On the components of the principal part of a manifold with a finite group action

by

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Abstract. For an effective smooth action of a finite group G on a closed connected manifold M the following questions are examined.

1. When is the closure of a component of a principal part of M a topological manifold with boundary?

2. When does there exist a closed set containing exactly one point from every orbit?

Let M be a closed connected smooth m-dimensional manifold with a smooth effective action of a finite group G. For every point $x \in M$ there is a slice V at x diffeomorphic to R^m with an orthogonal action of the isotropy group G_x. In the sequel V will be identified with R^m. It is known ([1] or [2]) that there is a smallest conjugacy class of isotropy groups called *principal groups*. In the case of an effective action of a finite group G on a connected manifold M the unique isotropy group is the trivial subgroup $\{e\}$ of G because for every $x \in M$ the points of a slice V at x with principal isotropy groups have the same isotropy group since the action of G_x on V is linear. The open and dense subset of M consisting of points with the trivial isotropy group is called the *principal part of M* and will be denoted by M_e. Its complement $M' = M \setminus M_e$ will be called the *singular part of M*.

Let M^g be the set of fixed points of the diffeomorphism of M corresponding to the element $g \in G$. The components of M^g are closed submanifolds of M and $M' = \bigcup_{g \in G\setminus\{e\}} M^g$. If $\dim M' < m-1$ or equivalently, for every $g \in G\setminus\{e\}$, the dimension of each component of M^g is less than $m-1$, then M_e is connected because M' does not separate any slice. If $\dim M^g = m-1$, then g is of order 2 because in a slice at a point of a component of M^g of dimension $m-1$ g acts as symmetry with respect to a hyperplane. If M is orientable, then such a g reverses the orientation of M. Therefore if M is orientable and G preserves orientation, then $\dim M' < m-1$ and M_e is connected.

It may happen that M_e is connected and $\dim M' = m-1$.

1. Example. Let M be the real projective plane P_2 with the action of Z_2 induced by the action of Z_2 in R^3 for which the generator g of Z_2 acts by symmetry with respect to a plane or equivalently by symmetry with respect to the orthogonal

line. M' is the union of the circle P_1 and an isolated point, and so $\dim M' = 1 = m-1$. M_e is homeomorphic to an open punctured disc, and therefore is connected. M^g has components of different dimensions. If we take $M = P_3$ instead of P_2, we get an example of an orientable manifold with the same properties.

Let C be any component of M_e. The space M_e/G of orbits of M_e is connected ([1] or [2]), and so $M_e = \bigcup_{g\in G} gC$. More precisely, we have

2. PROPOSITION. *If $\tilde{G}$ is the subgroup of G generated by the elements* (of order 2) *for which* $\dim M^g = m-1$, *then $\tilde{G}$ acts transitively on the family of components of M_e. For any component C of M_e, $M_e = \bigcup_{g\in\tilde{G}} gC$.*

Proof. We shall say that two components, C and C', of M_e are adjacent iff $\dim \bar{C} \cap \bar{C}' = m-1$ or equivalently iff there exists a point $x \in \bar{C} \cap \bar{C}'$ such that $G_x = \{e, g\}$ and g acts in the slice V at x by symmetry with respect to the hyperplane V^g, one open half-space is contained in C and the other in C'. For a given component C of M_e let $\mathscr{C}$ be the family of all components C' of M_e such that there exists a sequence of components C_i of M_e, $i = 0, 1, \ldots, k$ with C_{i-1} adjacent to C_i for $i = 1, \ldots, k$, $C_0 = C$ and $C_k = C'$. We shall prove that $\mathscr{C}$ contains all components of M_e. Suppose that this is not true. Let $P_1 = \bigcup_{C'\in\mathscr{C}} C'$, $P_2 = \bigcup_{C'\notin\mathscr{C}} C'$, and let N be the union of components of M^g of dimensions less than $m-1$ for all $g \in G$. Then $\bar{P}_1 \cap \bar{P}_2 \subset N$ and the connected set $M \setminus N$ is the disjoint union of nonvoid sets $\bar{P}_1 \setminus N$ and $\bar{P}_2 \setminus N$ closed in $M \setminus N$, which is impossible. Therefore for every component C' of M_e there exists a sequence C_i, $i = 0, 1, \ldots, k$ from the definition of $\mathscr{C}$ and a sequence of $g_i \in \tilde{G}$, $i = 1, \ldots, k$ such that $C_i = g_i C_{i-1}$. Consequently $C' = gC$ for $g = g_k \ldots g_1 \in \tilde{G}$, which completes the proof.

The closure of a component of M_e is not always a topological manifold with boundary.

3. EXAMPLE. Let G be the symmetry group of a regular n-polygon on the plane R^2 with n odd. G acts orthogonally also on $R^2 \times R$, the action on R being trivial, and on the projective plane $P_2 = M$. M_e has n components. Any component C is homeomorphic to an open punctured disc, but $\bar{C}$ is homeomorphic to a closed disc with two points from the boundary identified. The same G acts on the orientable manifold P_3 with similar conclusions.

