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INSTITUTE OF MATHEMATICS
POLISH ACADEMIE OF SCIENCES

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On elementary cuts in models of arithmetic

by

Henryk Kotlarski (Warszawa)

Abstract. Let $M \models P$ (= Peano arithmetic). We put $Y := \{N \underset{\text{end}}{\subset} M\colon N \prec M\}$. This family, as each family of initial segments of M, is simply ordered by inclusion. The order type of Y heavily depends on M; we shall compute this order type in the following cases: (a) M is countable and recursively saturated and (b) M is saturated. In both cases the proofs give fairly complete description of the situation.

We assume that the reader is familiar with saturated models (see Chang, Keisler [1]) and with recursively saturated models (see Schlipf [5]). We use standard model-theoretic terminology and notation.

§ 1. **The recursively saturated case.** Toward this section let $M \models P$ be recursively saturated. Our result is the following.

THEOREM 1. *If M is countable, then Y is of the order type of the Cantor set 2^ω with its lexicographical ordering*:

$$b^1 < b^2 \equiv (\exists\, n \in \omega)(b_n^1 = 0 \wedge b_n^2 = 1 \wedge (\forall\, m<n)(b_m^1 = b_m^2)).$$

Before proving this we shall prove some lemmas. For $a \in M$ we shall denote by $M(a)$ the closure under the initial segment of the Skolem closure of a; formally

$$M(a) := \{b \in M\colon \text{there exists a parameter-free term } t(v) \text{ such that } M \models b < t(a)\}.$$

By Gaifman [2, Theorem 4.1] $M(a) \underset{\text{end}}{\prec} M$.

LEMMA 2. *$M(a)$ is not recursively saturated.*

Proof. $M(a)$ omits the type $\{v > t(a)\colon t \text{ is a term}\}$. ■

Let $Y_1 = \{N \in Y\colon N \text{ is not recursively saturated}\}$. Our first aim is to prove the converse of Lemma 2, it will be our Lemma 4.

LEMMA 3 (with W. Marek). *If $D \subseteq Y$ has no greatest element, then $\bigcup D$ is recursively saturated.*

Proof. Let $p(v)$ be a recursive type in parameters $b_1, \ldots, b_k$. There exists an $N \in D$ such that $b_1, \ldots, b_k \in N$ (in fact, D is linearly ordered by inclusion), and so by the assumption there exists $N_1 \in D$ such that an $N \underset{\text{end}}{\prec} N_1$. Therefore pick $c \in N_1$ such that $\forall a \in N\; M \models a < c$. Now consider the type $p(v) \cup \{v < c\}$. This is still a recursive type and consistent, and so it is realized in M. But any of its realizations is in $\bigcup D$. ■

LEMMA 4. *Every $N\in Y_1$ is of the form $M(a)$ for some $a\in M$.*

Proof. Let $D=\{M(a): a\in N\}$. If the conclusion fails then D has no greatest element, and so, by Lemma 3, $N=\bigcup D\notin Y_1$. ∎

LEMMA 5. *Y_1 has the smallest element and no greatest element, and is densely ordered.*

Proof. $M(0)$ is the smallest element; if a certain $M(a)$ were the greatest element, then $M=M(a)$ and so it would not be recursively saturated by Lemma 2; so we only need to verify density. Let $M(a)<M(b)$. Consider the type

$$p(v)=\{t(a)<v:\ t \text{ is a term}\}\cup\{t(v)<b:\ t \text{ is a term}\}.$$

This type is clearly consistent (any of its finite subsets can be realized by an element of $M(a)$) and recursive; so let c realize p. But then $M(a)<M(c)<M(b)$. ∎

COROLLARY 6. *Y_1 is of the order type $1+\eta$, where η is the order type of rationals.* ∎

Let $E=\{b\in 2^{\omega}: \exists n\forall m>n\, b_m=0\}$.

The following fact is easily verified.

LEMMA 7. *E is of the order type $1+\eta$.* ∎

Proof of Theorem 1. Let j be an isomorphism of E onto Y_1. We extend j to $f: 2^{\omega}\to Y$ in the usual manner:

$$f(b)=\bigcup\{j(b'): b'\leqslant b,\ b'\in E\}.$$

One shows without difficulty that f is an isomorphism of 2^{ω} onto Y. We prove only that $b^1<b^2\to f(b^1)<f(b^2)$ and leave the rest to the reader.

Case 1. $b^1, b^2\in E$. Now $f(b^1)<f(b^2)$ because $f\restriction Y_1=j$.

