Proof. Use Theorem 3.4 and the observation that for any open expansion \( \{ U(L) : L \in \mathcal{L} \} \) of \( \mathcal{L} \), the set \( \{ x \in X : \{ L \in \mathcal{L} : x \in U(L) \} \text{ is finite} \} \) is an \( F_{\alpha} \)-set.

We close this paper with a question related to the results of Theorems 3.2 and 3.3 above.

**Question 3.6.** Does there exist in ZFC, a topological space that is not \( \sigma \)-discrete but whose every subset is a \( G_\delta \)-set?

Under MA+\( \neg \)CH, such spaces do exist, and they can even be metrizable (in fact, subspaces of \( R \); see [15] or [14]). By a result in [13], no space giving an affirmative answer to the above question could be normal and first countable.

Added in proof. Independently of the author, Z. Balogh has shown that, under PMEA, a space of character less than \( \aleph_1 \) is \( \sigma \)-discrete, if every subset of the space is an \( F_{\alpha} \)-set.

References


Reprints in the homotopy of multi-maps

by

A. Suszycki (Warsaw)

Abstract. By means of upper semi-continuous multi-functions defined on compacts and with values of trivial shape we introduce the notions of multi-retracts and multi-homotopies. We give some characterization of absolute multi-retracts and absolute neighborhood multi-retracts and apply the notion of multi-homotopy to the construction of groups, called multi-homotopy groups. In particular, we show that if \( Y \) is ANR, then the \( n \)-th multi-homotopy group of the space \( (Y, \varphi) \) is isomorphic to the \( n \)-th homotopy group of this space.

1. Introduction. Throughout this paper all spaces are compact and metric. By a multi-function \( \varphi \) from a space \( X \) to a space \( Y \) \( (\varphi : X \to Y) \) we mean one that assigns to every point \( x \in X \) a closed non-empty subset \( \varphi(x) \) of \( Y \). The upper semi-continuity (shortly u.s.c) of \( \varphi : X \to Y \) means that the graph \( \Phi \) of \( \varphi \) defined as

\[
\Phi = \{(x, y) \in X \times Y : y \in \varphi(x)\}
\]

is closed in \( X \times Y \). A map denotes, as usual, a continuous function. The notion of a multi-function is understood in the sense of Borsuk [3]. A u.s.c. multi-function \( \varphi : X \to Y \) is called a multi-map of \( X \) into \( Y \) if \( \varphi(x) \) is a set of trivial shape for every \( x \in X \). By an extension of a u.s.c. multi-function \( \varphi : X \to Y \) to \( M \supseteq X \) we mean a u.s.c. multi-function \( \varphi' : M \to Y \) such that \( \varphi'(x) = \varphi(x) \) and \( \varphi'(x) \) has the shape of a point for every \( x \in M \setminus X \). We say that a map \( f \) of \( X \) onto a cellular map \( (\varphi) \) if \( f^{-1}(x) \) has trivial shape for every \( x \in X \). Let us note that, if \( f \) is a cellular map of \( Y \) onto \( X \), then the multi-function \( \varphi : X \to Y \), defined by the formula \( \varphi(x) = f^{-1}(x) \), is a multi-map. Let us call such a multi-function an inverse of the map \( f \). We say that \( Y \) is countable-dimensional if it is the union of a countable family of finite-dimensional subspaces.

In the sequel we will need the following theorems:

1.1. **Theorem** (Kozlowski [9] thms 9 and 12). Let \( f \) be a cellular map of a space \( Y \) onto \( X \) such that the set \( \{ x \in X : f^{-1}(x) \text{ is a nondegenerate set} \} \) is contained in a compact and countable-dimensional subset of \( X \). Then for every closed subset \( A \) of \( X \) the map \( f : f^{-1}(A) : f^{-1}(A) \to A \) is a shape equivalence. Moreover, if \( Y \in \text{ANR} \), then \( X \in \text{ANR} \).

