e > 0$. It will be convenient to allow e to assume the value ∞ with the convention that Z/p^∞ is Z (not the p-adic integers). The various submodules of the form $\Delta(p^m)$ can be partially ordered in the following way: let e be the smallest integer (or ∞) such that reduction modulo p^e of $H^0(X(Z), L)$ (for appropriate $\underline{L}$) gives $\Delta(p^m)$. $X(Z)$ is the $G(Z)$-homogeneous space obtained by base extension as in the proof of Lemma 2. Use the ordering on the various e's.

For any submodule S, there is a minimal module of the type $\Delta(p^m)$ such that $S \subseteq \Delta(p^m) \times H^0(X, \underline{L}')$ for some $\underline{L}'$. This follows immediately upon noticing that $S \subseteq H^0(X, \underline{O}(1)) \times H^0(X, \underline{L}(-1))$ and that there are only finitely many $\Delta(p^m)$'s of degree $\leqslant l$.

The lemma follows from the proposition below by induction on l:

Proposition. *If $\Delta(p^m)$ is a minimal module for S as above, then there exists a submodule T of $H^0(X, \underline{L}')$ such that $S \cong \Delta(p^m) \times T$.*

Proof. Let e_1 be the largest integer (or ∞) such that $\Delta(p^m)$ is obtained by reduction mod p^{e_1}; let e_2 be the smallest such integer for S. It is clear that $e_2 \leqslant e_1$.

Note that if v is as in the proof of Lemma 2 and if $\sum z_i v_i$ is the translate of v under a generic element of G (where $z_i \in Z$ and v_i is a monomial with coefficients in the entries of a representative matrix of $g \in G$), then the monomials present in a submodule of $H^0(S, \underline{L})$ corresponding to reduction mod p^e are the monomials not vanishing in $(\sum z_i v_i)^l$ (after any reductions using the relations among the v_i's). Since

$$(\sum z_i v_i)^l = (\sum z_i v_i)^{p^m} (\sum z_i v_i)^{l-p^m}$$

reduction of the right-hand side mod p^{e_2} must reduce $(\sum z_i v_i)^{p^m}$ to an expression giving the monomials of $\Delta(p^m)$ precisely, by the minimal property of $\Delta(p^m)$. T is then obtained from the reduction of the second factor. This proves both the proposition and Lemma 3.

Remark. The product representation in Lemma 3 is by no means unique.

To apply Lemma 3 it is necessary to know the structure of the various Δ submodules. For $m \geqslant 1$, $\Delta(p^m)$ is the appropriate Frobenius power of a submodule of degree $< p$. Hence it suffices to calculate the latter.

Case I. $G = \mathrm{SL}(3)$. It suffices to calculate the expression $(\sum z_i v_i)^l$ as in the preceding proof. Since $\mathrm{SL}(3)/P_i \cong \boldsymbol{P}^2$, X is $\boldsymbol{P}^2$ and $H^0(X, \underline{O}(l))$ is the homogeneous elements of degree l in the symmetric algebra in three variables over k. If $i = 1$, these variables are p_1, p_2, p_3 (Grassmann coordinates). The case of $i = 2$, which is essentially the same as the following argument, will be left to the reader.

From [10] the transformation formula

$$P_I \to \Sigma_J \Delta_{I,J} P_J \tag{2}$$

is obtained, where I and J are multi-indices of the same rank and $\Delta_{I,J}$ is the function from $\mathrm{SL}(3)$ to k given by taking the cofactor of the submatrix consisting of the rows designated by I and the columns designated by J. It is an elementary fact that the $\Delta_{I,J}$'s are non-trivial functions.

The vector v may be taken to be p_1 (this is the highest weight if B is taken to be the upper triangular matrices). Hence $\sum z_i v_i = \Delta_{1,1} p_1 + \Delta_{1,2} p_2 + \Delta_{1,3} p_3$. Since the Δ's are algebraically independent, $(\sum z_i v_i)^l$ has no terms vanishing in char. p. Hence the module $H^0(X, \underline{O}(l))$ is irreducible. Note this means that only Δ's whose orders are of the form p^m $(m \geqslant 0)$ need be used in Lemma 3.

