On the winding number and equivariant homotopy classes of maps of manifolds with some finite group actions

by

C. Bowszyc (Warszawa)

Abstract. The paper considers equivariant maps of a closed connected $m$-dimensional manifold $M$ with an effective smooth action of a finite group $G$ into a punctured linear $(m+1)$-dimensional space $E, (0)$ with a smooth action of $G$ on $E$ such that $O$ is a fixed point and every isotropy group of the action on $M$ acts trivially on $E$. The following questions are investigated:

1. What numbers may be the winding numbers of such maps?
2. What are the equivariant homotopy classes of such maps?

The well-known Borsuk theorem asserts that any equivariant map of a sphere with the antipodal action of $Z_2$ into itself has an odd degree. In this paper we take up the question what winding numbers (degrees) have equivariant maps of a closed connected smooth $G$-manifold $M$ into a linear $G$-space $E$ of dimension greater by 1 with $O$ removed when every isotropy group of the action of a finite group $G$ on $M$ acts trivially on $E$ (Theorem 2.2).

Although these assumptions are very restrictive, they contain the case of free actions on $M$ and the case of the trivial action on $E$. Without the imposed assumptions the results may be false (Example 2.4).

Moreover, Theorems 3.1, 4.3 and 5.1 give a complete equivariant homotopy classification of such maps and may be viewed as a generalization of the Hopf theorem.

The methods used are similar to those in Krasnoselski’s paper [5]. Although the maps under consideration are continuous, they are treated by means of rather differential topology methods as in [3] or [6].

In the whole paper $G$ is a finite group. By a manifold we mean a paracompact smooth manifold without boundary. All actions of a group $G$ are assumed to be smooth.

1. Auxiliary results. We shall use a kind of mappings given by

1.1. Definition. Let $P$ be a $p$-dimensional manifold and $E$ a real $(m+1)$-dimensional vector space. A map $f: P \to E$ is called good if $f$ is continuous on $P$, $f$ is smooth on some open set $Q$ containing $f^{-1}(O)$ and $O$ is a regular value of $f | Q$. If, in addition, $G$ acts smoothly on $P$ and $E$, $f$ is a $G$-map and $f^{-1}(O)$ is contained...
in the part $P_s$ of $P$ consisting of all points with the trivial isotropy group $\{e\}$, then $F$ is called a $G$-good map.

For a good map, $f^{-1}(O)$ is a $(p-m-1)$-dimensional submanifold of $P$ (invariant if $f$ is $G$-good and $O$ is a fixed point of the action of $G$ on $E$) or is empty.

The following facts concern extensions of good maps to good maps.

1.2. Let $P$ be a closed set in $\mathbb{R}^p$ and $B = \bigcap_{t < 0} f(t)$ a compact set in $\mathbb{R}^p$. If $f: U \to E$ is a good map, then there exists an open set $W$ containing $U \cap D$ and a good map $h: W \to E$ such that $h|F = f|F$.

Proof. Choose open sets $U_{\alpha}, U_1$ and $U_2$ such that $\bigcup U_{\alpha} \subseteq U_1 \subseteq U_2 \subseteq U$ and a smooth function $\phi: P \to [0, 1]$ satisfying conditions $\phi(x) = 0$ for $x \in U_1 \setminus U_2$ and $\phi(x) = 1$ for $x \in U_2 \setminus U_3$. Choose open sets $V_0 \cap V_1$ and $V_2$ and smooth $\varphi: P \to [0, 1]$, $\tilde{\varphi}: P \to [0, 1]$ such that

$$
D = V_0 \cap V_1 \subseteq V \cap D \subseteq V_2 \subseteq U \cap D
$$

and $\tilde{\varphi}(x) = 0$ for $x \in U_1 \setminus U_2$ and $\tilde{\varphi}(x) = 1$ for $x \in U_2 \setminus U_3$. Choose open sets $U_{\alpha} \subseteq U_2 \subseteq U$ and a smooth function $\phi: P \to [0, 1]$ satisfying conditions $\phi(x) = 0$ for $x \in U_1 \cap U_2$ and $\phi(x) = 1$ for $x \in U_2 \cap U_3$. Choose open sets $W_0 \subseteq W_1 \subseteq U_1 \setminus U_2$ and a smooth function $\phi: P \to [0, 1]$ satisfying conditions $\phi(x) = 0$ for $x \in U_1 \cap U_2$ and $\phi(x) = 1$ for $x \in U_2 \cap U_3$. Choose open sets $W_0 \subseteq W_1 \subseteq U_1 \setminus U_2$ and a smooth function $\phi: P \to [0, 1]$ satisfying conditions $\phi(x) = 0$ for $x \in U_1 \cap U_2$ and $\phi(x) = 1$ for $x \in U_2 \cap U_3$.

Let $\alpha_0 = 0$ be the minimum of $\{\phi(x)\}$ for $x \in x$ belonging to the compact set $K = (\bigcap_{\alpha \leq \alpha_0} U_{\alpha}) \cap U_2 \cap O$. Let $f_\alpha: U \to E$ be a smooth map such that $\{f_\alpha(x) - f(x)\} < \varepsilon$ for $x \in K$. Define the map $f_{\alpha}: U_1 \cup U_2 \to E$ by

$$
f_{\alpha}(x) = \begin{cases} f(x) + \phi(x)(f_\alpha(x) - f(x)) & \text{if } x \in U_1 \cup (V \cap U) \\ f_\alpha(x) & \text{if } x \in U_2 \setminus U_1 \end{cases}
$$

Then $f_{\alpha}$ is continuous, $f_{\alpha}(U_1) = f|U_1$, $f_{\alpha}(x) \neq f(x)$ for $x \in K \cap V$ and therefore $f_{\alpha}$ is smooth in some open set $Z$ containing $f^{-1}(O)$ if $f^{-1}(O) \neq \emptyset$.