1.2. **Theorem** ([14], [15]). Let \( \psi \) be a multi-map of a space \( X \) into \( Y \in \text{ANR} \), where \( X \in M \). If \( \dim(M \setminus X) < \infty \) or \( X \) is countable-dimensional, then there exists a neighborhood \( U \) of \( X \) in \( M \) such that \( \psi \) has an extension onto \( U \). Moreover, if \( Y \in \text{AR} \), then \( \psi \) has an extension onto \( M \).
To prove the sufficiency let us put \( X = Y \subseteq Q = M, f = \text{id}_Y \). Then there is a multi-map \( \phi : Q \to Y \) such that \( y \in \phi(y) \) for every \( y \in Y \). It follows that \( \phi \) is a multi-retraction of \( Q \) to \( Y \), and by 2.5 we infer that \( Y \in m-\text{AR} \).

In a similar way one can prove the following

2.8. \( Y \in m-\text{ANR} \iff \forall \text{ for each closed subset } X \text{ of a space } M \text{ and for every map } f : X \to Y \text{ there exist a neighborhood } U \text{ of } X \text{ in } M \text{ and a multi-map } \phi : U \to Y \text{ such that } f(x) \in \phi(x) \text{ for every } x \in X \).

Notice that it is not possible to replace the condition \( f(x) \in \phi(x) \) in 2.7 and 2.8 by the condition \( \phi(x) = f(x) \). This results from the following known fact:

2.9. If \( \text{id}_Y \) has an extension \( \phi \) onto a locally connected space \( X \supseteq Y \), then \( Y \) is also locally connected.

Proof. Assume that \( Y \) is not locally connected at a point \( p \in Y \). Then there is a closed neighborhood \( U \) of \( p \) in \( Y \) and a sequence \( (p_n) \) of points of \( Y \) convergent to \( p \) and such that if \( i \neq j \), then \( p_i \) and \( p_j \) belong to different components of the neighborhood \( U \). Let \( 0 \leq \varepsilon < \text{diam} U \). Since \( \phi \) is u.s.c., there exists a number \( \eta \), \( 0 < \eta < \varepsilon \), such that if \( q(x, p) < \eta \), \( x \in X \), then \( \phi(x) \) is contained in the ball \( K \subseteq Y \) with center \( p \) and radius \( \varepsilon \). It follows from the local connectedness of \( X \) that for arbitrarily given \( \delta > 0 \) there exists a closed and connected neighborhood \( V \) of \( p \) in \( X \) with a diameter less than \( \delta \). Let us take \( \delta = \eta \). There is an index \( i \) such that \( p_i \in V \) and \( p_i \neq p \). Then the image \( \phi(V) \subseteq Y \) of a connected set is a connected set containing the points \( p \) and \( p_i \), which is impossible. Hence it follows that \( Y \) must be locally connected.

We now formulate several properties of \( m-\text{AR} \) (resp. \( m-\text{ANR} \))-spaces which give a partial answer to the following problem: Are \( m-\text{AR} \) (resp. \( m-\text{ANR} \))-spaces invariant under cellular maps?

2.10. Let \( \phi : X \to Y \) be a multi-map and \( g : Y \to X \) a map such that \( y \in \phi(g(y)) \) for every \( y \in Y \). If \( X \in m-\text{ANR} \), then \( Y \in m-\text{ANR} \). Moreover, if \( X \in m-\text{AR} \), then \( Y \in m-\text{AR} \).

Proof. Let \( Y \in m-\text{ANR} \). Since \( X \in m-\text{ANR} \), there exists a neighborhood \( V \) of \( Y \) in \( M \) and a map \( g' : V \to U \) such that \( \phi(g'(y)) = g(y) \). Then the composition \( \phi g' : V \to Y \) is a multi-map satisfying the condition \( y \in \phi g'(y) \) for every \( y \in Y \); therefore \( \phi g' \) is a multi-retraction of \( U \) to \( Y \). We infer by 2.6 that \( Y \in m-\text{ANR} \).

If \( X \in m-\text{AR} \), then putting \( U = M \) in the above proof we find that \( Y \in m-\text{AR} \).

In particular, if \( g \) is a cellular map of \( Y \) onto \( X \in m-\text{ANR} \), then by 2.10 \( Y \in m-\text{ANR} \) (resp. \( m-\text{AR} \)).

