On the components of the principal part of a manifold with a finite group action

by

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Abstract. For an effective smooth action of a finite group \( G \) on a closed connected manifold \( M \) the following questions are examined.
1. When is the closure of a component of a principal part of \( M \) a topological manifold with boundary?
2. When does there exist a closed set containing exactly one point from every orbit?

Let \( M \) be a closed connected smooth \( m \)-dimensional manifold with a smooth effective action of a finite group \( G \). For every point \( x \in M \) there is a slice \( V \) at \( x \) diffeomorphic to \( \mathbb{R}^n \) with an orthogonal action of the isotropy group \( G_x \). In the sequel \( V \) will be identified with \( \mathbb{R}^n \). It is known ([1]) or [2]) that there is a smallest conjugacy class of isotropy groups called principal groups. In the case of an effective action of a finite group \( G \) on a connected manifold \( M \) the unique isotropy group is the trivial subgroup \( \{ e \} \) of \( G \) because for every \( x \in M \) the points of a slice \( V \) at \( x \) with principal isotropy groups have the same isotropy group since the action of \( G_x \) on \( V \) is linear. The open and dense subset of \( M \) consisting of points with the trivial isotropy group is called the principal part of \( M \) and will be denoted by \( M' \). Its complement \( M'' = M \setminus M' \) will be called the singular part of \( M \).

Let \( M^g \) be the set of fixed points of the diffeomorphism of \( M \) corresponding to the element \( g \in G \). The components of \( M^g \) are closed submanifolds of \( M \) and \( M' = \bigcup_{g \in \mathbb{G} \setminus \{ e \}} M^g \). If \( \dim M' < m - 1 \) or equivalently, for every \( g \in G \setminus \{ e \} \), the dimension of each component of \( M^g \) is less than \( m - 1 \), then \( M' \) is connected because \( M' \) does not separate any slice. If \( \dim M^g = m - 1 \), then \( g \) is of order 2 because in a slice at a point of a component of \( M^g \) of dimension \( m - 1 \) \( g \) acts as symmetry with respect to a hyperplane. If \( M \) is orientable, then such a \( g \) reverses the orientation of \( M \). Therefore if \( M \) is orientable and \( G \) preserves orientation, then \( \dim M' < m - 1 \) and \( M' \) is connected.

It may happen that \( M' \) is connected and \( \dim M'' = m - 1 \).

Example. Let \( M \) be the real projective plane \( \mathbb{P} \) with the action of \( \mathbb{Z}_3 \) induced by the action of \( \mathbb{Z}_2 \) in \( \mathbb{R}^3 \) for which the generator \( g \) of \( \mathbb{Z}_3 \) acts by symmetry with respect to a plane or equivalently by symmetry with respect to the orthogonal
line. $M'$ is the union of the circle $P_1$ and an isolated point, and so $\dim M' = m - 1$. $M'_n$ is homeomorphic to an open punctured disc, and therefore is connected. $M^\circ$ has components of different dimensions. If we take $M = P_2$ instead of $P_3$, we get an example of an orientable manifold with the same properties.

Let $C$ be any component of $M'_n$. The space $M'_n \setminus C$ of orbits of $M'_n$ is connected ([1] or [2]), and so $M'_n = \bigcup g C$. More precisely, we have $M'_n = \bigcup g C$.

2. Proposition. If $G$ is the subgroup of $G$ generated by the elements of $C$ (of order 2) for which $\dim M' = m - 1$, then $G$ acts transitively on the family of components of $M'_n$. For any component $C$ of $M'_n$, $M'_n = \bigcup g C$.