Let $K_0$ and $K_1$ be open sets such that $f^{-1}(O) \cap (V \setminus U_2) \subseteq Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq Z$ with $Z$ compact. Let $\varphi: P \to [0, 1]$ be a smooth function such that $\varphi(x) = 0$ for $x \in U_1 \setminus U_2$ and $\varphi(x) = 1$ for $x \in Z_0$. There exists a compact set $K_0$ such that $f^{-1}(O) \cap (Z_0 \setminus K_0) \subseteq U \subseteq K_0 \subseteq U \subseteq Z \setminus U$ and the tangent map $d traced by the Sard lemma there exists a regular value $a \in E$ for $f_{\alpha}|Z$ arbitrary close to $O$. Define $W = U_0 \cap V_0$ and $h: W \to E$ by $h(x) = f_{\alpha}(x) - \varphi(x).$ If $|a| is sufficiently small, then $\partial h_a$ is an isomorphism for $x \in K_0$ and $\partial h_a(x) \neq 0$ for $x \in K_0$. Therefore $h$ is a good map and $h|F = f_{\alpha}|F = f|F$.

1.3. Let $G$ act on a manifold $P$ and on a vector space $E$ with the fixed point $O$. Suppose that $G$ acts smoothly on a connected manifold $P$ in such a way that each isotropy group of the action on $E$ acts continuously on $E$ and the manifold $P$ is an open invariant set $F$ containing a closed invariant set $F$ in such a way that the action of $G$ on $E$ is smooth, then there exists a $G$-good extension $h: P \to E$.

Proof. Set

$$
f_{\alpha}(x) = \begin{cases} f(x) & \text{for } x \in U_1 \\ \varphi(x) f_\alpha(x) & \text{for } x \in V \cap U_2 \end{cases}
$$

1.4. Let $B$ be a $b$-dimensional $G$-manifold with exactly one orbit corresponding to the conjugacy class of isotropy subgroups $(H)$. Let $G$ act on an $(m+1)$-dimensional vector space $E$ with the fixed point $O$ in such a way that the action of $H$ acts trivially on $E$ and $b \leq m$ or $H = \{e\}$. If $f: W \to E$ is a $G$-good map on an open invariant sub-
and if \( k+1 = n \) then \( V_1 \) is open in \( P \). By 1.4 there exists a \( G \)-good extension \( h_k: B \to E \) of \( f_k|_{V_0} \). If \( k+1 = n \) set

\[
f_{k+1}(x) = \begin{cases} f_k(x) & \text{if } x \in V_0, \\ h_k(x) & \text{if } x \in B = P_x. \end{cases}
\]

Suppose that \( k+1 < n \). There exists an invariant tubular neighbourhood \( N \) of \( B \), open in \( P \), which can be identified with the equivariant normal bundle of \( B \) in \( P \). Let \( \pi: N \to B \) be the projection. We can assume that:

1. In this bundle we have an equivariant Riemannian metric.
2. The part of the bundle of the unit open disks \( N_i \) over \( V_0 \) denoted by \( N_i|_{V_0} \) is contained in \( U_x \).
3. The diameters \( \delta(N_i) \) of the fibres of the vector bundle \( N \) over points \( y \in B \) in some metric on \( P \) tend to \( 0 \) when \( y \) tends to infinity — the point added to \( B \) in the one-point compactification of \( B \).

The set \( B \times V_0 = F_{k+1} \setminus V_0 \) is closed in \( P \) and the closure in \( P \) \( N_i|_{B \times V_0} \) is the part of the bundle of the unit closed disks of \( N \) over \( B \times V_0 \) by condition 3. Let \( \varphi: B \to [0,1] \) be an equivariant smooth map such that \( \varphi(x) = 0 \) for \( x \in V_0 \) and \( \varphi(x) = 1 \) for \( x \in B \setminus V_0 \). We can assume that \( f_k(x) \neq 0 \) for \( x \in N_i|_{V_0} \). Define

\[
U_{k+1} = (U_x \setminus N_i|_{B \times V_0}) \cup N_k \\

\text{and } f_{k+1}: U_{k+1} \to E \text{ by }

\begin{align*}
f_{k+1}(x) &= \begin{cases} f_k(x) & \text{for } x \in U_x \setminus N_i|_{B \times V_0}, \\ \varphi(x) \pi(x) & \text{for } x \in N_i|_{B \times V_0}, \\ h_k \pi(x) & \text{for } x \in N_i|_{B \times V_0}. \end{cases}
\end{align*}
\]

\( f_{k+1} \) is a well-defined \( G \)-good map on \( U_{k+1} \).

The last map \( h = f_n \) is a \( G \)-good extension of \( f \) on the whole manifold \( P = U \).

All maps in 1.2, 1.3, 1.4 and 1.5 are smooth if \( f \) was smooth.

1.6. Corollary. Let \( M \) be a closed connected manifold with an effective action of \( G \). Let \( G \) act on the vector space \( E \) with the fixed point \( 0 \) in \( E \) in such a way that each isotropy group of the action on \( M \) acts trivially on \( E = \mathbb{R} \). If \( \dim M > \dim E \), then there is a smooth \( G \)-map \( f: M \to E \).