Case 2. $b^1\in E$, $b^2\notin E$. Now $j(b^2)\notin Y_1$ and so, for any $a\in j(b^2)$ such that $j(b^1)<a$, $j^{-1}(M(a))$ is between b^1 and b^2; thus

$$f(b^1)=j(b^1)<M(a)\leqslant j(b^2)=f(b^2).$$

Case 3. $b^1\notin E$, $b^2\in E$. Obviously $f(b^1)\leqslant f(b^2)$. The inequality must be strict, since $f(b^2)$ is not the union of the family $\{j(b'): b'\leqslant b^1,\ b'\in E\}$, since this family has no greatest element and so its union is recursively saturated.

Case 4. $b^1, b^2\notin E$. We leave this case to the reader. ∎

COROLLARY 8. *Y_1 and Y are symbiotic, i.e., for all $a,b\in M$*

$$(\exists N\in Y_1\ a<N<b)\equiv(\exists N\in Y\ a<N<b).\ \blacksquare$$

COROLLARY 9. *$Y\setminus Y_1$ is of the order type of reals $+\,1$.* ∎

Our earlier (unpublished) argument leading to the above results was much more tricky, namely we used some tricks involving non-standard satisfaction (cf. Krajewski [3] for this notion) to prove Corollary 8, and then we derived Corollary 9 and Theorem 1. We found the argument in question while working on the saturated case.

§ 2. **The saturated case.** From now on let M be saturated and let μ be its cardinality. Consider the set 2^{μ} with its lexicographical ordering:

$$b^1<b^2\equiv\exists\alpha<\mu\, b^1_{\alpha}=0\wedge b^2_{\alpha}=1\wedge\forall\beta<\alpha\, b^1_{\beta}=b^2_{\beta}.$$

THEOREM 10. *Y is of the order type 2^{μ}.*

The proof of Theorem 10 is almost the same as that of § 1. We shall only indicate the differences.

Let $Y_1=\{M(a): a\in M\}$. Exactly as above, one verifies that Y_1 is a saturated dense linear ordering with first and without least element.

Let $E':=\{b\in 2^{\mu}$: there exists an $\alpha<\mu$, α is not limit and $b_{\alpha}=1\wedge\forall\beta>\alpha\, b_{\beta}=0\}$. One verifies that E' is a saturated dense linear ordering without first and last elements, so let E be $E'+$ the smallest element; E is isomorphic with Y_1, now one extends this isomorphism to an isomorphism $f: 2^{\mu}\to Y$. ∎

We shall now give a classification of the elements of Y. For an ordinal ξ we define

$Y_{\xi}=\{N\in Y$: N can be written as the union of a strictly increasing sequence $M(a_0)\subsetneq M(a_1)\subsetneq\ldots$ of length ξ and ξ is the smallest ordinal with this property$\}$.

This notation coincides with the notion Y_1 defined before. Elements of Y_{ξ} are called *cuts of cofinality* ξ. Observe that the isomorphism $f: 2^{\mu}\to Y$ given by Theorem 10 carries elements of E onto cuts of cofinality 1 and carries branches $b\in E_{\xi}$, $\xi>1$, ξ limit, onto cuts of cofinality ξ, where

$$E_{\xi}=\{b\in 2^{\mu}: \forall\eta\geqslant\xi\, b_{\eta}=0\wedge\forall\alpha<\xi\,\exists\beta\,\alpha<\beta<\xi\wedge b_{\beta}=1\}.$$

Let D be the set containing 1 and all infinite regular cardinals $\leqslant\mu$.

THEOREM 11. (i) $Y=\bigcup_{\xi\in D}Y_{\xi}$;

(ii) $Y_{\xi}\neq\emptyset\equiv\xi\in D$;

(iii) *all families Y_{ξ}, $\xi\in D$ are symbiotic, i.e.*

$$\forall\xi_1<\xi_2,\ \xi_1,\xi_2\in D\to\forall a,\ b\in M(\exists N\in Y_{\xi_1}\ a<N<b)\equiv(\exists N\in Y_{\xi_2}\ a<N<b);$$

(iv) *for all $\xi\in D$, if $\xi<\mu$ then card $Y_{\xi}=\mu$; so card $Y_{\mu}=2^{\mu}$;*

(v) *for $N\in Y$, N is saturated iff $N\in Y_{\mu}$.*

One can prove Theorem 11 directly or simply look at the ordering 2^{μ}. We leave the details to the reader. ∎

It follows that M has only μ non-saturated elementary cuts and 2^{μ} saturated ones (by (v) and (iv)).

Now we show that M has many resplendent elementary cuts even in a stronger sense of the word "many".

THEOREM 12. *For $N\in Y$, N is resplendent iff $N\notin Y_1$.*

Proof. → obvious, since if $N\in Y_1$ then it is even not recursively saturated.

