Proof. Use Theorem 3.4 and the observation that for any open expansion $\{U(L)|\ L\in\mathscr{L}\}$ of $\mathscr{L}$, the set $\{x\in X|\ \{L\in\mathscr{L}|\ x\in U(L)\}$ is finite$\}$ is an F_σ-set.

We close this paper with a question related to the results of Theorems 3.2 and 3.3 above.

QUESTION 3.6. Does there exist, in ZFC, a topological space that is not σ-discrete but whose every subset is a G_δ-set?

Under MA+$\neg$CH, such spaces do exist, and they can even be metrizable (in fact, subspaces of $\boldsymbol{R}$; see [15] or [14]). By a result in [13], no space giving an affirmative answer to the above question could be normal and first countable.

Added in proof. Independently of the author, Z. Balogh has shown that, under PMEA, a space of character less than $\underline{c}$ is σ-discrete, if every subset of the space is an F_σ-set.

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Retracts and homotopies for multi-maps

by

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Abstract. By means of upper semi-continuous multi-functions defined on compacta and with values of trivial shape we introduce the notions of multi-retracts and multi-homotopies. We give some characterizations of absolute multi-retracts and absolute neighborhood multi-retracts and apply the notion of multi-homotopy to the construction of groups, called multi-homotopy groups. In particular, we show that if $y\in Y\in \mathrm{ANR}$, then the nth multi-homotopy group of the space (Y, y) is isomorphic to the nth shape group of this space.

1. Introduction. Throughout this paper all spaces are compact and metric. By a multi-function φ from a space X to a space Y (φ: $X\to Y$) we mean one that assigns to every point $x\in X$ a closed non-empty subset $\varphi(x)$ of Y. The upper semi-continuity (shortly u.s.c.) of φ: $X\to Y$ means that the *graph* Φ of φ defined as

$$\Phi = \{(x, y)\in X\times Y|\ y\in\varphi(x)\}$$

is closed in $X\times Y$. A *map* denotes, as usual, a continuous function. The notion of shape is understood in the sense of Borsuk [3]. A u.s.c. multi-function φ: $X\to Y$ is called a *multi-map* of X into Y if $\varphi(x)$ is a set of trivial shape for every $x\in X$. By an *extension* of a u.s.c. multi-function φ: $X\to Y$ onto $M\supset X$ we mean a u.s.c. multi-function φ': $M\to Y$ such that $\varphi'|X=\varphi$ and $\varphi'(x)$ has the shape of a point for every $x\in M\setminus X$. We say that a map f of Y onto X is a *cellular map* (compare [12]) if $f^{-1}(x)$ has trivial shape for every $x\in X$. Let us note that, if f is a cellular map of Y onto X, then the multi-function φ: $X\to Y$, defined by the formula $\varphi(x)=f^{-1}(x)\subset Y$ is a multi-map. Let us call such a multi-function an *inverse* of the map f. We say that X is *countable-dimensional* if it is the union of a countable family of finite-dimensional subspaces.

In the sequel we will need the following theorems:

1.1. THEOREM (Kozlowski [9] thms 9 and 12). *Let f be a cellular map of a space Y onto X such that the set $\{x\in X|\ f^{-1}(x)$ is a nondegenerate set$\}$ is contained in a compact and countable-dimensional subset of X. Then for every closed subset A of X the map $f|f^{-1}(A)$: $f^{-1}(A)$: $f^{-1}(A)\to A$ is a shape equivalence. Moreover, if $Y\in\mathrm{ANR}$, then $X\in\mathrm{ANR}$.*

1.2. THEOREM ([14], [15]). *Let φ be a multi-map of a space X into $Y\in\mathrm{ANR}$, where $X\subset M$. If $\dim(M\setminus X)<\infty$ or X is countable-dimensional, then there exists a neighborhood U of X in M such that φ has an extension onto U. Moreover, if $Y\in\mathrm{AR}$, then φ has an extension onto M.*

By a *Z-set* in a space Y (compare [1], p. 366) we mean a closed set $X\subset Y$ such that for every $\varepsilon>0$ there exists a map $g\colon Y\to Y\setminus X$, ε-near to the identity map id_Y. We use $X\cong Y$ to indicate that X is homeomorphic to Y. Q denotes the Hilbert cube. Small Greek letters are reserved to denote multi-functions. I is the unit interval $\langle 0, 1\rangle$ of reals.

2. Multi-retracts. Suppose that Y is a subset of X. Then a multi-map $\sigma\colon X\to Y$ is said to be a *multi-retraction of* X *to* Y if $y\in\sigma(y)$ for every $y\in Y$. We say that a subset Y of a space N is a *neighborhood multi-retract in the space* N if there is a neighborhood U of Y in N and a multi-retraction $\sigma\colon U\to Y$. If $U=N$, then we call Y a *multi-retract of the space* N.

Obviously every retract of a space is a multi-retract of that space, and a neighborhood retract is a neighborhood multi-retract. Moreover, if $N\supset Y\in\mathrm{FAR}$, then Y is a multi-retract of N. Indeed, defining $\sigma\colon N\to Y$ so that $\sigma(y)=Y$ for every $y\in N$ we get a multi-retraction of N to Y. Hence we infer that not every multi-retract of a space is a retract of that space.

Let σ be a multi-retraction of X to Y. Denote by $\Sigma\subset X\times Y$ the graph of σ and by p and q the projections of Σ onto X and Y, respectively. Then $p\colon\Sigma\to X$ is a cellular map. Moreover, if X has finite dimension, then by ([13], p. 86) $\mathrm{Sh}(X)=\mathrm{Sh}(\Sigma)$. But $\mathrm{Sh}(Y)\leqslant\mathrm{Sh}(\Sigma)$, because Y is homeomorphic to the diagonal of the set $Y\times Y$, which is a retract of Σ. Hence we infer (compare [8], Theorem 1) that

2.1. *If* Y *is a multi-retract of a finite-dimensional space* X, *then* $\mathrm{Sh}(X)\geqslant\mathrm{Sh}(Y)$, and, by ([3], p. 151, (1.10))

2.2. *A multi-retract of a finite-dimensional and movable space is movable.*

Let us note that the analogous properties for infinite-dimensional compacta fail (see Example 2.11).

A space Y will be called an *absolute neighborhood multi-retract* ($Y\in m$-ANR) provided that, for every space $N\supset Y$, Y is a neighborhood multi-retract of N. If Y is a multi-retract of every space $N\supset Y$, then we cally Y an *absolute multi-retract* ($Y\in m$-AR).

We have the following properties

2.3. *A space homeomorphic to an* m-AR-*space is an* m-AR-*space.*

2.4. *A space homeomorphic to an* m-ANR-*space is an* m-ANR-*space.*

2.5. $Y\in m$-AR $\Leftrightarrow$ $Y\subset N\in$ AR *is a multi-retract of* N.

2.6. $Y\in m$-ANR $\Leftrightarrow$ $Y\subset N\in$ AR *is a neighborhood multi-retract in* N.

We show that

2.7. $Y\in m$-AR $\Leftrightarrow$ *for each closed subset* X *of a space* M *and for every map* $f\colon X\to Y$ *there is a multi-map* $\varphi\colon M\to Y$ *such that* $f(x)\in\varphi(x)$ *for every* $x\in X$.

Indeed, if $Y\in m$-AR and f is a map of X into $Y\subset Q$, then there exist a multi-retraction $\nu\colon Q\to Y$ and a map $f'\colon M\to Q$, such that $f'(x)=f(x)$ for every $x\in X$. Then the composition $\nu\circ f'$ is a multi-map of X into Y such that $(\nu\circ f')(x)=\nu(f(x))$ for every $x\in X$.

To prove the sufficiency let us put $X=Y\subset Q=M$, $f=\mathrm{id}_Y$. Then there is a multi-map $\varphi\colon Q\to Y$ such that $y\in\varphi(y)$ for every $y\in Y$. It follows that φ is a multi-retraction of Q to Y, and by 2.5 we infer that $Y\in m$-AR.

In a similar way one can prove the following

2.8. $Y\in m$-ANR $\Leftrightarrow$ *for each closed subset* X *of a space* M *and for every map* $f\colon X\to Y$ *there exist a neighborhood* U *of* X *in* M *and a multi-map* $\varphi\colon U\to Y$ *such that* $f(x)\in\varphi(x)$ *for every* $x\in X$.