If \( \dim M < \dim E \), then any two continuous (smooth) \( G \)-maps \( f, f': M \to E \) are \( G \)-homotopic (smoothly).

For the proof we take in 1.5 \( P = M \) and \( U = \emptyset \) in case a) and

\[
P = R \times M, \quad U = (R \times \{0\}) \times M, \quad F = \{0\} \times M
\]

and

\[
f(t, x) = \begin{cases} f_0(x) & \text{for } t \in (-\infty, 0), x \in M, \\ f_{0}(x) & \text{for } t \in (0, +\infty), x \in M. \end{cases}
\]

The following example shows that equivariant maps do not always exist.

1.7. Example. There is no equivariant map of the unit sphere \( S^2 \subset \mathbb{R}^3 \) with the antipodal action of \( Z_2 \) into an orientable surface \( S \) of genus \( g \geq 0 \) embedded symmetrically with respect to \( O \) in \( \mathbb{R}^3 \) with the action of \( Z_2 \) by symmetry with respect to \( O \).

If such a map \( f: S^2 \to S \) existed, then \( f \) is homotopic to a constant map because \( S^2 \) has a trivial fundamental group and the universal covering space of \( S \) is homeomorphic to an open disk. Therefore \( \deg f = 0 \). By 1.6 a) there exists an equivariant map \( g: S \to S^3 \) because \( S^3 \) is an equivariant deformation retract of \( \mathbb{R}^3 \setminus \{0\} \). The map \( g \circ f: S^2 \to S^3 \) is equivariant with \( \deg g \circ f = 0 \), which contradicts Borsuk's theorem.

1.8. Homogeneity Lemma. Let \( G \) act effectively on a connected manifold \( P \). If \( x, y \) belong to the same component \( C \) of the principal part \( P_x \), then there exists an equivariant diffeomorphism \( h: P \to P \) such that \( x \) is mapped to \( y \) by the diffeomorphism \( h \), which does not move points beyond some compact invariant set and beyond \( P_x \).

The proof is similar to that in the non-equivariant case (6). We have in \( C \) the equivalence relation: \( x \sim y \) iff the statement of 1.8 is true. Let \( V \) be a slice at \( x \) in \( C \) diffeomorphic to a Euclidean space and let \( y \in V \). By the non-equivariant homogeneity there exists a diffeotopy \( f_t: V \to V \) such that \( f_0 = 0 \), \( f_1(x) = y \) and \( f_1(x) = z \) beyond some compact set. We define the equivariant diffeotopy \( h_t: P \to P \) by

\[
h_t(x) = \begin{cases} z & \text{if } z \in P \setminus GV, \\ g_t(x) & \text{if } z \in GV, g \in G. \end{cases}
\]

Therefore the classes of the relation are open and \( C \) is the only class by connectivity.

1.9. Remark. If the component \( C \) of a nonorientable manifold and \( \omega_x \) and \( \omega_y \) are any orientations of the tangent spaces \( T_xP \) and \( T_yP \), respectively, for \( x, y \in C \), then the \( G \)-diffeomorphism \( h \) of 1.8 can be chosen in such a way that the tangent map \( dh_x \) maps \( \omega_x \) to \( \omega_y \).

1.10. There is a generalization of 1.8 (and 1.9) analogous to that in the non-equivariant case: If \( \dim P > 1 \) and \( x_1, y_1 \) for \( i = 1, \ldots, k \) are two \( k \)-tuples of points of a component \( C \) belonging to different orbits, then there is a \( G \)-diffeomorphism \( h: P \to P \) such that \( h(x) = y_i \) for \( i = 1, \ldots, k \).

This follows by induction on \( k \) because a finite set does not separate a manifold of dimension greater than 1.

1.11. Remark. If \( x \) and \( y \) belong to different components of \( P_x \), then a \( G \)-diffeomorphism \( h: P \to P \) such that \( h(x) = y \) does not always exist, e.g. if the subgroup of \( G \) preserving the component \( C \) of \( x \) denoted by \( G_x \) is different from the subgroup \( G_{x_2} = g_0G_{x_2}^{-1} \) for \( x_2 \) (as in Example 3 of [7]). If \( g \) belong to the centre of \( G \), then such an \( h \) exists. But there is no \( G \)-diffeotopy \( h_t \) from \( Id_x \) to \( h \) because each \( h_t \) would map \( P_t \) onto \( P_t \) and \( C \) onto \( C \).

2. Winding numbers of equivariant maps.

2.1. Let \( M \) be a closed connected manifold of dimension \( m > 1 \) with an effective smooth action of a finite group \( G \). Suppose that \( G \) acts smoothly on an \((m+1)\)-dimen-
sional Euclidean vector space $E$, $O$ is a fixed point and every isotropy group of the action on $M$ acts trivially on $E$. Denote $E_0 = E - \{O\}$. The above assumptions will always be observed in the sequel.

If $M$ is oriented, there are two possibilities:

a) Every $g \in G$ simultaneously preserves the orientations of $M$ and $E$ or simultaneously reverses them.

b) Some $g \in G$ preserves the orientation of $M$ and reverses the orientation of $E$ or vice versa.

In case a) we shall say that the actions of $G$ on $M$ and $E$ are concordant and in case b) that they are discordant.