$\leftarrow$ Assume that $N \in Y_\xi$, $\xi > 1$. If $\xi = \mu$ then N is saturated, hence resplendent, and so assume that $\xi < \mu$. Pick a sequence $\{a_\varrho\colon \varrho < \xi\}$ such that $N = \bigcup_{\varrho<\xi} M(a_\varrho)$ and $M(a_0) < M(a_1) < \ldots$

Let a sentence $\varphi(R, b)$ be given; $b \in N$, R is a new predicate symbol. We may assume that $b \leqslant a_0$. Consider the language

$$L \cup \{b\} \cup \{a_\varrho\colon \varrho < \xi\} \cup \{X_\varrho\colon \varrho < \xi\} \cup \{R\},$$

where X_ϱ are new unary predicate symbols. Let T be the following theory in this language:

$$\begin{aligned}&\mathrm{Th}(N, b, a_\varrho)_{\varrho<\xi} \cup \{\varphi(R, b)\} \cup \\ &\quad \cup \{\forall \bar{v}(\psi(\bar{v}) \equiv \psi^{X_\varrho}(\bar{v}))\colon \varrho < \xi,\ \psi \text{ is } L \cup \{R\}\text{-formula}\} \cup \\ &\quad \cup \{\forall v\, v \in X_{\varrho_1} \to v \in X_{\varrho_2}\colon \varrho_1 < \varrho_2 < \xi\} \cup \{a_\varrho \in X_\varrho\colon \varrho < \xi\} \cup \\ &\quad \cup \{\forall v\, v \in X_\varrho \to v < a_{\varrho+1}\colon \varrho < \xi\}.\end{aligned}$$

Here ψ^X denotes the relativisation of a formula ψ to X. We claim that T is consistent. In fact each finite subset of T is satisfiable.

Pick a saturated $N_0 \in Y$ so that $a_0 < N_0 < a_1$ (such an N_0 exists because of Theorem 11 (iii) and (v)); this N_0 will be the interpretation of X_0. As N_0 is saturated, there is an R_0 so that $(N_0, R_0) \models \varphi(R, b)$. Now the set of formulas

$$\{\eta(v, z)\colon N \models \eta(a_1, z)\colon \eta \in L,\ z \in N_0\}$$

is consistent with $\mathrm{Th}(N_0, R_0)$; therefore pick a saturated model which realizes it, such a model is isomorphic to some (N_1, a_0, a_1) with some R_1 and so on. That N_1 interprets X_1 and so on.

As the language of T is of cardinality $< \xi$, T has a saturated model of power μ (this follows from the existence of M; namely if the Peano arithmetic has a saturated model of power μ, then μ is regular and $\forall \varrho < \mu\ 2^\varrho \leqslant \mu$ thus saturated models exist in cardinal μ, cf. Chang–Keisler [1]).

Let $\mathfrak{A}$ be a saturated model of T of power μ. So $\mathfrak{A}$ is of the form:

$$\mathfrak{A} = \langle A, +, \cdot, <, R, b, a_\varrho, X_\varrho\rangle_{\varrho<\xi}.$$

Now the model $\langle A, +, \cdot, <, b, a_\varrho\rangle_{\varrho<\xi}$ is saturated, of power μ and elementarily equivalent to $\langle M, b, a_\varrho\rangle_{\varrho<\xi}$, and so these two models are isomorphic. But now this isomorphism carries $\{x \in A\colon \text{there is } \varrho < \mu \text{ such that } x < a_\varrho\}$ onto N. Moreover the last model satisfies $\exists R\, \varphi(R, b)$ because it is the union of $L \cup \{R\}$-elementary chain of models which have R as needed. ■

§ 3. Open problems.

1. Computations of order types of families of cuts carried out in this paper heavily depend on the order completeness of Y. What happens with non-complete families is not clear. That is why we pose the following problem. Let $M \models P$ be countable and non-standard. Let $Z = \{N \subset_e M\colon N \models P\}$.

What is the order type of Z? Does it depend on M at all?

1. Another problem is the following. Does there exist a resplendent $M \models P$ which has only card M elementary cuts? A positive answer would show not only that the use of types was necessary in § 2, but also that results of the Chang–Makkai type may fail for resplendent M (the existence of many elementary cuts can be derived from theorems of the Chang–Makkai type, see Schlipf [5]).

3. Several investigators have tried to describe possible lattices of elementary submodels of $M \models P$, see Gaifman [2] and Mills [4] for information. The following problem seems to be interesting. Let $M \models P$ be countable and recursively saturated. What is the lattice of elementary submodels of M? Does it depend on M?

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INST. OF APPLIED MATH. AND STATISTICS,
AGRICULTURAL UNIVERSITY
Warszawa

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