Notice that it is not possible to replace the condition $f(x)\in\varphi(x)$ in 2.7 and 2.8 by the condition $\varphi|X=f$. This results from the following known fact:

2.9. *If* id_Y *has an extension* φ *onto a locally connected space* $X\supset Y$, *then* Y *is also locally connected.*

Proof. Assume that Y is not locally connected at a point $p\in Y$. Then there is a closed neighborhood U of p in Y and a sequence $(p_n)_1^\infty$ of points of Y convergent to p and such that if $i\neq j$, then p_i and p_j belong to different components of the neighborhood U. Let $0<\varepsilon<\operatorname{diam}U$. Since φ is u.s.c., there exists a number η, $0<\eta<\varepsilon$, such that if $\varrho(x,p)<\eta$, $x\in X$, then $\varphi(x)$ is contained in the ball $K\subset Y$ with center p and radius ε. It follows from the local connectedness of X that for arbitrarily given $\delta>0$ there exists a closed and connected neighborhood V of p in X with a diameter less than δ. Let us take $\delta=\eta$. There is an index i such that $p_i\in V$ and $p_i\neq p$. Then the image $\varphi(V)\subset U$ of a connected set is a connected set containing the points p and p_i, which is impossible. Hence it follows that Y must be locally connected.

We now formulate several properties of m-AR (resp. m-ANR)-spaces which give a partial answer to the following problem: Are m-AR (resp. m-ANR)-spaces invariant under cellular maps?

2.10. *Let* $\varphi\colon X\to Y$ *be a multi-map and* $g\colon Y\to X$ *a map such that* $y\in\varphi(g(y))$ *for every* $y\in Y$. *If* $X\in$ ANR, *then* $Y\in m$-ANR. *Moreover, if* $X\in$ AR, *then* $Y\in m$-AR.

Proof. Let $Y\subset M\in$ AR. Since $X\in$ ANR, there exists a neighborhood U of Y in M and a map $g'\colon U\to X$ such that $g'|Y=g$. Then the composition $\varphi\circ g'\colon U\to Y$ is a multi-map satisfying the condition $y\in(\varphi\circ g')(y)$ for every $y\in Y$; therefore $\varphi\circ g'$ is a multi-retraction of U to Y. We infer by 2.6 that $Y\in m$-ANR.

If $X\in$ AR, then putting $U=M$ in the above proof we find that $Y\in m$-AR.

In particular, if g is a cellular map of Y onto $X\in$ ANR (resp. AR), then by 2.10 $Y\in m$-ANR (resp. m-AR).

2.11. Example. Consider a cellular map of a non-movable compactum Y^* onto the Hilbert cube Q (see [16]). Non-movability Y^* implies that $Y^*\notin$ FANR. However, by 2.10 $Y^*\in m$-AR.

Consequently, we infer by the fact that every FAR-space is an m-AR-space that the class of m-AR-spaces contains the class of FAR-spaces as its proper subclass.

2.12. *Let f be a cellular map of a space X onto $Y \in m$-ANR, $Y \subset N \in$ AR. If there exist a neighborhood V of Y in N and a multi-retraction $\sigma\colon V \to Y$ such that $\dim \sigma(y) < \infty$ for each $y \in V$, then $X \in m$-ANR. Moreover, if $Y \in m$-AR and $V = N$, then $X \in m$-AR.*

Proof. Let $X \subset Q$ and let $f'\colon Q \to N$ be a map such that $f'(x) = f(x)$ for $x \in X$. It follows by Theorem 1.1 that the composition $f^{-1} \circ \sigma\colon V \to X$ (f^{-1} being the inverse of f) is a multi-map. Denote $U = f'^{-1}(V)$. Then $f^{-1} \circ \sigma \circ f'\colon U \to X$ is also a multi-map and in addition satisfies the following condition:

$$f^{-1}[\sigma(f'(x))] = f^{-1}[\sigma(f(x))] \supset f^{-1}(f(x)) \ni x \quad \text{for every } x \in X.$$

Therefore $f^{-1} \circ \tilde{\sigma} \circ f'$ is a multi-retraction of U to X, and to finish the proof it suffices to apply 2.6. If $Y \in m$-AR and $V = N$, then the proof of Theorem 2.12 is similar.

Let us observe that if f is a cellular map of X onto Y^* (Y^* being as in Example 2.11), then by Theorem 2.12 $X \in m$-AR. Indeed, the cellular Taylor map of Y^* onto Q has finite-dimensional sets as its point-inverses (see [16]), and we infer by the proof of 2.10 that the point-inverses are the values of the multi-retraction of M to Y^*.

2.13. Theorem. *Let f be a cellular map of Y onto X and suppose that one of the sets X, Y is an FAR-space. Then the cylinder $M_f = Y \times I \cup_f X$ of the map f is an m-AR-space.*

Proof. Let $Y \in$ FAR. Every point $z \in M_f$ is of the form (y, t), where $y \in Y$ and $t \in I$. Let us fix a point $x_0 \in X$ and let $y_0 \in Y$ be a point in $f^{-1}(x_0)$. We show that there is a multi-map $\varphi\colon M_f \to M_f$ such that

(1) $$z \in \varphi(z) \quad \text{and} \quad x_0 \in \varphi(z) \quad \text{for every } z \in M_f.$$

Let $z = (y, t) \in M_f$. Denote by F_y the set $\{(y', t') \in M_f \mid f(y) = f(y')\}$. Then $Y \cap F_y = f^{-1}[f(y)] \in$ FAR and $F_y \in$ FAR as a cone with base $f^{-1}[f(y)]$. Hence, by ([3], (7.5), p. 321), $Y \cup F_y \in$ FAR and $(Y \cup F_y) \cup F_{y_0} \in$ FAR. We define $\varphi\colon M_f \to M_f$ as follows:

$$\varphi(z) = \varphi((y, t)) = Y \cup F_y \cup F_{y_0} \quad \text{for} \quad z \in M_f.$$

Then φ satisfies condition (1) and, what is easy to check, φ is a multi-map. Now, let $M_f \subset Q$. We define the multi-map $\sigma\colon Q \to M_f$ by the following conditions: $\sigma|M_f = \varphi$, $\sigma|Q \setminus M_f$ is a constant map into $\{x_0\}$. By (1), σ is a multi-retraction. Hence $M_f \in m$-AR.

If $X \in$ FAR, then it suffices to repeat the procedure, defining $\varphi(z) = X \cup \cup F_y \cup F_{y_0}$ for $z \in M_f$.

2.14. Corollary. *Let f be a cellular map of Q onto X such that the set*

$$B = \{f^{-1}(x) \mid x \in X \text{ and } f^{-1}(x) \text{ is a nondegenerate set}\}$$

is a Z-set in Q. Then $X \in m$-AR.

Proof. We may assume that $Q = Q_1 \times I$, where $Q_1 \cong Q$, and that $B \subset Q_1 \times \{1\}$. Define a map $f_1\colon Q_1 \to f(Q_1)$ by the formula $f_1(y) = f(y)$ for $y \in Q_1$, where we identify Q_1 with $Q_1 \times \{1\}$. Then X is the cylinder of the cellular map f_1; therefore we may apply Theorem 2.13.

We say that non-empty sets $F_1, \ldots, F_k$, $k \in N$, constitute a *simple decomposition* of a space X if their union is X and the intersection of any number of the sets is the empty set or an FAR-set.