Next proposition gives conditions under which the closures of components of M_e are topological manifolds.

4. PROPOSITION. *For a component C of M_e the following conditions are equivalent:*

a) *$\bar{C}$ is a topological manifold with the boundary $\partial\bar{C} = \mathrm{Fr}\,C$.*

b) *For every $x \in \mathrm{Fr}\,C$ the local graded singular homology group with integer coefficients $H(C \cup \{x\}, C)$ is trivial.*

c) *For every $x \in \mathrm{Fr}\,C$ if V is a slice at x, then $V' = V \setminus V_e$ is a union of hyperplanes (of dimension $m-1$) and $V \cap C$ is a component of V_e.*

d) *$M' = M \setminus M_e$ is a union of $(m-1)$-dimensional manifolds and for every $x \in M'$ and every neighbourhood U of x there is a neighbourhood of x $V \subset U$ such that different components of V_e are contained in different components of M_e.*

e) *$\mathrm{Int}\,\bar{C} = C$ and no point $x \in \mathrm{Fr}\,C$ separates any connected neighbourhood U of x in $C \cup \{x\}$.*

Proof. a) $\Rightarrow$ b). Every $x \in \partial\bar{C} = \mathrm{Fr}\,C$ has a neighbourhood U in $C \cup \{x\}$ homeomorphic to an open half-space with one point on the boundary added. By excision $H(C \cup \{x\}, C) \approx H(U, U \setminus \{x\})$ is trivial in all dimensions.

b) $\Rightarrow$ c). Suppose that V' is not a union of hyperplanes. Then there exist some $y \in V'$ and a slice U at y such that $U' = U \setminus U_e$ is a linear subspace of U of dimension $k \leqslant m-2$. We can assume that the connected set U_e is contained in C because all components of M_e are diffeomorphic by Proposition 2. Then by excision $H_{m-k}(C \cup \{y\}, C) \approx H_{m-k}(U_e \cup \{y\}, U_e) \approx Z$ because the pair $(U_e \cup \{y\}, U_e)$ has the homotopy type of $(R^{m-k}, R^{m-k} \setminus \{0\})$. This is a contradiction.

By excision, the exact homology sequence of the pair $((V \cap C) \cup \{x\}, V \cap C)$ and the contractibility of the cone $(V \cap C) \cup \{x\}$, it follows that

$$0 \approx H_1(C \cup \{x\}, C) \approx H_1((V \cap C) \cup \{x\}, V \cap C) \approx \tilde{H}_0(V \cap C),$$

and so $V \cap C$ is a component of V_e.

c) $\Rightarrow$ a). For $x \in \mathrm{Fr}\,C$ and a slice V at x, $\bar{C} \cap V$ is a polyhedral cone homeomorphic to a closed half-space.

c) $\Leftrightarrow$ d) is obvious because every neighbourhood of x contains a slice at x.

c) $\Leftrightarrow$ e) results from the following facts:

$\mathrm{Int}\,\bar{C} = C$ iff for any slice V the singular part V' is a union of hyperplanes.

For $x \in \mathrm{Fr}\,C$ and a slice V at x the point x does not separate $(V \cap C) \cup \{x\}$ iff $V \cap C$ is a component of V_e.

Every neighbourhood U of $x \in \mathrm{Fr}\,C$ in $C \cup \{x\}$ contains a neighbourhood $(V \cap C) \cup \{x\}$ for some slice V at x.

Thus the proof is completed.

5. DEFINITION. A subset F of M is called a *fundamental set* iff each orbit has in F exactly one point.

(This definition is not generally accepted).

It is evident that in the above sense fundamental sets always exist. A fundamental set cannot be an open set, by the existence of slices if M' is not void and by the connectedness of M if the action is free (unless G is trivial).

The question arises when there exists a closed (or equivalently compact) fundamental set. In the case of a free action such a fundamental set does not exist because M is connected.

6. PROPOSITION. *Let C be a component of M_e and let G_C be the subgroup of G preserving C. On M there is a closed fundamental set iff G_C is trivial. In this case the sets $g\bar{C}$ for $g \in G$ are all possible closed fundamental sets and $G = \tilde{G}$* (comp. Proposition 2), *i.e. G is generated by the elements $g \in G$ (of order 2 and reversing orientation if M is orientable) such that $\dim M^g = m-1$.*

Proof. Suppose that F is a closed (compact) fundamental set. It is homeomorphic by the canonical map to the space of orbits M/G. Because M_e/G is connected, the set $C = M_e \cap F$ is connected and closed in M_e, and M_e is the disjoint union of gC for $g \in G$. The set $\bigcup_{g \neq e} gC$ is closed in M_e because G is finite and its complement C in M_e is an open component of M_e. The set F is closed and M_e/G is dense in M/G, and so $\bar{C} = F$. This proves that $G_C = \{e\}$ and $g\bar{C}$ are all possible closed fundamental sets.