Case Ia. $G = \mathrm{SL}(2)$. This case occurs when dealing with the fibers of the maps $G/B \to G/P$. Since P will be of semi-simple rank 1 in the applications, it will be of type A_1, so may be assumed to be $\mathrm{SL}(2)$.

The argument is the same as case I (with two variables) and leads to the same conclusion.

Case II. $G = \mathrm{Sp}(4)$. As previously noted $\mathrm{Sp}(4)/P_1 \cong \boldsymbol{P}^3$. Since the proper Δ's remain non-trivial and algebraically independent on $\mathrm{Sp}(4)$ this is essentially the same argument as Case I: $H^0(\boldsymbol{P}^3, \underline{O}(l))$ is $\mathrm{Sp}(4)$-irreducible.

$\mathrm{Sp}(4)/P_2$ is more interesting. It is isomorphic to the subvariety $p_{13} + p_{24} = 0$ of Grass (2, 4). The relations defining $\mathrm{Sp}(4)$ in the 4×4 matrices can be written in terms of the Δ's:

$$\begin{aligned} \Delta_{12,13} + \Delta_{12,24} &= 0, & \Delta_{23,13} + \Delta_{23,24} &= 0, \\ \Delta_{13,13} + \Delta_{13,24} &= 1, & \Delta_{24,13} + \Delta_{24,24} &= 1, \\ \Delta_{14,13} + \Delta_{14,24} &= 0, & \Delta_{34,13} + \Delta_{34,24} &= 0. \end{aligned} \tag{3}$$

The highest weight vector v may be chosen to be p_{34}; applying (2) and (3) yields:

$$p_{34} \to \Delta_{34,12} p_{12} + 2\Delta_{34,13} p_{13} + \Delta_{34,14} p_{14} + \Delta_{34,23} p_{23} + \Delta_{34,34} p_{34}. \tag{4}$$

Since the p's and Δ's appearing in (4) are algebraically independent (the only relation on the p's is $p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23}=0$), $H^0(\mathrm{Sp}(4)/P_2, \underline{O}(l))$ is irreducible if $\mathrm{char}(k)>2$. If $\mathrm{char}(k)=2$, $H^0(\mathrm{Sp}(4)/P_2, \underline{O}(1))$ has a codimension 1 submodule and only Frobenius powers of these two modules need be used in Lemma 3.

Case III. In this case direct calculation becomes much more difficult and the discussion will be limited to quoting some results of Jantzen ([15], [16]). I am indebted to H. H. Andersen for bringing these results to my attention. Only G/P_1 will be considered as Lemma 1 cannot be applied to G/P_2 anyways. If $p\geq 7$, $H^1(G/P_1, \underline{O}(l))$ is irreducible unless $2l+5>p>l+4$, in which case it has a proper irreducible submodule.

First note that $H^{\dim G/B-1}(\underline{L}^{-1}\otimes\Omega_d)$ can be studied by Serre duality, once $H^1(\underline{L})$ is analyzed (Ω_d is the sheaf of differentials. By a theorem of G. Kempf [11] it easily follows that $H^1(\underline{L})\neq(0)$ only if exactly one degree of $\underline{L}$ is negative. (The converse is false, even in char. 0.) Let f be the canonical map $G/B\to G/P$, with P being the maximal parabolic subgroup such that the restriction of $\underline{L}$ to any fiber of f has negative degree. Let Ω be the relative sheaf of differentials of f. By Kempf's theorem $R^0f_*(\underline{L})=(0)$, so $H^1(\underline{L})=H^0(R^1f_*(\underline{L}))$ by the Leray Spectral Sequence. By Serre duality

$$R^1f_*(\underline{L})\otimes f_*(\underline{L}^{-1}\otimes\Omega)\to R^1f_*(\Omega)\cong \underline{O}_{G/P}$$

is a perfect $O_{G/P}$-pairing. Hence $H^1(\underline{L})\cong \mathrm{Hom}_{\underline{O}_{G/P}}(f_*(\underline{L}^{-1}\otimes\Omega), O_{G/P})$ as a G-module.

Assume $i<0, j\geq 0$. The proofs below work for $i\geq 0, j<0$ with only the obvious modifications. Assume G is of type A_2 or C_2 only, as these hypotheses conflict with the previous conditions imposed with the case of type G_2.