2.15. Theorem. *If X has a simple decomposition $F_1, \ldots, F_k$, then there exist closed neighborhoods U_i of F_i, $i = 1, \ldots, k$, in $M \supset X$ and a multi-retraction $\nu\colon U = \bigcup_{i=1}^{k} U_i \to X$ such that, if $x \in U$ and $U_x = U_{i_1} \cup \ldots \cup U_{i_m}$ is the union of all sets U_{i_j}, $1 \leqslant i_j \leqslant k$, that contain the point x, then*

$$\nu(x) = F_{i_1} \cup \ldots \cup F_{i_m}.$$

Proof. We show that there exist closed neighborhoods U_i of the sets F_i, $i = 1, 2, \ldots, k$ in $M \supset X$ such that for every subsequence $(i_j)_{j=1}^{n}$ of the sequence $(1, 2, \ldots, k)$ the following implication is satisfied

(1) $$U_{i_1} \cap \ldots \cap U_{i_n} \neq \emptyset \Rightarrow F_{i_1} \cap \ldots \cap F_{i_n} \neq \emptyset.$$

To this end suppose the contrary. Let us take decreasing sequences $(U_i^l)_{l=1}^{\infty}$ of closed neighborhoods of the sets F_i, $i = 1, 2, \ldots, k$ such that $\bigcap_{l=1}^{\infty} U_i^l = F_i$ for every $i = 1, 2, \ldots, k$. Then there exist a subsequence $(i_j)_{j=1}^{m}$ of the sequence $(1, 2, \ldots, k)$ and a sequence $(p_n)_{n=1}^{\infty}$ of points of $U = \bigcup_{i=1}^{k} U_i$ convergent to a point $p \in X$ such that

(2) $$p_n \in \bigcap_{j=1}^{m} U_{i_j}^n \quad \text{for every } n \in N,$$

(3) $$\bigcap_{j=1}^{m} F_{i_j} = \emptyset.$$

It follows by (2) that $p \in \bigcap_{j=1}^{m} U_{i_j}^n$ for every $n \in N$. Therefore $p \in \bigcap_{j=1}^{m} F_{i_j}$, which contradicts (3). This proves the existence of neighborhoods U_i satisfying (1). Now, let ν be a multi-function from $U = \bigcup_{i=1}^{k} U_i$ to X, defined as in the theorem. Notice that if $x \in X$ then the union of all sets of the sequence $(F_1, F_2, \ldots, F_k)$ containing the point x is an FAR-set. Hence (1) implies that all values of ν have a trivial shape. Moreover, since the sets U_i, $i = 1, 2, \ldots, k$ constitute a closed covering of U, we see that ν is u.s.c. Thus ν is a multi-retraction of U to X.

2.16. Corollary. *A set having a simple decomposition is an m-ANR-set.*

2.17. Remark. The set Y^* in Taylor's example (see 2.11) is an m-AR-space, but does not have a simple decomposition. Indeed, if Y^* had a simple decomposition, then by the theorem on the union of two FANR-spaces (see [6]), Y^* would be an FANR-space, which is impossible, because Y^* is non-movable.

There is an example ([4], (4.18), p. 156) of a 2-dimensional set $X \in$ ANR which is not decomposable into a finite number of AR-sets. It follows that there exist points of the set X no neighborhood of which is an AR-set. If the set X had a simple decomposition, then for each $x \in X$ the union F_x of all sets of the decomposition containing the point x would be a neighborhood of x in X and an FAR-set. Using the fact that FAR-sets are contractible in each of their neighborhoods (in an ANR-space) and applying the argument as in ([4], (4.17), p. 155) (in particular using [4], (4.1), p. 153), one derives a contradiction of the fact that $F_x \in$ FAR. Hence we obtain the following fact:

2.18. *There exist* ANR-*spaces having no simple decompositions.*

Since every $Y \in$ FAR has a simple decomposition, we have

2.19. *Every* FAR-*set is an* m-AR-*set.*

For finite-dimensional multi-retracts we have the following properties:

2.20. *If* Y *has finite dimension, then*

$$Y \in m\text{-AR} \Leftrightarrow Y \in \text{FAR}.$$

Indeed, let Y be a multi-retract of a finite-dimensional space $N \in$ AR. It follows by 2.1 that $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(N)$. Hence $Y \in$ FAR. The converse follows from 2.19.

2.21. *Every finite-dimensional* $Y \in m$-ANR *is an* FANR-*space.*

To prove 2.21 assume that Y is a subset of a finite-dimensional cube I^n. Then there exist a neighborhood U of Y in I^n and a multi-retraction $\sigma\colon U \to Y$. Moreover, we may assume $U \in$ ANR. Then by 2.1 $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(U)$, and we infer by ([3], (1.2), p. 254) that $Y \in$ FANR.

If $Y \in m$-AR has infinite dimension, then it need not be an FANR-space (see 2.11).

Let us remark that the fundamental dimension $\mathrm{Fd}(Y)$ for $Y \in m$-AR can be infinite. To prove this it suffices to observe that if $\mathrm{Fd}(Y^*)$ (for Y^* see 2.11) were finite, then, by the movability of Q and the result of Bogatyi ([2], p. 261) on the shape equivalence of compacta under cellular maps, we would have $\mathrm{Sh}(Y^*) = \mathrm{Sh}(Q)$. But it is known that $\mathrm{Sh}(Y^*) \neq \mathrm{Sh}(1)$.

2.22. Remark. By Theorem 1.1 properties 2.20 and 2.21 remain true for countable-dimensional spaces.

Let us denote by $H_n(X)$ the nth Vietoris homology group of X with an abelian coefficient group $\mathscr{U}$. If f is a cellular map of Y onto X, then by ([11], (2.3) and (4.7)) f induces an isomorphism between the fundamental groups $\underline{\pi}_n(Y, y)$ and $\underline{\pi}_n(X, f(y))$ for every $y \in Y$ and f induces an isomorphism of the homology groups $H_n(X)$ and $H_n(Y)$, $n \in N$.

Let $\sigma\colon X \to Y$ be a multi-retraction and $\Sigma \subset X \times Y$ the graph of σ. Fix a point $(x, y) \in \Sigma$ and denote by p and q the projections of Σ onto X and Y, respectively.

Since p is a cellular map of Σ onto X, by the theorems mentioned above p induces the isomorphisms

$$p_*\colon H_n(\Sigma) \to H_n(X), \quad n = 0, 1, 2, \ldots$$

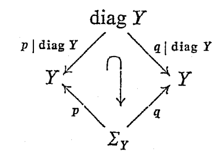
and

$$p_f\colon \underline{\pi}_n(\Sigma, (x, y)) \to \underline{\pi}_n(X, x), \quad n \in N.$$

Let p_*^{-1} and p_f^{-1} denote the isomorphisms inverse to p_* and p_f, respectively, and q_* and q_f the homomorphisms of homology and fundamental groups induced by the projection q. Then we say that the compositions $\sigma_* = q_* \circ p_*^{-1}\colon H_n(X) \to H_n(Y)$ and $\sigma_f = q_f \circ p_f^{-1}\colon \underline{\pi}_n(X, x) \to \underline{\pi}_n(Y, y)$ are the *homomorphisms induced by* σ.

2.23. *If* $\sigma\colon X \to Y$ *is a multi-retraction, then the homomorphisms* $(\sigma|Y)_*$ *and* $(\sigma|Y)_f$ *induced by* $\sigma|Y\colon Y \to Y$ *are the identity homomorphisms.*

Indeed, let $\Sigma_Y \subset Y \times Y$ denotes the graph of $\sigma|Y$ and let p and q denote the projections of Σ_Y onto the first and the second factor, respectively. Then 2.23 follows by the commutativity of the diagram



where $\operatorname{diag} Y$ means the diagonal of the set $Y \times Y$.

Since the multi-retract Y of X is homeomorphic to the $\operatorname{diag} Y$ which is a retract of Σ and since p_* and p_f are isomorphisms, it follows by ([3], (5.3) and (5.4), p. 191) that

2.24. *If* Y *is a multi-retract of* X, *then for every* $n = 0, 1, \ldots$ *the group* $H_n(Y)$ *is a direct divisor of the group* $H_n(X)$ *and for* $n > 1$ *the group* $\underline{\pi}_n(Y, y)$ *is a direct divisor of* $\underline{\pi}_n(X, x)$.

By 2.24 we have

2.25. *If* $Y \in m$-AR, *then for* $n \neq 0$ *and every* $y \in Y$ *the groups* $H_n(Y)$ *and* $\underline{\pi}_n(Y, y)$ *are trivial.*

2.26. *If* $Y \in m$-ANR, *then almost all groups* $H_n(Y)$ *are trivial; moreover, if* $\mathscr{U}$ *is finitely generated, then all groups* $H_n(Y)$ *are finitely generated.*

It follows from 2.26 and the example of a locally contractible compactum with a positive nth Betti number for every $n = 0, 1, \ldots$ ([4], (11.1), p. 124) that there exist locally contractible compacta that are not m-ANR-spaces.