2.11. Example. Consider a cellular map of a non-movable compactum \( Y \) onto the Hilbert cube \( Q \) (see [16]). Non-movability \( Y \) implies that \( Y^* \notin \text{FANR} \). However, by 2.10 \( Y^* \in m-\text{AR} \).

Consequently, we infer by the fact that every FAR-space is an \( m-\text{AR} \)-space that the class of \( m-\text{AR} \)-spaces contains the class of FAR-spaces as its proper subclass.
2.12. Let \( f \) be a cellular map of a space \( X \) onto \( Y \in m\text{-ANR} \), \( Y \subseteq N \subseteq \text{AR} \). If there exists a neighborhood \( V \) of \( Y \) in \( N \) and a multi-retraction \( \sigma : V \to Y \) such that \( \dim \sigma(y) < \infty \) for each \( y \in V \), then \( X \in m\text{-ANR} \). Moreover, if \( Y \in m\text{-AR} \) and \( V = N \), then \( X \in m\text{-AR} \).

Proof. Let \( X \in Q \) and let \( f' : Q \to N \) be a map such that \( f'(z) = f(z) \) for \( z \in X \). It follows by Theorem 1.1 that the composition \( f^{-1} \circ \sigma : V \to X \) (\( f^{-1} \) being the inverse of \( f \)) is a multi-map. Denote \( U = f^{-1}(V) \). Then \( f^{-1} \circ \sigma \circ f : U \to X \) is also a multi-map and in addition satisfies the following condition:

\[
f^{-1}(\sigma(f'(z))) = f^{-1}(\sigma(f(z))) = f^{-1}(f(z)) = x \quad \text{for every} \quad z \in X.
\]

Therefore \( f^{-1} \circ \sigma \circ f' \) is a multi-retraction of \( U \) to \( X \), and to finish the proof it suffices to apply 2.6. If \( Y \in m\text{-AR} \) and \( V = N \), then the proof of Theorem 2.12 is similar.

Let us observe that if \( f \) is a cellular map of \( X \) onto \( Y \in Y^* \) (\( Y^* \) being as in Example 2.11), then by Theorem 2.12 \( X \in m\text{-AR} \). Indeed, the cellular Taylor map \( f^* \) of \( Y^* \) onto \( Q \) has finite-dimensional sets as its point-inverses (see [16]), and we infer by the proof of 2.10 that the point-inverses are the values of the multi-retraction of \( M \) to \( Y \).

2.13. Theorem. Let \( f \) be a cellular map of \( X \) onto \( Y \) and suppose that one of the sets \( X, Y \) is an \( m\text{-FAR} \)-space. Then the cylinder \( M_f = Y \times X \cup f(X) \) of the map \( f \) is an \( m\text{-AR} \)-space.

Proof. Let \( Y \in \text{FAR} \). Every point \( z \in M_f \) is of the form \((y, t), \) where \( y \in Y \) and \( t \in I \). Let us fix a point \( x_0 \in X \) and let \( y_0 \in Y \) be a point in \( f^{-1}(x_0) \). We show that there is a multi-map \( \phi : M_f \to M_f \) such that

\[
\phi(z) = \phi((y, t)) = (y \cup F_y \cup F_\infty) \quad \text{for} \quad z \in M_f.
\]

Then \( \phi \) satisfies condition (1), and it is easy to check that \( \phi \) is a multi-map. Now, let \( M_f = Q \). We define the multi-map \( \sigma : Q \to M_f \) by the following conditions:

\[
\sigma(z) = x_0 \quad \text{for} \quad z \in M_f.
\]

Hence \( M_f \in m\text{-AR} \).

If \( X \in \text{FAR} \), then it suffices to repeat the procedure, defining \( \phi(z) = X \cup F_y \cup F_\infty \) for \( z \in M_f \).