$H^1(\underline{L})\cong \mathrm{Hom}_{\underline{O}_{G/P}}(f_*(\underline{L}^{-1}\otimes\Omega), \underline{O}_{G/P})$ can be written as a set of polynomials in the generators of $H^0(G/P_2, \underline{O}(j))$ and $H^s(G/P_1, \underline{O}(i))$, where $s=\dim G/P_1$. This follows essentially because $H^s(G/P_1, \underline{O}(i))$ is $\mathrm{Hom}(H^0(\underline{O}(e-i)), K)$, where e is the degree of the sheaf of differentials on G/P_1.

Given submodules $A\subseteq H^0(G/P_2, \underline{O}(j))$ and $C\subseteq H^s(G/P_1, \underline{O}(i))$, the associated module $H(A, C)$ is defined as $(A\times C)\cap H^1(\underline{L})$, $H^1(\underline{L})$ being understood as in the previous paragraph.

THEOREM 1. *Assume G is of type A_2 or C_2 and that $\underline{L}$ is a line bundle on G/B of degrees i,j with $i<0$, $j\geq 0$. The submodules of $H^1(\underline{L})$ are all of the form*

$$+_l H(A_l, C_l)$$

where i) $A_{l(\max)}\subset A_{l(\max)-1}\subset \ldots \subset A_1$,
$C_{l(\max)}\supset C_{l(\max)-1}\supset \ldots \supset C_1$,

ii) *A_l is a submodule of $H^0(G/P_2, \underline{O}(j))$,
C_l is a submodule of $H^s(G/P_1, \underline{O}(i))$.*

Proof. Let h be a submodule of $H^1(\underline{L})$. Let A_1 be the largest submodule of $H^0(G/P_2, \underline{O}(j))$ generated by products of the Grassmann coordinates of degree j on G/P_2 appearing in terms of elements of h. Let C_1 be the largest submodule of $H^s(G/P_1, \underline{O}(i))$ so that $H(A_1, C_1)\subseteq h$.

Proceeding inductively, assume $A_1, \ldots, A_r$, $C_1, \ldots, C_r$ are defined. Let A_{r+1} be the largest submodule of $H^0(G/P_2, \underline{O}(j))$ so that: a) $A_{r+1}\subset A_r$, b) there exists a maximal C_{r+1}, $C_r\subsetneqq C_{r+1}$, with $H(A_{r+1}, C_{r+1})\subseteq h$. If A_{r+1} does not exist, $r=l(\max)$.

It is only necessary to show $h\subseteq +H(A_l, C_l)$. Let F be any fiber of f. There is a restriction r_F: $H^1(\underline{L})\to H^1(\underline{L}_F)$. r_F takes non-zero G-submodules to non-zero P_F-submodules (P_F = stabilizer of F).

Let $c\in h$. There exists a dense open set of fibers of f for which $r_F(c)$ generates some submodule $C'\subseteq H^s(G/P_1, \underline{O}(i))$. Take C' to be as large as possible.

There are special fibers of f defined by the vanishing of all but one Grassmann coordinate, say $p_a\neq 0$, $p_b=\ldots=p_u=0$. Then any fibre of f can be defined by equations $p_b^g=\ldots=p_u^g=0$ for some $g\in G$ (this is not canonical as the choice of g is only defined up to the stabilizer of the fiber).

c must contain an expression $(p_a^g)^j b_F$, where b_F is a generator of C which lies in $H^1(F, \underline{O}(i))$; in fact $c\equiv(p_a^g)^j b_F \bmod I_F$, where I_F is the ideal of F in G/B. Hence it follows that if A' is the P_F-submodule generated by $(p_a^g)^j$, then $A'\times\{b_F\}$ is in $h \bmod I_F$. Hence $c\in H(A', C')\subseteq h$, because all the monomials in $A'\times C'$ which vanish on the ideal of the fiber (which is generic) appear. By construction $H(A', C')\subseteq H(A_r, C_r)$ for some r, so $c\in H(A_r, C_r)$. This proves Theorem 1.

As an application of Theorem 1, a method for computing the number of composition factors will be given. In order to do this, more information on $H(A, C)$ is needed. (Throughout the rest of the paper, G is of types A_2 or C_2 only.)

A, being a submodule of $H^0(G/P_2, \underline{O}(j))$, must contain a unique irreducible G-submodule with a highest weight χ_1. Let C' be the annihilator of C in $H^0(G/P_1, \underline{O}(e-i))$. It follows from the proof of Lemma 2 that C' has a unique irreducible quotient G-module, with highest weight χ_2.