2.27. Example. Let L be a curve lying in the plane E^2 and defined in polar coordinates (r, a) as follows:

$$r = 1/a + 1 \quad \text{for} \quad a \in \langle 1, \infty).$$

Denote by Y the closure (in E^2) of L. Then $Y \in$ FANR. We show that $Y \notin m$-ANR.

To this end assume the converse. Then there is a compact neighborhood U of Y in E^2 and a multi-retraction $\sigma\colon U \to Y$. Let y be a point of the circle $S = Y\setminus L$. Then $\sigma(y)$ is contained in S, and moreover it is an arc. Let y' be a point of the set $S\setminus\sigma(y)$. We take a closed neighborhood V of $\sigma(y)$ in Y such that $y' \notin V$. Then V is the set with countable many components. We infer by the upper semi-continuity of σ that there exists a closed neighborhood U_0 of y in U such that $\sigma(U_0)\subset V$. Let $B_0\subset U_0$ be a closed ball in U with center y. Denote by y_0 a point of $B_0\setminus S$. Then $\sigma(y_0)$ and $\sigma(y)$ lie in different components of V, which is impossible by the fact that the image of any connected set (in our case B_0) under any multi-map is connected. Thus must be $Y\notin m$-ANR.

2.28. EXAMPLE. Denote by X_0 the subset of the plane E^2 which is the closure of the graph L_0 of the function $f(x) = \sin(\pi/x)$ for $0<x\leqslant 1$. Let $a_0 = (0, -1)$, $b_0 = (0, 1)$. Write $X_i = X_0\times\{i\}$, $(a_0, i) = a_i \in X_i$, $(b_0, i) = b_i \in X_i$, $i = 1, 2, 3$. Assume now that $X\subset E^2$ is the set formed from the sum $X_1 \cup X_2 \cup X_3$ by the identification of the points a_{i+1} with b_i for $i = 1, 2$ and b_3 with a_1. Then X has a simple decomposition, hence $X\in m$-ANR. Since X is not locally connected, it is not the image of an ANR-set under any map. We show that

(a) There exists no cellular map of X onto any ANR-set.

To prove (a) suppose, conversely, that there exists a cellular map g of X onto an ANR-set Y. Then there are a compact neighborhood U of X in E^2 and a map $g'\colon U \to Y$ such that $g'|X = g$. Therefore the composition $\sigma = g^{-1}\circ g'$, where g^{-1} is the inverse of g, is a multi-retraction of U to X such that, for $x \in X$, $\sigma(x)$ is a point-inverse under g. Let us take a point $x\in A = X\setminus(L_0\times\{i\})$, $i = 1, 2, 3$. Then we must have $\sigma(x)\supset A$ (otherwise there would be a point $x'\in A\setminus\sigma(x)$ and a closed neighborhood V of $\sigma(x)$ in X such that $x'\notin V$, which, by a similar argument as in Example 2.27, is impossible). However, the condition $A\subset\sigma(x)$ fails, because $\sigma(x)$ is a point-inverse under the cellular map g and hence has the shape of a point. The contradiction proves condition (a).

Let us formulate some problems concerning multi-retracts.

2.29. PROBLEM. Is it true that every plane m-ANR-set has a simple decomposition?

2.30. PROBLEM. Is it true that every retract (deformation retract) of an m-AR-space (resp. m-ANR-space) is an m-AR-space (resp. m-ANR-space)?

In particular,

2.31. PROBLEM. Is the image of Q under any cellular map an m-AR-space?

2.32. PROBLEM. Is it true that finite-dimensional m-ANR-spaces have the shape of a compact polyhedron?

3. Multi, homotopies. Let (X, x_0) and (Y, y_0) be spaces with any given basic points x_0 and y_0. We write $\varphi\colon (X, x_0)\to(Y, y_0)$ if φ is a multi-map of X into Y such that $y_0\in\varphi(x_0)$. If f and g are maps of X into Y such that $f(x_0) = g(x_0) = y_0$, then by a *homotopy* of the maps f and g we understand a map $h\colon X\times I\to Y$ satisfying the conditions $h|X\times\{0\} = f$, $h|X\times\{1\} = g$ and $h(x_0, t) = y_0$ for every $t\in I$. We call a homotopy h an *isotopy* if $h|X\times\{t\}$ is a homeomorphism onto $h(X\times\{t\})$ for each $t\in I$.

Let $\varphi_1, \varphi_2\colon (X, x_0)\to(Y, y_0)$ be multi-maps. We say that a multi-map $\chi\colon X\times I\to Y$ is a *multi-homotopy connecting* φ_1 *with* φ_2 if $\chi(x, 0) = \varphi_1(x)$, $\chi(x, 1) = \varphi_2(x)$ and $y_0\in\chi(x_0, t)$ for every $x\in X$ and $t\in I$. In this case the multi-maps φ_1 and φ_2 are said to be *multi-homotopic* (notation: $\varphi_1 \overset{m}{\simeq} \varphi_2$).

3.1. The relation of multi-homotopy is an equivalence relation; hence the collection of all multi-maps $(X, x_0)\to(Y, y_0)$ decomposes into disjoint classes of multi-maps multi-homotopic to one another, called *multi-homotopy classes*.

Let us note that, if $f_1, f_2\colon (X, x_0)\to(Y, y_0)$ are homotopic maps, then they are multi-homotopic. The converse is not true.

3.2. EXAMPLE. Let X be a non-contractible FAR-set. Let us select a point $x_0\in X$. Assume that $f_1 = \mathrm{id}_X$ and $f_2\colon X\to X$ is the constant map of X into x_0. Then f_1 and f_2 are not homotopic, but they are multi-homotopic under the multi-homotopy $\chi\colon X\times I\to X$ defined by the conditions

$$\chi(X\times\langle 0, \tfrac{1}{2})) = \{x_0\}, \quad \chi(x, \tfrac{1}{2}) = X, \quad \chi(\{x\}\times(\tfrac{1}{2}, 1\rangle) = \{x\}, \quad x\in X.$$

If $\varphi\colon (X, x_0)\to(X, x_0)$ is a multi-map of X into itself with the property that $x\in\varphi(x)$ for every $x\in X$, then we write $\varphi = m\text{-}\mathrm{id}_X$. If $\varphi\colon (X, x_0)\to(Y, y_0)$ is a multi-map, then $\varphi \overset{m}{\simeq} \theta$ will mean that φ is multi-homotopic to the map θ of X into Y such that $\theta(x) = y_0$ for every $x\in X$. A set (X, x_0) is said to be *multi-contractible*, if $\mathrm{id}_X \overset{m}{\simeq} \theta$.

3.3. *If* $\varphi = m\text{-}\mathrm{id}_X$, *then* $\varphi \overset{m}{\simeq} \mathrm{id}_X$.

Indeed, the conditions $\chi(\{x\}\times\langle 0, 1)) = \{x\}$, $\chi(x, 1) = \varphi(x)$ define the multi-homotopy connecting φ with id_X.

3.4. *If* $\varphi\colon (X, x_0)\to(Y, y_0)$ *is a multi-map of* X *into* Y *such that* $\varphi(x) = \varphi(x_0)$ *for every* $x\in X$, *then* $\varphi \overset{m}{\simeq} \theta$.

3.5. *If* f *is a map of* (X, x_0) *into a multi-contractible space* (Y, y_0), *then* $f \overset{m}{\simeq} \theta$.

In fact, if $\chi\colon Y\times I\to Y$ realizes the relation $\mathrm{id}_Y \overset{m}{\simeq} \theta$, then $\tilde{\chi}\colon X\times I\to Y$ defined by the formula $\tilde{\chi}(x, t) = \chi(f(x), t)$ is a multihomotopy realizing $f \overset{m}{\simeq} \theta$.

3.6. *Let* $\varphi\colon (X, x_0)\to(Y, y_0)$ *be a multi-map of* X *into* $Y\in$ FAR. *Then* $\varphi \overset{m}{\simeq} \theta$.

To show this, it suffices to define the multi-homotopy $\chi\colon X\times I\to Y$ by the following conditions: $\chi(x, t) = \varphi(x)$ for $t\in\langle 0, \tfrac{1}{2})$, $\chi(x, \tfrac{1}{2}) = Y$, $\chi(X\times(\tfrac{1}{2}, 1\rangle) = \{y_0\}$.