Proof. We shall say that two components, $C$ and $C'$, of $M'_n$ are adjacent iff $\dim C \cap C' = m - 1$ or equivalently there exists a point $x \in C \cap C'$ such that $G = \{ e, g \}$ and $g$ acts in the slice $V$ at $x$ by symmetry with respect to the hyperplane $V'$, one open half-space is contained in $C$ and the other in $C'$. For a given component $C$ of $M'_n$ let $G$ be the family of all components $C'$ of $M'_n$ such that there exists a sequence of components $C_i$ of $M'_n$, $i = 0, 1, \ldots, k$ with $C_{i-1}$ adjacent to $C_i$ for $i = 1, \ldots, k$, $C_0 = C$ and $G_i = C_i$. We shall prove that $G$ contains all components of $M'_n$. Suppose that this is not true. Let $P_1 = \bigcup G_i$, $P_2 = \bigcup G_i$, and let $N$ be the union of components of $M'_n$ of dimensions less than $m - 1$ for all $g \in G$. Then $P_1 \cap P_2 \in N$ and the connected set $M \setminus N$, the disjoint union of nonvoid sets $P_1 \setminus N$ and $P_2 \setminus N$ closed in $M \setminus N$, is impossible. Therefore for every component $C'$ of $M'_n$ there exists a sequence $C_i$ of $C_i$, $i = 0, 1, \ldots, k$ from the definition of $G$ and a sequence of $g_i \in G$, $g_1 = g_2, \ldots, g_1 \in G$, $C_i = g_i C_{i-1}$. Consequently $C_i = g C$ for $g = g_1 \ldots g_k$. Complete the proof.

3. Example. Let $G$ be the symmetry group of a regular $n$-polygon on the plane $\mathbb{R}^2$, $n$ odd. $G$ acts orthogonally also on $\mathbb{R}^2 \times \mathbb{R}$, the action on $\mathbb{R}$ being trivial, and on the projective plane $P_2 = M'_n$. $M'_n$ has $n$ components. Any component $C$ is homeomorphic to an open punctured disc, but $C$ is homeomorphic to a closed disc with two points from the boundary identified. The same $G$ acts on the orientable manifold $P_2$ with similar conclusions.

Next proposition gives conditions under which the cycles of components of $M'_n$ are topological manifolds.

4. Proposition. For a component $C$ of $M'_n$ the following conditions are equivalent:

a) $C$ is a topological manifold with the boundary $\partial C = \text{Fr} C$.

b) For every $x \in \text{Fr} C$ the local graded singular homology group with integer coefficients $H^i(C) \cup \{ x \}$ is trivial.

c) For every $x \in \text{Fr} C$ if $V$ is a slice at $x$, then $V' = V \setminus V$ is a union of hyperplanes of dimension $m - 1$ and $V \cap C$ is a component of $V$.

d) $M' = M'_nM'_n$ is a union of $(m-1)$-dimensional manifolds and for every $x \in M'$ and every neighbourhood $U$ of $x$ there is a neighbourhood of $x \subset U$ such that different components of $V_x$ are contained in different components of $M'_n$.

e) $\text{Int} C = C$ and no point $x \in \text{Fr} C$ separates any connected neighbourhood $U$ of $x$ in $C \cup \{ x \}$.

Proof. a) $\Rightarrow$ b). Every $x \in \text{Fr} C = \text{Fr} C$ has a neighbourhood $U$ in $C \cup \{ x \}$ homeomorphic to an open half-space with one point on the boundary added. By excision $H(C \cup \{ x \}) \approx H(U \setminus \{ x \})$ is trivial in all dimensions.

b) $\Rightarrow$ c). Suppose that $V'$ is not a union of hyperplanes. Then there exist some $x \in V'$ and a slice $U$ at $x$ such that $U \setminus U = U'$ is a linear subspace of $U$ of dimension $k \leq m - 2$. We can assume that the connected set $U_2$ is contained in $C$ because all components of $M'_n$ are diffeomorphic by Proposition 2. Then by excision $H(U_2 \cup \{ x \}, C) \approx H(U_2 \cup \{ x \}, U_2) \approx Z$ because the pair $(U_2 \cup \{ x \}, U_2)$ has the homotopy type of $(\mathbb{R}^{m-k}, \mathbb{R}^{m-k} \setminus \{ 0 \})$. This is a contradiction.