Since the Weyl group of P_2 is a subgroup of W, the element of greatest length (one!) in the Weyl group of P_2, call it w_P^2, is an element of W. Similarly there is $w_P^1\in W$. Let r be the sheaf of differentials on G/B relative to f: $G/B\to G/P_2$

LEMMA 4. $H(A, C)\neq 0$ *iff* $X_1-w_P^2(X_2)$ *is dominant.*

Proof. This follows from the proof of Lemma 3 in [9]. The choice of G as $\mathrm{SL}(n)$ in that proof is irrelevant for present purposes. (The essential point of the cited proof is that if $H(A, C)\neq(0)$, it must contain an irreducible G-submodule with $X_1-w_P^2(X_2)$ as highest weight.) Restatements of this lemma may be found in [1].

Assume $A\subsetneqq A_1$, $C\subsetneqq C_1$, with A_1/A, C_1/C irreducible.

LEMMA 5. a) $H(A, C)\subsetneqq H(A, C_1)$ *if the following condition holds: let A^v be* *the annihilator of A in $[H^0(\underline{O}(j))]^v$ (the Serre dual); similarly define C^v, C_1^v. The condition is that $\chi_1(A)-w_P^2(\chi_2(C_1))$ and $\chi_1(C^v)-w_P^1(\chi_2(A^v))$ are dominant.*

b) $H(A, C)\subsetneqq H(A_1, C)$ *if* $\chi_1(A_1)-w_P^2(\chi_2(C))$ *and* $\chi_1(C^v)-w_P^1(\chi_2(A^v))$ *are dominant.*

Proof. Let $t = \dim G/B-1$. Then the annihilators of $H(A, C)$, etc. lie in $H^t(\underline{L}^{-1}\otimes\Omega_a)$, where Ω_a is the absolute sheaf of differentials. Since $H^t(\underline{L}^{-1}\otimes\Omega)$ $\cong \mathrm{Hom}_k(H^1(\underline{L}), k)$ the arguments of [9] still work. In fact the elements of the various annihilators clearly can be written in terms of elements of A^v, C^v, etc.

$H(A, C)\subsetneqq H(A, C_1)$ if their annihilators have a proper inclusion (in reverse order). Suppose $H(A, C_1)^v \neq (0)$. Then $H(A, C_1)^v\otimes C^v\subseteq H(A, C)^v$, so the inclusion must be proper. Hence it suffices to check $H(A, C)^v \neq (0)$ (if $H(A, C_1)^v = (0)$, there certainly would be a proper inclusion). Using Lemma 4 gives one of the conditions in a). The other expresses the (vacuous) condition that $H(A, C_1)^v \neq H^1(\underline{L})^v$. This condition is included to show the symmetry between a) and b), although it is vacuous by the preceding argument (or by direct analysis).

The proof of b) is entirely similar.

A composition series for $H^1(\underline{L})$ can now be constructed. Let C_1 be the irreducible submodule of $H^s(G/P_1, \underline{O}(i))$. Let $A_1\subseteq \ldots \subseteq A_r$ be a sequence of submodules of $H^0(G/P_2, \underline{O}(j))$ such that 1) A_{m+1}/A_m is irreducible, 2) $H(A, C_1) = 0$ if $A\subsetneqq A_1$, 3) $H(A, C_1) = H(A_r, C_1)$ if $A_r\subseteq A$. Let $C_1\subseteq \ldots \subseteq C_u$ be a sequence of submodules of $H^s(G/P_1, \underline{O}(i))$ such that 1) C_{m+1}/C_m is irreducible. 2) $H(A_r, C_u) = H^1(\underline{L})$.

Then by Lemma 5, $H(A_{m+1}, C_1)/H(A_m, C_1) \neq (0)$, but by Theorem 2 it is irreducible. Similarly $H(A_r, C_{m+1})/H(A_r, C_m)$ is a non-trivial irreducible G-module.

Hence a composition series

$$H(A_r, C_u)\supset H(A_r, C_{u-1}) \supset \ldots \supset H(A_r, C_1)\supset \ldots \supset H(A_1, C_1)\supset (0) \tag{4}$$

is obtained.

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