If $M$ and $E$ are oriented, then for a continuous map $f: M \to E_0$, the winding number $W(f)$ is defined by the degree of the continuous map $f/|f|: M \to S^n \subset E_0$, where $S^n$ is the unit sphere in $E$ oriented as the boundary of the unit ball in $E$. If $M$ is non-oriented, then the winding number modulo 2 denoted by $W_2(f)$ is defined similarly.

2.2. THEOREM. Let $G$, $M$, $E$, $E_0$ be as in 2.1 and let $M$ be oriented.

a) If the actions of $G$ are concordant, then for any continuous equivariant map $f_0$, $f_1: M \to E_0$, $W(f_0) - W(f_1) \equiv \sum_{g \in G} \deg g \cdot \langle h \rangle$ and $W(f_0) \equiv W(f_1) \mod |G|$. 

b) If the actions of $G$ are discordant, then for any continuous equivariant map $f: M \to E_0$, $W(f) = 0$ (even without the assumptions about isotropy groups).

Proof. Let $\theta_g$ and $\psi_g$ denote the diffeomorphisms of $M$ and $E$, respectively, corresponding to $g$. The local degree at 0, $\deg \psi_g$, is equal to 1 if $\psi_g$ preserves the orientation of $E$ and equals to $-1$ otherwise. Since the action of $G$ is discordant, there exists a $g \in G$ such that $\deg \theta_g = \deg \psi_g$. The map $f$ is equivariant, and so $f \circ \theta_g = \psi_g \circ f$. Therefore $W(f) \equiv \deg \psi_g \cdot W(f)$, $W(f) = -W(f)$ and $W(f) = 0$.

a) By the extension Lemma 1.5 applied to the manifold $P = R \times M$, the sets $U = (R - \{1\}) \times M$, $F = (R - \{0, 1\}) \times M$ and the mapping $f: U \to E$ defined by

\[
 f(t, x) = \begin{cases} 
 f_0(x) & \text{if } t < \frac{1}{2}, \\
 f_1(x) & \text{if } t > \frac{1}{2} 
\end{cases}
\]

there is a $G$-good homotopy $h: I \times M \to E$ from $f_0$ to $f_1$. (If $M$ is a manifold with boundary, but $h$ can be extended to a $G$-good map on the manifold $P$ without boundary. Similarly we shall use the notion of $G$-good map in the sequel.) $h^{-1}(0)$ is a finite equivariant subset of $\{0\} \times M$, because $\dim I = M = m + 1 = \dim E$. Choose one point $x_0$ in each orbit of $h^{-1}(0)$ for $i = 1, \ldots, k$.

It is known (cf. [3] or [6]) that $W(f_0) - W(f_1) \equiv \sum_{x \in h^{-1}(0)} \deg x \cdot h$, where $\deg x \cdot h$ is the local degree at isolated zero $x$ of $h$. If $\theta_g$ and $\psi_g$ denote the diffeomorphisms of $I \times M$ and $E$, respectively, corresponding to $g \in G$, then $\deg \theta_g = \deg \psi_g$ because the actions of $G$ are concordant. From the equality $h \circ \psi_g = \psi_g \circ h$ we get $\deg \psi_g \cdot \deg \theta_g = \deg \psi_g \cdot \deg h$ and $\deg \theta_g \cdot \deg h$ for every $x \in h^{-1}(0)$ and $g \in G$. So the local degrees of $h$ at all points of one orbit of $h^{-1}(0)$ are equal. Therefore $W(f_0) - W(f_1) \equiv \sum_{x \in h^{-1}(0)} \deg h \cdot x$.

2.3. REMARK. If the action of $G$ on $E$ is equivariant, we can consider equivariant maps $M \to S^n$ instead of $M \to E_0$ and the degrees of such maps instead of winding numbers. Since the sphere $S^n$ is an equivariant deformation retract of $E_0$, this concerns also the results in sections 3-5.

Theorem 2.2 a) may be false if the assumptions on the isotropy groups are not satisfied.

2.4. EXAMPLE. Consider the action of $Z_4$ on the unit circle $S^1 \subset R^2$ and $E = R^3$, in which the generator of $Z_4$ acts by symmetry with respect to a line. Those actions are concordant. The maps $f_0 = \text{id}_{R^3}$ and $f_1$ is constant map into one of two fixed points on $S^1$ is equivariant, but $\deg f_0 = 1$ and $\deg f_1 = 0$.

2.5. COROLLARY. If the action of $G$ on $E$ is trivial, then for any action of $G$ on $M$ and mapping $f: M \to E_0$ constant on orbits $W(f) \equiv 0 \mod |G|$.

2.6. REMARK. Theorem 2.2 can always be applied if the action of $G$ on $M$ is free. The proof in this case may be considerably simplified.

2.7. EXAMPLE. Suppose that $G$ acts on an $(m+1)$-dimensional Euclidean vector space with the fixed point $O$, $N$ is a compact $(m+1)$-dimensional invariant submanifold of $E$ with boundary $M = \partial N \subset E_0$ and the induced action of $G$ on $M$ is free. Then, for any equivariant map $f: M \to E_0$, $W(f) \equiv 0 \mod |G|$ if $O \in N$ and $W(f) \equiv 1 \mod |G|$ if $O \notin N$.

Indeed, if $f_0$ is the inclusion $M \to E_0$, then it is equivariant and has an extension to the inclusion $f_1: N \to E$ without zeros if $O \notin N$ and with exactly one zero $0$ with the local degree $\deg f_0 = 1$ if $O \in N$. Since $W(f_0) = \deg f_0$, this follows from 2.2 a).