By excision, the exact homology sequence of the pair $(V' \setminus C \cup \{ x \}, V \cap C)$ and the contractibility of the cone $(V \cap C) \cup \{ x \}$, it follows that $0 \approx H_2(C \cup \{ x \}, C) \approx H_2((V \cap C) \cup \{ x \}, V \cap C) \approx B_2(V \cap C)$, and so $V \cap C$ is a component of $V_x$.

c) $\Rightarrow$ d). For $x \in \text{Fr} C$ and a slice $V$ at $x$, $C \cap V$ is a polyhedral cone homeomorphic to a closed half-space.

d) $\Rightarrow$ e). Obvious because every neighbourhood of $x$ contains a slice at $x$.

e) $\Rightarrow$ a) results from the following facts:

\begin{itemize}
  \item $\text{Int} C = C$ iff for any slice $V'$ the singular part $V'$ is a union of hyperplanes.
  \item For $x \in \text{Fr} C$ and a slice $V$ at $x$ the point $x$ does not separate $(V \cap C) \cup \{ x \}$ iff $V' \cap C$ is a component of $V_x$.
\end{itemize}

Every neighbourhood $U$ of $x \in \text{Fr} C$ in $C \cup \{ x \}$ contains a neighbourhood $(V \cap C) \cup \{ x \}$ for some slice $V$ at $x$.

Thus the proof is completed.

5. Definition. A subset $F$ of $M$ is called a fundamental set iff each orbit has in $F$ exactly one point.

(Thus, this definition is not generally accepted.)

It is evident that in the above sense fundamental sets always exist. A fundamental set cannot be an open set, by the existence of slices if $M$ is not void and by the connectedness of $M$ if the action is free (unless $G$ is trivial).

The question arises when there exists a closed (or equivalently compact) fundamental set. In the case of a free action such a fundamental set does not exist because $M$ is connected.

6. Proposition. Let $C$ be a component of $M'_n$ and let $G$ be the subgroup of $G$ preserving $C$. On $M$ there is a closed fundamental set iff $G$ is trivial. In this case the sets $G \setminus g G$ for $g \in G$ are all possible closed fundamental sets and $G = G$ (comp. Proposition 2), i.e. $G$ is generated by the elements $g \in G$ (of order 2 and reversing orientation if $M$ is orientable) such that $\dim M' = m - 1$. 

Proof. Suppose that $F$ is a closed (compact) fundamental set. It is homeomorphic by the canonical map to the space of orbits $M/G$. Because $M/G$ is connected, the set $C = M_e \cap F$ is connected and closed in $M_e$, and $M_e$ is the disjoint union of $gC$ for $g \in G$. The set $\cup gC$ is closed in $M_e$ because $G$ is finite and its complement $C^*$ is open in $M_e$. The set $F$ is closed and $M_e/G$ is dense in $M/G$, and so $C = F$. This proves that $C^* = \{e\}$ and $gC$ are all possible closed fundamental sets.

Suppose that $G_e$ is trivial. Let $x$ be any point of $C$. If $x \in C$, then for $g \in G \setminus \{e\}$, $g x$ belongs to another component of $M_e$ and $g x \notin C$. Let $x \in Fr C$. Suppose that there exists a $g_0 \in G$ such that $g_0 x \neq x$ and $g_0 x \in C$. Let $V$ be a slice at $x$. By Proposition 2 applied to the unit sphere in $V$ with the action of $G_e$, it follows that for any two components of $V$, some element of $G_e$ maps one of them onto the other. Therefore for $C \cap V$ and $g_0^{-1}(C \cap V)$ (which are unions of components of $V$) there is a $g_1 \in G_e$ such that $g_1 (C \cap V) \cap g_0^{-1}(C \cap V) \neq \emptyset$. It follows that $g_1 g_0 (C \cap V) \cap (C \cap g_0 V) \neq \emptyset$ and $g_0 g_1 = e$ because $G_e = \{e\}$. On the other hand, $x = g_0 g_1 x = g_0 x \neq x$, which is a contradiction. We have proved that $C$ does not contain two points of one orbit. Because $M_e$ is dense and $G$ is finite, we have $M = M_e = \cup gC$ and hence $C$ contains exactly one point from each orbit.