3.7. *Let* $\varphi\colon (S^n, x_0)\to(Y, y_0)$ *be a multi-map of the n-dimensional sphere* S^n *into* Y. *Then the following conditions are equivalent*

(a) φ *has an extension onto the ball* B^{n+1}.

(b) $\varphi \overset{m}{\simeq} \theta$.

Proof. Let φ' be the extension of φ onto B^{n+1}. We define $\chi\colon S^n\times I\to Y$ by the formula

$$\chi(x,t)=\varphi'[(1-t)x+tx_0].$$

Since $\chi(x,0)=\varphi'(x)$, $\chi(x,1)=\varphi'(x_0)=\varphi(x_0)$ and $\chi(x_0,t)=\varphi(x_0)$ for every $t\in I$ and $x\in S^n$, then χ is a multi-homotopy connecting φ with $\chi|S^n\times\{1\}$. So, it suffices to apply 3.4. To prove the implication (b) $\Rightarrow$ (a), we define the extension φ' of φ by setting $\varphi'(x)=\{y_0\}$ for $0\leqslant ||x||\leqslant\frac{1}{2}$ and $\varphi'(x)=\chi[x/||x||, 2-2||x||]$ for $\frac{1}{2}\leqslant ||x||\leqslant 1$, where χ realizes the relation $\varphi\overset{m}{\simeq}\theta$.

3.8. *Every m-AR-set is multi-contractible.*

Proof. Let $X\subset Q$ be an m-AR-set. Let us fix a point x_0 in X. Then there exists a multi-retraction $\sigma\colon Q\to X$ such that $x\in\sigma(x)$ for every $x\in X$. We define the multi-homotopy $\chi\colon X\times I\to X$ connecting $\sigma|X=m\text{-id}_X$ with the multi-map ψ, which assigns to every $x\in X$ a set $\sigma(x_0)$:

$$\chi(x,t)=\sigma[(1-t)x+tx_0].$$

By 3.3 and 3.4 we get

$$\mathrm{id}_X\overset{m}{\simeq}\sigma|X\overset{m}{\simeq}\psi\overset{m}{\simeq}\theta.$$

This implies the following

3.9. FAR-*sets are multi-contractible.*

There exist multi-contractible spaces not having the shape of a point. For instance the non-movable compactum Y^* (see 2.11) is an m-AR-set and, by 3.8, is multi-contractible.

We now prove the following theorem on the extension of a multi-homotopy:

3.10. THEOREM. *Let $\chi\colon X\times I\to Y\in$ ANR be a multi-map, where $X\subset M$ and $\dim(M\setminus X)<\infty$ or M is countable-dimensional. If $\chi|X\times\{0\}\colon X\to Y$ has an extension $\chi_0\colon M\to Y$, then there exists a multi-map $\tilde\chi\colon M\times I\to Y$, such that $\tilde\chi|X\times I=\chi$ and $\tilde\chi|M\times\{0\}=\chi_0$.*

Proof. Put $P=M\times\{0\}\cup X\times I$. By Theorems 1.1 and 1.2, there is a neighborhood U of P in $M\times I$ such that the multi-map $\psi\colon P\to Y\in$ ANR defined by the formulae

$$\psi(x,0)=\chi_0(x)\quad\text{for}\quad x\in M,$$
$$\psi(x,t)=\chi(x,t)\quad\text{for}\quad x\in X\text{ and }t\in I$$

has an extension ψ' onto U. There exists a map $f\colon M\times I\to U$ which is the identity onto P (see [4], (8.2), p. 94). Then it suffices to define $\tilde\chi\colon M\times I\to Y$ as the composition

$$\tilde\chi=\psi'\circ f.$$

For $(x,t)\in X\times I$ we get

$$\tilde\chi(x,t)=\psi'(f(x,t))=\psi'(x,t)=\chi(x,t).$$

For $x\in M$

$$\tilde\chi(x,0)=\psi'(f(x,0))=\psi'(x,0)=\chi_0(x).$$

Thus $\tilde\chi$ is the required multi-map.

One can easily see that the classical homotopy extension theorem cannot be directly transferred to the case of multi-maps into m-ANR-sets. Instead, the supposition $Y\in m$-ANR leads to the following

3.11. THEOREM. *Let h be a map of $P=M\times\{0\}\cup X\times I$ into $Y\in m$-ANR, where X is a subset of M. Then there exists a multi-map $\varphi\colon M\times I\to Y$, such that for every $(x,t)\in P$, $h(x,t)\in\varphi(x,t)$.*

Proof. Let Y be a subset of the Hilbert cube Q. Then there are a neighborhood V of Y in Q and a multi-retraction $\sigma\colon V\to Y$. Let $h'\colon M\times I\to Q$ be a map satisfying the condition $h'(x,t)=h(x,t)$ for every $(x,t)\in P$. Put $U=h'^{-1}(V)$. There exists (see [4], (8.2), p. 94) a map $f\colon M\times I\to U$ which is the identity onto P. We define $\varphi\colon M\times I\to Y$ as the composition $\varphi=\sigma\circ h'\circ f$. For $(x,t)\in P$ we get

$$\varphi(x,t)=\sigma\big(h'(f(x,t))\big)=\sigma\big(h'(x,t)\big)=\sigma\big(h(x,t)\big).$$

Since $y\in\sigma(y)$ for every $y\in Y$, we have $h(x,t)\in\sigma\big(h(x,t)\big)$. This completes the proof.

3.12. *Every multi-contractible* ANR-*space is an m-AR-space.*

Proof. Let (X,x_0) be a multi-contractible ANR-set contained in $M\in$ AR. Let us denote by $\chi\colon X\times I\to X$ the multi-homotopy connecting id_X with the map $\chi|X\times\{0\}$ of X into x_0. Setting

$$\varphi(x,0)=\{x_0\}\text{ for }x\in M,\qquad\varphi(x,t)=\chi(x,t)\text{ for }x\in X\text{ and }t\in(0,1\rangle,$$

we get the multi-map φ of $P=M\times\{0\}\cup X\times I$ into X. Since $P\in$ ANR, there exists a neighborhood U of P in $M\times I$ and a retraction $r\colon U\to P$. There exists a map $f\colon M\times I\to U$ which is the identity onto P (see [4], (8.2), p. 94). Therefore $\psi=\varphi\circ r\circ f|M\times\{1\}\colon M\times\{1\}\to X$ is a multi-map such that $\psi(x)=\varphi\big(r(f(x))\big)=\varphi\big(r(x)\big)=\varphi(x)=\chi(x,1)=\{x\}$ for every $x\in X\times\{1\}$. Thus ψ is a multi-retraction of M to X and we conclude that $X\in m$-AR.

From this it follows the following corollary

3.13. *The sphere S^n is not multi-contractible.*

4. Multi-homotopy groups. Let S^+ and S^- denote disjoint sets, homeomorphic to the closed halfspheres in the n-dimensional sphere S^n, $n\in N$. By S we denote the set obtained from the disjoint union $S^+\cup S^-\cup(S^{n-1}\times I)$ by identifying $S^{n-1}\times\{0\}$ with the boundary $\dot S^+=(S^+\setminus\mathrm{Int}\,S^+)$ of S^+ and $S^{n-1}\times\{1\}$ with the boundary $\dot S^-=(S^-\setminus\mathrm{Int}\,S^-)$ of S^-. Clearly $S\cong S^n$. Let x_0 be a selected point of S belonging to $S^{n-1}\times\{\frac{1}{2}\}$. If Y is a space with the basic point y_0, then the multi-homotopy class (see 3.1) containing the multi-map $\varphi\colon(S,x_0)\to(Y,y_0)$ we denote by $[\varphi]$. In the set of the multi-homotopy classes we introduce the group operation. To this end we first define the operation assigning to every two multi-maps $\varphi,\psi\colon(S,x_0)\to(Y,y_0)$ a multi-map $\varphi\nabla\psi\colon(S,x_0)\to(Y,y_0)$.