If, in addition, the action of $G$ on $E$ is orthogonal, then the Gauss map $f_1: M \to S^{n-1} \subset E_0$ which assigns to a point $x \in M$ the unit normal to $M$ at $x$ directed outward of $N$, is equivariant. The degree of $f_1$ is equal to the Euler-Poincaré characteristic $\chi(N) = \dim \mathcal{P}$, where $\mathcal{P}$ is the fixed point index of $T$ (cf. [2]). For any equivariant map $f: M \to E_0$, $W(f) \equiv \chi(N) \mod |G|$. If the number $m$ is even, then $\chi(N) = \chi(M)$ by considering the double of $N$.

2.8. EXAMPLE. Let $E$ be an $(m+1)$-dimensional linear space, and $N$ a compact $(m+1)$-dimensional manifold in $E$ with boundary $M = \partial N$. Let $T: M \to M$ be a fixed point free smooth involution. $T$ defines an action of $Z_2$ on $M$. Consider $E$ with the action of $Z_2$ generated by symmetry with respect to $O$. Let $\mathcal{T}: \mathcal{T} \to E$ be any continuous extension of $T$. The map $f_2: M \to E_0$ defined by $f_2(x) = x - T(x)$ is equivariant and $W(f_2) = \dim \mathcal{T}$, where $\dim \mathcal{T}$ is the fixed point index of $T$ (cf. [2]). For any equivariant map $f: M \to E_0$, $W(f) = \dim \mathcal{T} \mod 2$.

2.9. EXAMPLE. Let $Z_n$ act on $E = C^n$ with the action of a generator $g$ of $Z_n$ defined by $\psi(z) = e^{2\pi i/k}z$ for $z \in C^n$, where $n$ and $k$ are natural numbers. Let $Z_n$
act also on the unit sphere \( M = S^{2n-1} \subset \mathbb{C}^n \), the action defined by \( \theta(x) = e^{2\pi i n x} \) for \( z \in S^{2n-1} \), where the natural numbers \( n \) and \( r \) are relatively prime. The action on \( S^{2n-1} \) is free. The class of \( \text{imod} \) denoted by \( [f] \) in an irreducible element of \( Z_n \). Let \( [a] = [b] \) in \( Z_n \), i.e. \( a \equiv b \mod n \). The map \( f: S^{2n-1} \rightarrow \mathbb{C} \) defined by \( f_0(z_1, ..., z_n) = (z_1^2, ..., z_n^2) \) is equivaraint and \( W(f) = q^2 \) because \( f_0 \) has an extension \( f_2: C \rightarrow \mathbb{C} \) given by the formula, is the unique zero of \( f_0 \) and \( \text{deg}_{\mathbb{C}} f_0 = q^2 \).

By 2.1, for any equivariant map \( f: S^{2n-1} \rightarrow \mathbb{C} \), \( W(f) = q^2 \). 

2.10. Remark. In case 2.2 a) if \( G \) is a group, \( j = 1, ..., k \), and \( r_j \) is a number such that \( G_{r_j} \) are subgroups of \( G \) and \( r_j \) is a number such that \( G_{r_j} \rightarrow E_0 \) have \( W(f) = r_j \mod |G| \) for \( j = 1, ..., k \), then the number \( r_j \) of 2.2 a) is uniquely \( \text{mod}|G| \) determined by the numbers \( r_j \). 

2.11. Corollary. Under the conditions of Theorem 2.2, if \( f \) in addition the action of \( G \) on \( E \) is linear, then, for every continuous map \( f: M \rightarrow E_0 \) with \( W(f) = r \mod |G| \) in the concordant case and \( W(f) \neq 0 \) in the discordant case, there is a point \( x \in M \) such that \( O \in \text{conv} \{ f(g^{-1} x) \}_{g \in G} \).

Indeed, if \( O \notin \text{conv} \{ f(g^{-1} x) \}_{g \in G} \), then the map \( f_0: M \rightarrow E_0 \) defined by

\[
f_0(x) = \frac{1}{|G|} \sum_{g \in G} f(g^{-1} x)
\]

is equivariant and homotopic to \( f \) (by the standard homotopy). Therefore \( W(f) = W(f_0) = r \mod |G| \) in the concordant case or \( W(f) = W(f_0) = 0 \) in the discordant case, which contradicts the assumptions.

In particular, if the action of \( G \) on \( E \) is trivial and if \( W(f) = 0 \mod |G| \), then there is a point \( x \in M \) such that \( O \in \text{conv} \{ f(g^{-1} x) \}_{g \in G} \).

In the case of the action of \( Z_r \) on \( M \) we get

2.12. Corollary. Let \( T \) be a fixed point free smooth involution on \( M \) and \( f: M \rightarrow S^r \) a continuous map into the unit sphere in \( E \).

a) If \( f \) has an odd degree, then there is a point \( x \in M \) such that \( f(Tx) = -f(x) \).

b) If \( \text{deg} f \neq \text{mod} 2 \) in the concordant case or \( \text{deg} f \neq 0 \) in the discordant case, then there is a point \( x \in M \) such that \( f(Tx) = f(x) \).

From a) it follows that in the concordant case if \( r = 1 \), then every continuous map \( \psi: M \rightarrow \mathbb{R}^n \) has a point \( x \in M \) such that \( \psi(Tx) = \psi(x) \) because \( \mathbb{R}^n \) is homeomorphic to \( S^n \).