2.14. Corollary. Let \( f \) be a cellular map of \( Q \) onto \( X \) such that the set \( B = \{ f^{-1}(x) \mid x \in X \} \) is a nondegenerate set.

\[
\text{is a} \ Z\text{-set in} \ Q. \text{Then} \ X \in m\text{-AR} \text{-space.}
\]

Proof. We may assume that \( Q = Q_1 \times I \), where \( Q_1 \cong \mathbb{Q} \), and that \( B \subseteq Q_1 \times \{1\} \).

Define a map \( f_1 : Q_1 \to f(Q_1) \) by the formula \( f_1(y) = f(y) \) for \( y \in Q_1 \), where we identify \( Q_1 \) with \( \mathbb{Q} \times \{1\} \). Then \( X \) is the cylinder of the cellular map \( f_1 \); therefore we may apply Theorem 2.13.

We say that non-empty sets \( F_1, \ldots, F_k, k \subseteq N \), constitute a simple decomposition of a space \( X \) if their union is \( X \) and the intersection of any number of the sets is the empty set or an \( \text{FAR} \)-set.

2.15. Theorem. If \( X \) has a simple decomposition \( F_1, \ldots, F_k \), then there exist closed neighborhoods \( U_i \) of \( F_i \), \( i = 1, \ldots, k, \) in \( M \supseteq X \) and a multi-retraction \( v : U = \bigcup_{i=1}^k U_i \to X \) such that, if \( x \in U \) and \( U_i = U_{i_1} \cup \cdots \cup U_{i_m} \) is the union of all sets \( U_j, 1 \leq i_j \leq k, \) that contain the point \( x \), then

\[
v(x) = F_{i_1} \cup \cdots \cup F_{i_m}.
\]

Proof. We show that there exist closed neighborhoods \( U_i \) of the sets \( F_i \), \( i = 1, 2, \ldots, k \) such that for every sequence \( (j)_{j=1}^m \) of the sequence \( (1, 2, \ldots, k) \) the following implication is satisfied

\[(1) \quad U_{j_1} \cap \cdots \cap U_{j_n} \neq \varnothing \Rightarrow F_{j_1} \cap \cdots \cap F_{j_n} \neq \varnothing
\]

To this end suppose the contrary. Let us take decreasing sequences \( (U_{i,j})_{j=1}^\infty \) of closed neighborhoods of the sets \( F_i \), \( i = 1, 2, \ldots, k \) such that \( U_{j,i} \cap F_i = F_i \) for every \( i = 1, 2, \ldots, k \). Then there exist a subsequence \( (j)_{j=1}^m \) of the sequence \( (1, 2, \ldots, k) \) and a sequence \( \sigma_{j=1}^m \) of points of \( U = \bigcup_{i=1}^k U_i \) convergent to a point \( p \in X \) such that

\[(2) \quad p \in \bigcup_{j=1}^m F_{i,j} \quad \text{for every} \ n \in N,
\]

\[(3) \quad U_{i,j} = \varnothing.
\]

It follows by (2) that \( p \in \bigcup_{i,j=1}^\infty U_{i,j} \) for every \( n \in N \). Therefore \( p \in \bigcup_{i,j=1}^\infty F_{i,j} \), which contradicts (3). This proves the existence of neighborhoods \( U_i \) satisfying (1). Now, let \( v \) be a multi-function from \( U = \bigcup_{i=1}^k U_i \) to \( X \), defined as in the theorem. Notice that if \( x \in X \) then the union of all sets of the sequence \( (F_1, F_2, \ldots, F_k) \) containing the point \( x \) is an \( \text{FAR} \)-set. Hence (1) implies that all values of \( v \) have a trivial shape. Moreover, since the sets \( U_i, i = 1, 2, \ldots, k \) constitute a closed covering of \( U \), we see that \( v \) is u.s.c. Thus \( v \) is a multi-retraction of \( U \) to \( X \).

2.16. Corollary. A set having a simple decomposition is an \( m\text{-ANR} \)-set.

2.17. Remark. The set \( Y^* \) in Taylor's example (see 2.11) is an \( m\text{-AR} \)-space, but does not have a simple decomposition. Indeed, if \( Y^* \) had a simple decompo-
sition, then by the theorem on the union of two FANR-spaces (see [6]), $Y^*$ would be an FANR-space, which is impossible, because $Y^*$ is non-movable.