It is known that one can define isotopies $h_i\colon(S\setminus\{x_0\}\times I)\to S\setminus\{x_0\}$, $i=1,2,$

such that $h_i|(S\setminus\{x_0\})\times\{0\} = \mathrm{id}_{S\setminus\{x_0\}}$, $i = 1, 2$, $h_1((S\setminus\{x_0\})\times\{1\}) = S^+\setminus\dot{S}^+$ and $h_2((S\setminus\{x_0\})\times\{1\}) = S^-\setminus\dot{S}^-$. If we set

$$(\varphi \nabla \psi)(x) = \begin{cases} \varphi(h_1^{-1}(x, 1)) & \text{for } x \in S^+\setminus\dot{S}^+, \\ \varphi(x_0) & \text{for } x \in \dot{S}^+, \\ \{y_0\} & \text{for } x \in S^{n-1}\times(0, 1), \\ \psi(x_0) & \text{for } x \in \dot{S}^-, \\ \psi(h_2^{-1}(x, 1)) & \text{for } x \in S^-\setminus\dot{S}^-, \end{cases}$$

then we get the multi-map $\varphi \nabla \psi$: $(S, x_0) \to (Y, y_0)$. In particular, $(\varphi \nabla \psi)(x_0) = \{y_0\}$.

For classes $[\varphi]$ and $[\psi]$ we define their group operation as follows

(i) $$[\varphi][\psi] = [\varphi \nabla \psi].$$

Let us verify that the multi-homotopy class $[\varphi \nabla \psi]$ does not depend on the choice of the multi-maps φ, ψ, i.e. that if φ', ψ': $(S, x_0) \to (Y, y_0)$ are multi-maps and $\varphi \overset{m}{\simeq} \varphi'$, $\psi \overset{m}{\simeq} \psi'$, then

(ii) $$[\varphi \nabla \psi] = [\varphi' \nabla \psi'].$$

Let χ_1, χ_2: $S\times I \to Y$ denote multi-homotopies connecting φ with φ' and ψ with ψ', respectively. Then χ: $S\times I \to Y$ defined by the formula

$$\chi(x, t) = \begin{cases} \chi_1(h_1^{-1}(x, 1), t) & \text{for } x \in S^+\setminus\dot{S}^+, \\ \chi_1(x_0, t) & \text{for } x \in \dot{S}^+, \\ \{y_0\} & \text{for } x \in S^{n-1}\times(0, 1), \\ \chi_2(x_0, t) & \text{for } x \in \dot{S}^-, \\ \chi_2(h_2^{-1}(x, 1), t) & \text{for } x \in S^-\setminus\dot{S}^- \end{cases}$$

for every $t \in I$, is a multi-homotopy connecting $\varphi \nabla \psi$ with $\varphi' \nabla \psi'$. Therefore (ii) is satisfied.

It is not hard to prove (cf. the definition of homotopy groups) that the set of all multi-homotopy classes of multi-maps $(S, x_0) \to (Y, y_0)$ with the operation defined by (i) constitutes a group, abelian for $n > 1$. The neutral element of this group is the multi-homotopy class containing a map θ: $(S, x_0) \to (Y, y_0)$ such that $\theta(x) = y_0$ for every $x \in S$. Moreover, the group does not depend on the choice of the point $x_0 \in S$. Let us call that group *n-th multi-homotopy group of Y with base point* y_0. We shall denote this group by $m\text{-}\pi_n(Y, y_0)$.

4.1. *If Y has the property that for every two of its points there is a subset $F \in$ FAR of Y containing these points (in particular, if Y is arcwise connected), then the group $m\text{-}\pi_n(Y, y_0)$ is isomorphic to the group $m\text{-}\pi_n(Y, y_0')$, where y_0' is another basic point of Y.*

Indeed, it suffices to remark that if $F \in$ FAR contains y_0 and y_0' then for every multi-map φ: $(S, x_0) \to (Y, y_0)$ there exists a multi-map ψ: $(S, x_0) \to (Y, y_0')$, multi-homotopic with φ (relative y_0), namely

$$\varphi \overset{m}{\simeq} \varphi \nabla \theta \overset{m}{\simeq} \psi,$$

where $[\theta]$ is the neutral element of $m\text{-}\pi_n(Y, y_0)$ and ψ is defined as follows:

$$\psi|S\setminus\{x_0\} = \varphi \nabla \theta, \quad \psi(x_0) = F.$$

The isomorphism of the groups $m\text{-}\pi_n(Y, y_0)$ and $m\text{-}\pi_n(Y, y_0')$ is then established by the assignment

$$[\varphi] \mapsto [\psi].$$

If h is a homeomorphism of spaces (Y, y_0) and (Y', y_0'), then assigning to every element $[\varphi] \in m\text{-}\pi_n(Y, y_0)$ the element $[h \circ \varphi] \in m\text{-}\pi_n(Y', y_0')$ we define a homomorphism of the multi-homotopy groups of (Y, y_0) and (Y', y_0'). Moreover, this homomorphism is an isomorphism; hence

4.2. *The groups $m\text{-}\pi_n(Y, y_0)$ are topological invariants of the space (Y, y_0).*

It follows from 3.6 that

4.3. *All the multi-homotopy groups of an* FAR-*set are trivial.*

We now define a homomorphism of the group $m\text{-}\pi_n(Y, y_0)$ into the nth fundamental group $\underline{\pi}_n(Y, y_0)$ of a space (Y, y_0) (see [3], p. 132 for the definition of fundamental groups).

Let φ: $(S, x_0) \to (Y, y_0)$ be a multi-map. We may assume (taking, if necessary, the multi-map $\theta \nabla \varphi$ instead of φ) that $\varphi(x_0) = \{y_0\}$. Assume that S is a Z-set in $Q_1 \cong Q$ and that $Y \subset Q_2 \cong Q$. Let us take the upper semi-continuous decomposition $\mathscr{D}$ of the space $Q_1 \times Q_2$, whose elements are the sets $\{x\}\times\varphi(x)$ for $x \in S$ and the remaining one-point sets. Denote by r the projection of $Q_1 \times Q_2$ onto the decomposition space $\mathscr{M} = Q_1\times Q_2/\mathscr{D}$ and by Φ the graph of φ. It follows by Theorem 1.1 that $\mathscr{M} \in$ AR. Since r is a cellular map of Q onto an AR-set and the set of non-degenerated point-images of r is a Z-set in Q, by ([17], Theorem 3), we get $\mathscr{M} \cong Q$. From ([7], also [12], Lemma 2.3) we infer that $r(\Phi)$ is a Z-set in $\mathscr{M}$. Let us denote by p and q the projections of $Q_1 \times Q_2$ onto the first and second factor, respectively. Setting

$$h: x \mapsto r[(p|\Phi)^{-1}(x)] \quad \text{for} \quad x \in S,$$

we get a homeomorphism of S onto $r(\Phi)$. Denote by Q_3 the space formed from $Q_1 \times I$ by contracting $\{x_0\}\times I$ to a point. Let r_1: $Q_1 \times I \to Q_3$ the projection. Then $Q_3 \cong Q$ and by ([1], Theorem 10.1) the homeomorphism h between Z-sets $S \subset Q$ and $r(\Phi) \subset \mathscr{M}$ extends to a homeomorphism $\bar{h}$: $Q_3 \to \mathscr{M}$.

We define χ^φ: $S\times I \to Q_1 \times Q_2$ by the formula

$$\chi^\varphi(x, t) = r^{-1}[\bar{h}(r_1(x, t))].$$

Then χ^φ is a multi-map and $\chi^\varphi(x_0, t) = \{(x_0, y_0)\}$ for every $t \in I$. Hence χ^φ is a multi-homotopy connecting $\chi^\varphi|S\times\{1\}$ with the multi-map $\chi^\varphi|S\times\{0\}$. Moreover, $\chi^\varphi(x, 0) = \{x\}\times\varphi(x)$ for every $x \in S$ and $\chi^\varphi|S\times\{t\}$ is a homeomorphism onto $\chi^\varphi(S\times\{t\})$ for each $t \in (0, 1\rangle$. Therefore, defining the sequence of maps

$$\underline{a}^\varphi = \{a_k^\varphi: (S, x_0) \to (Q_2, y_0)\}$$

by the formula

$$a_k^\varphi(x) = q(\chi^\varphi(x, 1/k)) \quad \text{for} \quad k = 1, 2, \ldots,$$

we obtain an approximative map from (S, x_0) towards (Y, y_0) (for the definition of approximative maps see [3], p. 128).