3. Concordant actions. The following theorem gives the equivariant homotopy classification of maps in the concordant case.

3.1. Theorem. Let \( G, M, E, E_0, r \) be as in 2.1 and 2.2 a), i.e. \( M \) is oriented and the actions of \( G \) on \( M \) and \( E \) are concordant. Then the function \( W: (M, E_0) \rightarrow \{ n = r \cdot k \cdot G \} \) assigning to an equivariant homotopy class \( f \) a representative by a continuous equivariant map \( f: M \rightarrow E_0 \), its winding number \( W(f) \), is bijective.

Proof. a) Surjectivity. Let \( f_0: M \rightarrow E_0 \) be equivariant continuous. Such a map exists by 1.6 a). We may assume that \( W(f_0) = r \). Let \( k \) be any integer different from \( 0 \) and \( 0 < a < 1 \). Let \( p_i, i = 1, ..., k \), be points of \( (a, 1) \subset C \) belonging to different orbits of the \( G \)-manifold \( P = \mathbb{R} \times M \), where \( C \) is a component of \( M \). Let \( V_i \) be a slice at \( p_i \) contained in \( (a, 1) \times C \) such that \( gV_i \) are disjoint for \( i = 1, ..., k \) and \( g \in G \). \( V_i \) can be mapped onto \( E \) by a diffeomorphism preserving the orientation if \( k > 0 \) and reversing the orientation if \( k < 0 \). This diffeomorphism can be extended by 1.3 to a \( G \)-good map \( f: U \rightarrow E \), where

\[
U = (R \times [a, 2]) \cup \bigcup_{i=1}^{k} V_i\ 
\]

such that \( f(x) = f_0(x) \) for \( x \in [a, 2] \times M \cup \bigcup_{i=1}^{k} V_i \), and \( x \in M \). By the extension Lemma 1.5 applied to \( P = \mathbb{R} \times M \) and \( F = [0, 2] \times M \cup \bigcup_{i=1}^{k} V_i \), where \( D_i \) are closed discs about \( p_i \) in \( V_i \) there exists a \( G \)-good map \( h: [0, 2] \times M \rightarrow E \) such that \( h(f) = f \).

\( h^{-1}(O) \) consists of points \( p_i \), \( i = 1, ..., k \), \( g \in G \) and additional points \( q_i \), \( j = 1, ..., l \). We may assume by 1.1 that \( q_j \in (1/2 \times M) \). Define \( f_j: M \rightarrow E_0 \) by \( f_j(x) = h(1, x) \). Then the restriction of \( h_i \) to \( f_i \) gives a \( G \)-good homotopy from \( f_0 \) to \( f_i \). For \( i = 1, ..., k \), \( \text{deg} f_j = \text{sgn} k \), and for all \( g \in G \) \( \text{deg} f = \text{sgn} k \), because the actions of \( G \) are concordant. Therefore \( W(f_j) = W(f_0) = k \mod |G| \) and \( W(f_j) = r + k \mod |G| \).

b) Injectivity. Suppose that for equivariant continuous maps \( f_0, f_1: M \rightarrow E_0 \). By the extension Lemma 1.5 there exists a \( G \)-good homotopy \( h: \mathbb{R} \times M \rightarrow E \) from \( f_0 \) to \( f_1 \). If \( h^{-1}(O) \) is nonvoid, let \( h^{-1}(O) \) consist of points \( g \), \( i = 1, ..., k \), \( g \in G \), where \( p_i \in (0, 1) \times C \) and \( C \) is a component of \( M \). From the equalities \( O = W(f_0) - W(f_0) = \sum_{i=1}^{k} \text{deg} f_j \) and \( \text{deg} f_j = \pm 1 \) it follows that \( k \) is even and the points \( p_i \) can be arranged in such a way that \( g \), \( h = -1 \). Let \( V_i \subset (0, 1) \times C \) be a slice at \( p_i \) and \( D \) an open ball about \( p_i \) in \( V_i \). By 1.10 we may assume that \( V_i \cap h^{-1}(O) = D \) and \( h^{-1}(O) = \{ p_i \} \). By the Hopf theorem \( h^{-1}(O) \) can be extended to a continuous map \( f: V \rightarrow E \). By 1.3 there is a \( G \)-good map \( f: \mathbb{R} \times M \rightarrow E \) extending \( f \) and \( h^{-1}(O) \) consists of the orbits of \( p_i \) for \( i > 2 \). Proceeding further similarly, we get an equivariant homotopy \( h: \mathbb{R} \times M \rightarrow E_0 \) from \( f_0 \) to \( f_1 \).

4. Discordant actions. Before formulating the general result in this case we give some examples. We still observe the assumptions of 2.1.

4.1. Example. Let \( M \) be an orientable manifold with a free action of \( G \) not preserving the orientation and let \( G \) act trivially on a linear space \( E \). In this case the space of orbits \( M/G \) is a nonorientable manifold. There is a bijective correspondence between the set of equivariant homotopy classes \( [M, E_0]_G \) and the set of non-equivariant homotopy classes \( [M/G, E_0] \). The Hopf theorem the degree \( \text{mod} 2 \) gives the bijective correspondence \( [M/G, E_0] = Z_2 \) and there are two different equivariant homotopy classes \( [M, E_0]_G \) although the winding number of any equivariant map \( f: M \rightarrow E_0 \) is 0 by 2.2 b). The same is true for equivariant maps \( f: M \rightarrow S^n \).
In particular, there are two equivariant homotopy classes if $M$ is an even-dimensional sphere with the action of $\mathbb{Z}_2$ by antipodism ($\mathbb{M}/\mathbb{Z}_2$ is the nonorientable projective space) or if $M$ is an orientable surface of genus $g$ lying symmetrically with respect to $O$ in $\mathbb{R}^3$ with the action of $\mathbb{Z}_2$ by symmetry with respect to $O$ ($\mathbb{M}/\mathbb{Z}_2$ is the nonorientable surface of genus $g+1$).