There is an example ([3], (4.18), p. 156) of a 2-dimensional set $X$ in ANR which is not decomposable into a finite number of AR-sets. It follows that there exist points of the set $X$ no neighborhood of which is an AR-set. If the set $X$ had a simple decomposition, then for each $x \in X$ the union $F_x$ of all sets of the decomposition containing the point $x$ would be a neighborhood of $x$ in $X$ and an FAR-set. Using the fact that FAR-sets are contractible in each of their neighborhoods (in an ANR-space) and applying the argument as in ([4], (4.17), p. 155) (in particular using [4], (4.1), p. 153), one derives a contradiction of the fact that $F_x \subset F_{X \subset FAR}$.

Hence we obtain the following fact:

2.18. There exist ANR-spaces having no simple decompositions.

Since every $Y \subset FAM$ has a simple decomposition, we have:

2.19. Every FAR-set is an m-AR-set.

For finite-dimensional multi-retracts we have the following properties:

2.20. If $Y$ has finite dimension, then

$$Y \subset FAM \iff Y \subset FAR.$$  

Indeed, let $Y$ be a multi-retract of a finite-dimensional space $N \subset AR$. It follows by 2.1 that Sh($Y$) $\leq$ Sh($N$). Hence $Y \subset FAR$. The converse follows from 2.19.

2.21. Every finite-dimensional $Y \subset m-ANR$ is an FANR-space.

To prove 2.21 assume that $Y$ is a subset of a finite-dimensional cube $I^n$. Then there exist a neighborhood $U$ of $Y$ in $I^n$ and a multi-retraction $v: U \to Y$. Moreover, we may assume $U \subset ANR$. Then by 2.1 Sh($Y$) $\leq$ Sh($U$), and we infer by (12), (1.3), p. 254) that $Y \subset FANR$.

If $Y \subset m-ANR$ has infinite dimension, then it need not be an FANR-space (see 2.11).

Let us remark that the fundamental dimension $Fd(Y)$ for $Y \subset m-ANR$ can be infinite. To prove this it suffices to observe that if $Fd(Y)$ for $Y$ see 2.11 were finite, then, by the movability of $Q$ and the result of Bogatyi ([2], p. 261) on the shape equivalence of compacta under cellular maps, we would have Sh($Y$) = Sh($Q$). But it is known that Sh($Y$) $\neq$ Sh($Q$).

2.22. Remark. By Theorem 1.1 properties 2.20 and 2.21 remain true for countable-dimensional spaces.

Let us denote by $H_n(Y)$ the $n$th Vietoris homology group of $Y$ with the abelian coefficient group $\mathbb{Q}$. If $f$ is a cellular map of $Y$ onto $X$, then by ([4], (2.3) and (4.7)) $f$ induces an isomorphism between the fundamental groups $\pi_n(Y, x)$ and $\pi_n(X, f(x))$ for every $y \in Y$ and $f$ induces an isomorphism of the homology groups $H_n(Y)$ and $H_n(X)$, $n \in N$.

Let $\sigma: X \to Y$ be a multi-retraction and $\Sigma \subset X \times Y$ the graph of $\sigma$. Fix a point $(x, y) \in \Sigma$ and denote by $p$ and $q$ the projections of $\Sigma$ onto $X$ and $Y$, respectively.

Since $p$ is a cellular map of $X$ onto $X$, by the theorems mentioned above, $p$ induces the isomorphisms

$$p^*: H_n(X) \to H_n(X), \quad n = 0, 1, 2, \ldots$$

and

$$p^+_n: \pi_n((X, x)) \to \pi_n((X, x)), \quad n \in N.$$  

Let $p_*^t$ and $p^*_t$ denote the isomorphisms inverse to $p_*$ and $p^*$, respectively, and $g_0$ and $g_1$ the homomorphisms of homology and fundamental groups induced by the projection $q$. Then we say that the compositions $f_0 = g_0 \circ p_*^0: H_n(X) \to H_n(Y)$ and $f_1 = g_1 \circ p^*_1: \pi_n((X, x)) \to \pi_n((Y, x))$ are the homomorphisms induced by $\sigma$.