4.4. THEOREM. *Assigning to every element of the group $m\text{-}\pi_n(Y, y_0)$ with a representative $\varphi\colon (S, x_0) \to (Y, y_0)$ the approximative class with*

$$\underline{a}^\varphi = \{a_k^\varphi\colon (S, x_0) \to (Q_2, y_0)\}$$

as a representative, we get a homomorphism $\mathfrak{J}$ of the multi-homotopy group $m\text{-}\pi_n(Y, y_0)$ into the fundamental group $\underline{\pi}_n(Y, y_0)$. Moreover, if $Y \in$ ANR, then $\mathfrak{J}$ is an isomorphism.

Proof. First we show that

$$\text{(iii)} \qquad \mathfrak{J}([\varphi]\cdot[\psi]) = [\underline{a}^{\varphi\nabla\psi}] = [\underline{a}^\varphi]\cdot[\underline{a}^\psi],$$

where $\varphi, \psi\colon (S, x_0) \to (Y, y_0)$ are multi-maps. With this aim we remark that a representative of the class $[\underline{a}^\varphi]\cdot[\underline{a}^\psi]$ is to be found in the approximative map $\underline{b} = \{b_k\colon (S, x_0) \to (Y, y_0)\}_{Q_2}$ defined by the formula

$$b_k(x) = \begin{cases} a_k^\varphi(x) & \text{for} \quad x \in S^+, \\ y_0 & \text{for} \quad x \in S\setminus(S^+ \cup S^-), \\ a_n^\psi(x) & \text{for} \quad x \in S^-. \end{cases}$$

Thus, to prove (iii) it suffices to show that the approximative map

$$\{a_k^{\varphi\nabla\psi}, (S^+, \dot S^+) \to (Y, y_0)\}_{Q_2}$$

is homotopic to the approximative map

$$\{a_k^\varphi, (S^+, \dot S^+) \to (Y, y_0)\}_{Q_2} \quad \text{and} \quad \{a_k^{\varphi\nabla\psi}, (S^-, \dot S^-) \to (Y, y_0)\}_{Q_2},$$

homotopic to the approximative map $\{a_k^\psi, (S^-, \dot S^-) \to (Y, y_0)\}_{Q_2}$ ($\dot S^+$ and $\dot S^-$ are regarded here as points). We only show the former condition. The proof of the latter is similar.

Let us observe that χ^φ and $\chi^{\varphi\nabla\psi}$, restricted to the set $S^+\times I$, are multi-homotopies such that $\chi^\varphi(x, 0) = \{x\}\times\varphi(x) = \chi^{\varphi\nabla\psi}(x, 0)$ for $x \in S^+$. Let V_2 be a neighborhood of Y in Q_2 and let $U \in$ ANR, $U \subset r(Q_1 \times V_2)$ be a neighborhood of $r(\Phi)$ in $\mathcal{M}$. Since $U\setminus r(\Phi) \in$ ANR, there is an $\varepsilon > 0$ such that every two ε-near maps into $U\setminus r(\Phi)$ are homotopic in $U\setminus r(\Phi)$. Choose $k_0 \in N$ so large that for every $k \geqslant k_0$ and $x \in S^+$

$$\varrho(r[\{x\}\times\varphi(x)], r[\chi^{\varphi\nabla\psi}(x, 1/k)]) + \varrho(r[\{x\}\times\varphi(x)], r[\chi^\varphi(x, 1/k)]) < \varepsilon.$$

Then for every $k \geqslant k_0$ the maps $r\circ\chi^{\varphi\nabla\psi}|S^+\times\{1/k\}$ and $r\circ\chi^\varphi|S^+\times\{1/k\}$ of the set $S^+\times\{1/k\}$ into $U\setminus r(\Phi)$ are homotopic in $U\setminus r(\Phi)$. Let g_k be a homotopy joining these maps. Setting

$$\tilde g_k(x, t) = (q\circ r^{-1}\circ g_k)(x, 1/k, t) \quad \text{for} \quad (x, t) \in S^+\times I,$$

we get a homotopy $\tilde g_k\colon S^+\times I \to V_2$ joining in V_2 the map $a_k^{\varphi\nabla\psi}|S^+$ with the map $a_k^\varphi|S^+$. Thus (iii) is satisfied and hence $\mathfrak{J}$ is a homomorphism. We now show that, in the case of $Y \in$ ANR, $\mathfrak{J}$ is an isomorphism.

By ([3], (2.5), p. 129) if $\underline{c} = \{c_k, (S, x_0) \to (Y, y_0)\}_{Q_2}$ is an approximative map and $Y \in$ ANR, then the class $[\underline{c}]$ is generated by a map $c\colon (S, x_0) \to (Y, y_0)$. But every map is a multi-map; therefore we may obtain the approximative map $\underline{a}^c = \{a_k^c, (S, x_0) \to (Y, y_0)\}_{Q_2}$ for the map c. Moreover, χ^c is then a map, and for every neighborhood W of Y in Q_2 there is a $k_0 \in N$ such that, for $k \geqslant k_0$, $q\circ\chi^c|S\times\langle 0, 1/k\rangle$ is a homotopy joining in W the maps a_k^c and c. Hence $\underline{a}^c$ and $\underline{c}$ are homotopic and $\mathfrak{J}$ is an epimorphism. In order to prove that $\mathfrak{J}$ is a monomorphism, let $\varphi\colon (S, x_0) \to (Y, y_0)$ be a multi-map and assume that an approximative map $\underline{a}^\varphi$ is null-homotopic. The assumption $Y \in$ ANR implies the existence of a neighborhood V_2' of Y in Q_2 and a retraction $b\colon V_2' \to Y$. Besides there is a $k_1 \in N$ such that

$$\chi^\varphi(S\times\langle 0, 1/k_1\rangle) \subset Q_1 \times V_2'.$$

Since $\underline{a}^\varphi$ is null-homotopic, there exists a homotopy $g\colon S\times I \to V_2'$ joining in V_2' the map $a_{k_1}^\varphi$ with the map $g|S\times\{1\}$, which is the constant map of S into y_0. Then $\tilde\chi^\varphi\colon S\times I \to Y$ defined by the formula

$$\tilde\chi^\varphi(x, t) = \begin{cases} b\{q(\chi^\varphi(x, t))\} & \text{for} \quad x \in S \text{ and } t \in \langle 0, 1/k_1\rangle, \\ b\left\{g\left(x, \dfrac{tk_1 - 1}{k_1 - 1}\right)\right\} & \text{for} \quad x \in S \text{ and } t \in \langle 1/k_1, 1\rangle \end{cases}$$

is a multi-homotopy connecting in Y the multi-map φ with the constant map of S into y_0. Hence $\mathfrak{J}$ is a monomorphism and the proof of 4.4 is finished.

By Theorem 4.4 and ([3], (4.1), p. 133) we have the following corollary:

4.5. *If $y_0 \in Y \in$ ANR, then the group $m\text{-}\pi_2(Y, y_0)$ is isomorphic to the n-th homotopy group $\pi_n(Y, y_0)$.*

Let us remark that if the group $\pi_n(Y, y_0)$ of a space (Y, y_0) is isomorphic to the group $\underline{\pi}_n(Y, y_0)$, e.g., when Y is locall n-connected (see [11], also [10]), then by an analogous argument as in the proof of 4.4 we infer that $\mathfrak{J}$ is an epimorphism.

Notice that a certain kind of classes of multi-homotopy was investigated in [5], namely multi-functions from finite-dimensional compacta to S^n with values which are cellular subsets of S^n. Our Theorem 4.4 is a certain analogue of Theorems 2 and 3 in [5].

4.6. THEOREM. *Let $f\colon (Y', y_0') \to (Y, y_0)$ be a cellular map of Y' onto $Y \in$ ANR. Then, assigning*

$$[g] \mapsto [f^{-1}\circ g]$$

to every map $g\colon (S^n, x_0) \to (Y, y_0)$, we get a monomorphism of the group $\pi_n(Y, y_0)$ into the group $m\text{-}\pi_n(Y', y_0')$.