4.2. Example. Let $M$ be a $G$-manifold having a compact fundamental set in the sense of [7] and let $G$ act trivially on $E$. There is a bijective correspondence between $[M, E]_G$ and $[F, E]_G$. If $F$ is contractible, then there is only one homotopy class in $[F, E]$ and in $[M, E]_G$.

In particular, there is one equivariant homotopy class if $M$ is the unit sphere of any orthogonal representation of $G$ on $V$ whose singular part is a union of hyperplanes (Corollary 9 in [7]). This is the case for the symmetry group of any Platonic polyhedron.

In the case of discordant actions of $G$ on $M$ and $E$ the group $G$ is the disjoint union of the subgroup $G_+$ and its coset $G_-$, where the actions of $G_+$ on $M$ and $E$ are concordant. The number $|G|$ is even. The equivariant homotopy classification of maps in this case gives

4.3. Theorem. Let $G$, $M$, $F$, $E_0$ be as in 2.1, $M$ being orientable, and let the actions of $G$ on $M$ and $E$ be discordant. Denote by $M' = M 	imes M$, the singular part of $M$. Let $\dim M' = m - 1$ then $[M, E]_G$ consists of one class.

b) If $\dim M' = m - 1$ then $[M, E]_G$ consists of two classes.

Proof. a) Let $f_0, f_1: M 	o E_0$ be any equivariant maps (such maps exist by (1.6)). By the extension Lemma 1.5 there is a $G$-good homotopy $h: I \times M \to E$ from $f_0$ to $f_1$. The singular part $(0, 1) \times M$ of $h^m$-dimensional manifold $P = (0, 1) \times M$ has dimension $m$. Therefore there exists a $g_0 \in \mathcal{O}_G$ such that the fixed set $P_{g_0}$ of $g_0$ has a component $Q$ which is an $m$-dimensional submanifold of $P$. On a slice at any point from $Q$, $g_0$ acts by symmetry with respect to a hyperplane, and so $g_0^2 = e$. For any $g \in G$ different from $e$ and $g_0$ the intersection $Q \cap P^G$ is a finite union of manifolds of dimensions less than $m$. Therefore there exists a point $x_0 \in Q$ with the isotropy group $G_{x_0} = \{e, g_0\}$. Let $V$ be a slice at $x_0$. $V$ may be identified with $\mathbb{R}^{m-1} \times S^1$, and there is a $G$-invariant $h^i$-plane $H$ by the equation $x_{m-1} = 0$. $g_0$ acts on $V$ by symmetry with respect to this hyperplane. Let $D$ be the unit open ball in $V$, $V' = \{x \in V : x_{m-1} > 0\}$, $V'' = \{x \in V : x_{m-1} < 0\}$ and let $C$ be the component of $D$ containing Int $V'$. If $h^i$ is not a nonempty, it is a finite invariant subset of $P$. There exists a point $p \in C \cap h^i$. By 1.10 we can assume that $V' \cap h^i = D \cap C \cap h^i = \{p\}$. There exists a continuous retraction $r: V' \to V' \backslash D$. Define the map $f: V' \to E_0$ by

$$f(x) = h^i(r(x))$$

for $x \in V'$, $f(x) = h^i(rg_0(x))$ for $x \in V''$.

$f$ is $G_{x_0}$-equivariant because $g_0$ acts trivially on $E$. By 1.3 there is a $G$-good map $k: I \times M \to E$ extending $f$ and $h(f, k) = h^i$. The number of orbits in $h^i$ is less by 1 than that in $h^i$. By a similar procedure we get an equivariant homotopy $h^i: I \times M \to E_0$ from $f_0$ to $f_1$.

b) The condition $\dim M' < m - 1$ implies that $E_0$ is connected. Fix some equivariant map $f_0: M \to E_0$ and a $G$-good homotopy $H: I \times M \to E$ from $f_0$ to $f_1$, such that $H^i$ consists of exactly one orbit. We shall prove that $[M, E]_G$ consists of two different classes $[f_0]$ and $[f_1]$.

Let $f: M \to E_0$ be any continuous equivariant map. As in b) of the proof of 3.1, there is a $G$-good homotopy $h: I \times M \to E$ from $f_0$ to $f_1$. Suppose that $h^i$ contains more than one orbit. Since the actions of $G$ are discordant, for any $p \in h^i$ of $|G| = \deg h = \deg g$, $g \in G \backslash G_+$, $\deg h = -\deg h$ if $g \in G_+$, and $\deg h = \pm 1$. Let $V$ be a slice at $p_1$ and let $D$ be an open unit ball in $V$. By 1.10 we may assume that $V \cap h^i = D \cap h^i = \{p_1, p_2\}$.

because $P_2 = (0, 1) \times M_2$ is connected. As in b) of the proof of 3.1, we can modify $h$ to a $G$-good homotopy $h$ from $f_0$ to $f_1$ so that $h^i$ consists of two orbits. Similarly, $h$ can be modified to a $G$-good homotopy $h': I \times M \to E_0$ from $f_1$ to $f_2$, so that $h' = [f_1]$.