2.23. If $\sigma: X \to Y$ is a multi-retraction, then the homomorphisms $(e|Y)_n$ and $(e|Y)_n$ induced by $e|Y$: $Y \to Y$ are the identity homomorphisms.

Indeed, let $\Sigma \subset Y \times Y$ denote the graph of $e|Y$ and let $p$ and $q$ denote the projections of $\Sigma$ onto the first and the second factor, respectively. Then 2.23 follows by the commutativity of the diagram

$$
\begin{array}{c}
\text{diag} Y \\
\downarrow \text{diag} Y \\
\text{diag} Y \\
\downarrow \text{diag} Y
\end{array}
$$

where $\text{diag} Y$ means the diagonal of the set $Y \times Y$.

Since the multi-retract $Y$ of $X$ is homeomorphic to the diag $Y$ which is a retract of $X$ and since $p_*$ and $p^*$ are isomorphisms, it follows by ([3], (5.3) and (5.4), p. 191) that

2.24. If $Y$ is a multi-retract of $X$, then for every $n = 0, 1, \ldots$ the group $H_n(Y)$ is a direct divisor of the group $H_n(X)$ and for $n > 1$ the group $\pi_n(Y, x)$ is a direct divisor of $\pi_n(X, x)$.

By 2.24 we have

2.25. If $Y \subset m-ANR$, then for $n \neq 0$ and every $y \in Y$ the groups $H_n(Y)$ and $\pi_n(Y, y)$ are trivial.

2.26. If $Y \subset m-ANR$, then almost all groups $H_n(Y)$ are trivial; moreover, if $Y$ is finitely generated, then all groups $H_n(Y)$ are finitely generated.

It follows from 2.26 and the example of a locally contractible compactum with a positive $n$th Betti number for every $n = 0, 1, \ldots$ ([6], (1.1), p. 124) that there exist locally contractible compacta that are not $m$-ANR-spaces.

2.27. Example. Let $L$ be a curve lying in the plane $E^2$ and defined in polar coordinates $(r, \theta)$ as follows:

$$r = 1/|a+1| \quad \text{for} \quad a \in (1, \infty).$$

Denote by $Y$ the closure (in $E^2$) of $L$. Then $Y \subset FANR$. We show that $Y \notin m-ANR$. 


To this end assume the converse. Then there is a compact neighborhood $U$ of $Y$ in $E^2$ and a multi-retraction $r: U \to Y$. Let $y$ be a point of the circle $S^1 = \partial Y$. Then $s(y)$ is contained in $S$, and moreover it is an arc. Let $y'$ be a point of the set $S \setminus y$. We take a closed neighborhood $V$ of $s(y)$ in $Y$ such that $y' \notin V$. Then $V$ is the set with countable many components. We infer by the upper semi-continuity of $s$ that there exists a closed neighborhood $U_0$ of $y$ in $U$ such that $s(U_0) \subseteq V$. Let $B_0 \subseteq U_0$ be a closed ball in $U$ with center $y$. Denote by $y_0$ a point of $B_0 \cap S$. Then $s(y_0)$ and $s(y)$ lie in different components of $V$, which is impossible by the fact that the image of any connected set (in our case $U_0$) under any multi-map is connected. Thus must be $y' \notin m$-ANR.

2.28. Example: Denote by $X_0$ the subset of the plane $E^2$ which is the closure of the graph $L_0$ of the function $f(x) = \sin(\pi x)$ for $0 < x < 1$. Let $a_0 = (0, -1), b_0 = (0, 1)$. Write $X_0 = X_0 \times \{1\}$, $(a_0, t) = (a_1, t) = b_1, b_2 \in X_0$, $i = 1, 2, 3$. Assume now that $X \subseteq E^2$ is the set formed from the sum $X_1 \cup X_2 \cup X_3$ by the identification of the points $a_1, b_1, b_2, b_3, b_4, a_4$ with $a_1, b_1, b_2, b_3, b_4, a_4$. Then $X$ has a simple decomposition, hence $X \subseteq m$-ANR. Since $X$ is not locally connected, it is not the image of an ANR-set under any map. We show that

(a) There exists no cellular map of $X$ onto any ANR-set.