Proof. We show that the function defined in the theorem is a homomorphism. Let $g_1, g_2\colon (S, x_0) \to (Y, y_0)$ be maps ($S = S^n$). Then

$$[g_1]\cdot[g_2] = [g_1 \nabla g_2] \to [f^{-1} \circ (g_1 \nabla g_2)].$$

But

$$f^{-1} \circ (g_1 \nabla g_2) \overset{m}{\simeq} (f^{-1} \circ g_1) \nabla (f^{-1} \circ g_2)$$

under the multi-homotopy $\chi\colon S \times I \to Y'$ defined by the conditions

$$\chi | S^+ \times I = f^{-1} \circ g_1, \quad \chi | S^- \times I = f^{-1} \circ g_2$$

and, for $x \in S \setminus (S^+ \cup S^-)$,

$$\chi(x, 0) = f^{-1}(y_0) \ni y_0', \quad \chi(x, t) = \{y_0'\} \quad \text{for} \quad t \in (0, 1\rangle.$$

Hence $[f^{-1} \circ (g_1 \nabla g_2)] = [f^{-1} \circ g_1]\cdot[f^{-1} \circ g_2]$. We now prove that the homomorphism is a monomorphism. Let $Y' \subset Q$. Since $Y \in$ ANR, there exist a neighborhood $U \in$ ANR of Y' in Q and a map $f'\colon U \to Y$ such that $f'|Y' = f$. The assumption $f^{-1} \circ g \overset{m}{\simeq} \theta$ for a map $g\colon (S, x_0) \to (Y, y_0)$ implies the existence of a multi-map $\chi\colon S \times I \to Y'$ such that $\chi | S \times \{0\} = f^{-1} \circ g$ and $\chi | S \times \{1\}$ is the constant map of S into y_0'. It follows from Theorem 4.4 that χ is multi-homotopic in U to a map $g_1\colon S \times I \to Y'$, and moreover the multi-homotopy χ_1 connecting χ with g_1 is a map for $0 < t \leqslant 1$. Hence we infer that $h\colon S \times I \to Y$ defined as

$$h(x, t) = \begin{cases} f'\{\chi_1[(x, 0), 2t]\} & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ f'\{\chi_1[(x, 2t-1), 1]\} & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is a homotopy joining the map $g\colon (S, x_0) \to (Y, y_0)$ with the constant map of S into y_0. This completes the proof.

4.7. Corollary. *If $f\colon (Y', y_0') \to (S^n, x_0)$ is a cellular map of Y' onto S^n, then the group m-$\pi_n(Y', y_0')$ is isomorphic to $\pi_n(S^n)$.*

4.8. Example. Let Y be the same set as in Example 2.27. Put $a = (2, 1)$, $b = (1, 1)$. Then $\mathrm{Sh}(Y, a) = \mathrm{Sh}(Y, b)$ (see [3], (15.1), p. 240). If $x \in S = S^n$ and φ is a multi-map of S into Y, then

(1) $$\varphi(x) \subset Y \setminus L \quad \text{or} \quad \varphi(x) \subset L$$

Furthermore,

(2) $$\varphi(S) \subset Y \setminus L \quad \text{or} \quad \varphi(S) \subset L.$$

Condition (1) is clear. We show that also condition (2) is satisfied. It follows from the upper semi-continuity of φ and condition (1) that the inverse image of $Y \setminus L$ under φ is closed in S. If (2) is not true, then there is a point $x \in S$ such that in each of its neighborhoods in S there exists a point $x' \in S$ with the property that $\varphi(x') \subset L$. Let $s \in (Y \setminus L) \setminus \varphi(x)$. Take a closed neighborhood G of $\varphi(x)$ in Y, disjoint with the point s. Then there is a neighborhood V of x in S, such that $\varphi(V) \subset G$ (we may assume V to be an arc).

Let $x' \in V$ be the point described above. Then the image $\varphi(V) \subset G$ of the connected set V contains points lying in different components of the set G, which contradicts the property of preserving connectedness under multi-maps. Thus, condition (2) is proved.

From (2) we infer that m-$\pi_1(Y, a) = 0$ and that m-$\pi_1(Y, b)$ is a cyclic infinite group. Hence we infer that multi-homotopy groups are not the invariants of shape of pointed spaces.

4.9. Example. Let $\hat{S}$ be the union of the circles S_i lying in the plane, $i = 0, 1, 2, \ldots$ with centres 0 and radii r_i, such that

$$r_0 = 1, \quad r_i = \frac{i}{i+1} \quad \text{for} \quad i = 1, 2, \ldots$$

In the set $\hat{S} \times I$ we identify each circle $S_i \times \{1\}$ with the circle $S_{i+1} \times \{0\}$, $i = 1, 2, \ldots$, the circle $S_0 \times [0]$ with $S_0 \times \{1\}$ and the circle S_1 with a point. Let $a \in S_0$. Denote by T the union of the resulting set and the disc whose boundary is the circle $\{a\} \times I$ in T. Then T is approximatively 1-connected, hence it follows that the group $\underline{\pi}_1(T, a)$ is trivial. However, m-$\pi_1(T, a) \neq 0$, because there is an embedding of the sphere S^1 into T (for instance with values in $S_0 \times \{0\}$) which is not multi-homotopic to the constant map of S^1 into a (the proof of this fact is similar to that given in Example 4.8). Moreover, one can prove that $T \notin m$-ANR (cf. Example 2.27).

Example 4.9 and Theorems 4.4 and 4.6 suggest a positive answer to the following problem:

4.10. Problem. Is it true that homomorphism $\mathfrak{J}$ defined in Theorem 4.4 is an isomorphism if $Y \in m$-ANR?

4.11. Problem. Is it true that the multi-homotopy groups are invariant under cellular maps?

4.12. Problem. Are the multi-homotopy groups of m-ANR-sets finitely generated?

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On elementary cuts in models of arithmetic

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Abstract. Let $M \models P$ (= Peano arithmetic). We put $Y := \{N \underset{\text{end}}{\subset} M\colon N \prec M\}$. This family, as each family of initial segments of M, is simply ordered by inclusion. The order type of Y heavily depends on M; we shall compute this order type in the following cases: (a) M is countable and recursively saturated and (b) M is saturated. In both cases the proofs give fairly complete description of the situation.

We assume that the reader is familiar with saturated models (see Chang, Keisler [1]) and with recursively saturated models (see Schlipf [5]). We use standard model-theoretic terminology and notation.

§ 1. **The recursively saturated case.** Toward this section let $M \models P$ be recursively saturated. Our result is the following.

THEOREM 1. *If M is countable, then Y is of the order type of the Cantor set 2^ω with its lexicographical ordering*:

$$b^1 < b^2 \equiv (\exists\, n \in \omega)(b_n^1 = 0 \wedge b_n^2 = 1 \wedge (\forall\, m<n)(b_m^1 = b_m^2)).$$

Before proving this we shall prove some lemmas. For $a \in M$ we shall denote by $M(a)$ the closure under the initial segment of the Skolem closure of a; formally

$$M(a) := \{b \in M\colon \text{there exists a parameter-free term } t(v) \text{ such that } M \models b < t(a)\}.$$

By Gaifman [2, Theorem 4.1] $M(a) \underset{\text{end}}{\prec} M$.

LEMMA 2. *$M(a)$ is not recursively saturated.*

Proof. $M(a)$ omits the type $\{v > t(a)\colon t \text{ is a term}\}$. ∎

Let $Y_1 = \{N \in Y\colon N \text{ is not recursively saturated}\}$. Our first aim is to prove the converse of Lemma 2, it will be our Lemma 4.

LEMMA 3 (with W. Marek). *If $D \subseteq Y$ has no greatest element, then $\bigcup D$ is recursively saturated.*

Proof. Let $p(v)$ be a recursive type in parameters $b_1, \ldots, b_k$. There exists an $N \in D$ such that $b_1, \ldots, b_k \in N$ (in fact, D is linearly ordered by inclusion), and so by the assumption there exists $N_1 \in D$ such that an $N \underset{\text{end}}{\prec} N_1$. Therefore pick $c \in N_1$ such that $\forall a \in N\ M \models a < c$. Now consider the type $p(v) \cup \{v < c\}$. This is still a recursive type and consistent, and so it is realized in M. But any of its realizations is in $\bigcup D$. ∎