It remains to prove that the classes $[f_0]$ and $[f_1]$ are different. We have the $G$-good homotopy $H_0: I \times M \to E_0$ from $f_0$ to $f_1$ with $h^i$ of $|G|$ consisting of one orbit. Suppose, on the contrary, that there exists also a continuous equivariant homotopy $H_1: I \times M \to E_0$ from $f_0$ to $f_1$. The homotopies $H_0$ and $H_1$ may be considered as $G$-good maps on the manifold without boundary $K \times M$ and we can suppose that there are numbers $0 < a < b < 1$ such that $H_0(t, x) = H_1(t, x) = f_0(x)$ for $t < a$ and $H_0(t, x) = H_1(t, x) = f_1(x)$ for $t > b$. The $G$-manifold $P = K \times K \times M$ has the singular part $P = K \times K \times M$ and $\dim P = m$ by the assumption of $\dim E < m - 2$. Extension Lemma 1.5 applied to $P, F = P \times [0, 1) \times (0, 1) \times M, U = P \times [0, 1) \times [a, b] \times M$ and to the $G$-good map $H: U \to E$ defined by

$$H(s, t, x) = \begin{cases} H_0(t, x) & \text{if } s < a, \\ H_1(t, x) & \text{if } s > b, \\ f_0(x) & \text{if } t < a, \\ f_1(x) & \text{if } t > b. \end{cases}$$

gives a $G$-good map $H: P \to E$ extending $H|F$.

The set $L = \{H^i \cap I \times (0, 1) \times M\}$ is a compact 1-dimensional invariant submanifold of $P$, whose boundary is the orbit $[O] \times H^i$. So $L$ is the disjoint union of arcs $L_i$, $i = 1, \ldots, |G|/2$ and a finite number of closed curves. The union $\hat{L}$ of arcs $L_i$ is invariant. The subgroup $G_1$ of $G$ consisting of elements preserving $L_1$ consists of two elements. Let $g \in G_1 \backslash G$. By the Brouwer fixed point theorem there
exists an \( x \in L \subset P_0 \) such that \( gx = x \). But this is impossible because the action of \( G \) on \( P_0 \) is free.

5. The nonorientable case. For a nonorientable manifold \( M \) we have the following equivariant homotopic classification of maps.

5.1. Theorem. Let \( G, M, E, E_0 \) be as in 2.1 and let \( M \) be nonorientable. Let \( M' \) be the singular part of \( M \).

a) If \( G \) is odd, then the function \( W_2: [M, E_0]_0 \to \mathbb{Z}_2 \), assigning to an equivariant homotopy class \([f]\) represented by a continuous equivariant map \( f: M \to E_0 \) its winding number \( \text{mod} 2 \), is bijective.

b) If \( |G| \) is even and \( \dim M' = m - 1 \), then \( W_2(f) = 0 \) for every equivariant map \( f: M \to E_0 \) and \([M, E_0]_0\) consists of one class.

c) If \( G \) is even and \( \dim M' < m - 1 \), then, for all equivariant maps \( f: M \to E_0 \), \( W_2(f) \) is the same and \([M, E_0]_0\) consists of two classes.

The proof of a) is similar to the proof of 3.1, using 1.9 and the fact that \( M_0 \) is connected.

In cases b) and c) \( W_2(f) \) are independent of \( f \) by arguments as in the proof of 2.2 a).

In case b) \( G \) contains an isotropy group \( G_0 \) of the action on \( M \) of rank 2, which acts trivially on \( E \). So the constant map is \( G_0 \)-equivariant and \( W_2(f) = W_2(\text{const}) = 0 \) by the preceding remark. The proof of the rest of b) is analogous to that of 4.3a).

The proof of c) is similar to that of 4.3b).

It can be seen by examples that all the cases in Theorem 5.1 are possible (in c) the winding number \( \text{mod} 2 \) may be 0 and 1).

5.2. Let \( G, M, E, E_0 \) be as in 5.1. In addition let \( G \) act on \( E \) preserving the orientation. Denote by \( \tilde{M} \) the double orientation covering manifold of \( M \). The points of \( \tilde{M} \) can be thought of as the orientations of the tangent spaces \( TM \). The action of \( G \) on \( M \) lifts to the orientation-preserving action of \( G \) on \( \tilde{M} \). For \( g \in G \) and an orientation \( o \) of \( TM \), \( g o \) is the image of the orientation \( o \) by the tangent map \( dg \). (Comp. [1], I, 9.4). Let \( T: \tilde{M} \to \tilde{M} \) be the involution on \( \tilde{M} \) mapping an orientation \( o \) of \( TM \) into the opposite orientation \( -o \) of \( TM \). \( T \) commutes with the action of \( G \) on \( \tilde{M} \) and reverses the orientation of \( \tilde{M} \). Let \( \pi: \tilde{M} \to M \) be the covering projection.

The concordant actions of \( G \) on \( \tilde{M} \) and \( E \) satisfy the assumption of Theorem 2.2a) and every equivariant map \( f: \tilde{M} \to E_0 \) has the winding number \( W(f) \equiv 0 \text{mod} |G| \).

For the proof let \( g_0: M \to E_0 \) be any equivariant map. Set \( f_0 = f_0 \circ T \). We have \( W(f_0) = -W(f_0) \) and therefore \( W(f_0) = 0 \). Then the result is a consequence of 2.2a).

References