Suppose that G_C is trivial. Let x be any point of $\bar{C}$. If $x \in C$, then for $g \in G \setminus \{e\}$ gx belongs to another component of M_e and $gx \notin \bar{C}$. Let $x \in \operatorname{Fr} C$. Suppose that there exists a $g_0 \in G$ such that $g_0 x \neq x$ and $g_0 x \in \bar{C}$. Let V be a slice at x. By Proposition 2 applied to the unit sphere in V with the action of G_x it follows that for any two components of V_e some element of G_x maps one of them onto the other. Therefore for $C \cap V$ and $g_0^{-1}(C \cap g_0 V)$ (which are unions of components of V_e) there is a $g_1 \in G_x$ such that $g_1(C \cap V) \cap g_0^{-1}(C \cap g_0 V) \neq \emptyset$. It follows that $g_0 g_1 (C \cap V) \cap (C \cap g_0 V) \neq \emptyset$ and $g_0 g_1 = e$ because $G_C = \{e\}$. On the other hand, $x = g_0 g_1 x = g_0 x \neq x$, which is a contradiction. We have proved that $\bar{C}$ does not contain two points of one orbit. Because M_e is dense and G is finite, we have $M = \bar{M}_e = \bigcup_{g \in G} g\bar{C}$ and hence $\bar{C}$ contains exactly one point from every orbit. Therefore $\bar{C}$ is a closed fundamental set. Because gC are different for $g \in G$, from Proposition 2 it follows that $G = \tilde{G}$. Thus the proof is completed.

In the case of a free action or, more generally, when M_e is connected (e.g. if $\dim M' < m-1$ or if M is orientable and G preserves orientation) there are no closed fundamental sets (unless G is trivial).

7. Corollary. *If there exists a closed fundamental set F, then it is a topological manifold with boundary $\partial F = \operatorname{Fr} F$. ($F = \bar{C}$ and $\operatorname{Fr} C = \operatorname{Fr} F$ for some component C of M_e).*

This follows from Propositions 6 and 4 because conditions 4c) are satisfied.

The converse of Corollary 7 may be false e.g. for free actions. But there is a case where the converse is true.

8. Proposition. *If a component C of M_e has the closure $\bar{C}$ homeomorphic to a disk and $\partial\bar{C} = \operatorname{Fr} C$, then $\bar{C}$ is a closed fundamental set (the word disk can be replaced also by any topological manifold with boundary which has the fixed point property with respect to homeomorphisms).*

Proof. By Proposition 6 it is sufficient to prove that G_C is trivial. We shall proceed by induction with respect to dimension m of M.

Assume that the theorem is true for manifolds of dimensions less than m. Suppose that G_C is not trivial. Let $g \in G_C \setminus \{e\}$. By the Brouwer fixed point theorem there is an $x \in \bar{C}$, such that $gx = x$. We have $x \notin C$ because $C \subset M_e$, and so $x \in \operatorname{Fr} C = \partial\bar{C}$. Let V be a slice at x and S the unit sphere in V. By Proposition 4, condition c) it follows that $C \cap S$ is a component of the principal part S_e of the $(m-1)$-dimensional sphere S with the action of G_x: $\overline{C \cap S}$ is homeomorphic to a disk and $\partial\overline{C \cap S} = \operatorname{Fr}(C \cap S)$ in S. Since $g(C \cap S) = C \cap S$, by inductive hypothesis $g = e$ and this is a contradiction.

9. Corollary. *Let V be an orthogonal effective representation of a finite group G and S the unit sphere in V. For the action of G on S there is a closed fundamental set iff the singular part $V' = V \setminus V_e$ of V is a union of hyperplanes. In that case any fundamental set is homeomorphic to a disk and the action of G on the family of $|G|$ fundamental sets is transitive and free.*

The assumption $\partial\bar{C} = \operatorname{Fr} C$ in Proposition 8 is essential.

10. Example. Let $G = Z_2 \times Z_2$ with the standard generators g_1, g_2 act on the 2-dimensional unit sphere $M = S^2 \subset R^3$ in such a way that g_1 is the antipodism and g_2 the symmetry with respect to a plane. Then there are two components of the principal part of M and their closures are hemi-spheres. But there are no closed fundamental sets because $|G_C| = 2$. (G_C contains e and symmetry with respect to a line $g_1 g_2$).

In general, closed fundamental sets are not necessarily homeomorphic to disks, as is seen on the torus in R^3 with the action of Z_2, the generator acting by symmetry with respect to a plane. In this case the closed fundamental set is homeomorphic to a ring.

If there exists a closed fundamental set, then it is homeomorphic to the space of orbits. In Examples 1 or 10 the space of orbits is homeomorphic to a disk, but there are no closed fundamental sets.

11. Remark. In fact if a component C of M_e is homeomorphic to R^m then $\bar{C}$ is a fundamental set.

This follows from the known result that a nontrivial finite group cannot act freely on R^m (which is a consequence of the Smith theorem [1]). Indeed, G_C acts freely on C, and so G_C is trivial and $\bar{C}$ is fundamental set by Proposition 6.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[2] K. Jänich, *Differenzierbare G-Mannigfaltigkeiten*, Lect. Notes in Math. 59, Springer-Verlag, 1968.

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