Therefore $C$ is a closed fundamental set. Because $gC$ are different for $g \in G$, from Proposition 2 it follows that $G = G_e$. Thus the proof is completed.

In the case of a free action or, more generally, when $M_e$ is connected (e.g. if $dim M_e < m-1$ or if $M$ is orientable and $G$ preserves orientation) there are no closed fundamental sets (unless $G$ is trivial).

7. Corollary. If there exists a closed fundamental set $F$, then it is a topological manifold with boundary $\partial F = Fr F$. ($F = C$ and $Fr C = Fr F$ for some component $C$ of $M_e$).

This follows from Propositions 6 and 4 because conditions 4e) are satisfied.

The converse of Corollary 7 may be false e.g. for free actions. But there is a case where the converse is true.

8. Proposition. If a component $C$ of $M_e$ has the closure $C$ homeomorphic to a disk and $\partial C = Fr C$, then $C$ is a closed fundamental set (the word disk can be replaced also by any topological manifold with boundary which has the fixed point property with respect to homeomorphisms).

Proof. By Proposition 6 it is sufficient to prove that $G_e$ is trivial. We shall proceed by induction with respect to dimension $m$ of $M$.

Assume that the theorem is true for manifolds of dimensions less than $m$. Suppose that $G_e$ is not trivial. Let $y \in G_e \setminus \{e\}$. By the Brouwer fixed point theorem there is an $x \in C$, such that $y x = x$. We have $x \notin C$ because $C \subseteq M_e$, and so $x \notin Fr C$. Let $V$ be a slice at $x$ and $S$ the unit sphere in $V$. By Proposition 4, condition e) it follows that $C \cap S$ is a component of the principal part $S_0$ of the $(m-1)$-dimensional sphere $S$ with the action of $G_e$: $C \cap S$ is homeomorphic to a disk and $\partial C \cap S = Fr (C \cap S)$ in $S$. Since $g (C \cap S) = C \cap S$, by inductive hypothesis $g = e$ and this is a contradiction.

9. Corollary. Let $V$ be an orthogonal effective representation of a finite group $G$ and $S$ the unit sphere in $V$. For the action of $G$ on $S$ there is a closed fundamental set iff the singular part $V' = \emptyset$ of $V$ is a union of hyperplanes. In that case any fundamental set is homeomorphic to a disk and the action of $G$ on the family of $|G|$ fundamental sets is transitive and free.

The assumption $\partial C = Fr C$ in Proposition 8 is essential.

10. Example. Let $G = Z_2 \times Z_2$ with the standard generators $g_1, g_2$ act on the 2-dimensional unit sphere $M = S^2 \subset \mathbb{R}^3$ such a way that $g_1$ is the antipodism and $g_2$ the symmetry with respect to a plane. Then there are two components of the principal part of $M$ and their closures are hemi-spheres. But there are no closed fundamental sets because $|G| = 4$. ($G_e$ contains $e$ and symmetry with respect to a line $g_1 g_2$).

In general, closed fundamental sets are not necessarily homeomorphic to disks, as is seen on the torus in $\mathbb{R}^2$ with the action of $Z_2$, the generator acting by symmetry with respect to a plane. In this case the closed fundamental set is homeomorphic to a ring.

If there exists a closed fundamental set, then it is homeomorphic to the space of orbits. In Examples 1 or 10 the space of orbits is homeomorphic to a disk, but there are no closed fundamental sets.

11. Remark. In fact if a component $C$ of $M_e$ is homeomorphic to $\mathbb{R}^m$ then $C$ is a fundamental set.

This follows from the known result that a nontrivial finite group cannot act freely on $\mathbb{R}^m$ (which is a consequence of the Smith theorem [1]). Indeed, $G_e$ acts freely on $C$, and so $G_e C$ is trivial and $C$ is fundamental set by Proposition 6.

References


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