To prove (a) suppose, conversely, that there exists a cellular map $g$ of $X$ onto an ANR-set $Y$. Then there are a compact neighborhood $U$ of $X$ in $E^2$ and a map $g: U \to Y$ such that $g| = \varphi$. Therefore the composition $\sigma = g^{-1} g'$, where $g^{-1}$ is the inverse of $g$, is a multi-retraction of $U$ to $X$ such that, for $x \in X, \sigma(x)$ is a point-inverse under $g$. Let us take a point $x \in A = X \times \{0, 1\}$, $i = 1, 2, 3$. Then we must have $\sigma(x) \in A$ (otherwise there would be a point $x' \in A \setminus \sigma(x)$ and a closed neighborhood $Y$ of $\sigma(x)$ in $X$ such that $x' \notin Y$, which, by a similar argument as in Example 2.27, is impossible). However, the condition $A \subseteq \sigma(x)$ fails, because $\sigma(x)$ is a point-inverse under the cellular map $g$ and hence has the shape of a point. The contradiction proves condition (a).

Let us formulate some problems concerning multi-retracts.

2.29. Problem: Is it true that every plane $m$-ANR-set has a simple decomposition?

2.30. Problem: Is it true that every retract (deformation retract) of an $m$-space (resp. $m$-ANR-space) is an $m$-AR-space (resp. $m$-ANR-space)?

In particular,

2.31. Problem: Is the image of $O$ under any cellular map an $m$-AR-space?

2.32. Problem: Is it true that finite-dimensional $m$-ANR-spaces have the shape of a compact polyhedron?

3. Multi-homotopies. Let $(X, x_0)$ and $(Y, y_0)$ be spaces with any given basic points $x_0$ and $y_0$. We write $\phi: (X, x_0) \to (Y, y_0)$ if $\phi$ is a multi-map of $X$ into $Y$ such that $\phi(y_0) = \phi(x_0) = y_0$, and then by a homotopy of the maps $f$ and $g$ we understand a map $h: X \times I \to Y$ satisfying the conditions $h|X \times \{0\} = f$, $h|X \times \{1\} = g$ and $h(x_0, t) = y_0$ for every $t \in I$. We call a homotopy $h$ an isotopy if $h|X \times \{t\}$ is a homeomorphism onto $h(X \times \{t\})$ for each $t \in I$.

Let $\phi_1, \phi_2: (X, x_0) \to (Y, y_0)$ be multi-maps. We say that a multi-map $\chi: X \times I \to Y$ is a multi-homotopy connecting $\phi_1$ with $\phi_2$ if $\chi(x, 0) = \phi_1(x), \chi(x, 1) = \phi_2(x)$ and $\chi(x, t) \in (x_0, t)$ for every $x \in X$ and $t \in I$. In this case the multi-maps $\phi_1$ and $\phi_2$ are said to be multi-homotopic (notation: $\phi_1 \equiv \phi_2$).

3.1. The relation of multi-homotopy is an equivalence relation; hence the collection of all multi-maps $(X, x_0) \to (Y, y_0)$ decomposes into disjoint classes of multi-maps multi-homotopic to one another, called multi-homotopy classes.

Let us note that, if $f_1, f_2: (X, x_0) \to (Y, y_0)$ are homotopic maps, then they are multi-homotopic. The converse is not true.

3.2. Example. Let $X$ be a non-contractible FAR-set. Let us select a point $x_0 \in X$. Assume that $f_1 = \text{id}_X$ and $f_2: X \to X$ is the constant map of $X$ into $x_0$. Then $f_1$ and $f_2$ are not homotopic, but they are multi-homotopic under the multi-homotopy $\chi: X \times I \to Y$ defined by the conditions

$\chi(x, 0) = \{x\}, \chi(x, 1) = \{x\}, \chi(x, t) \in (x_0, t), x \in X$.

If $\phi: (X, x_0) \to (Y, y_0)$ is a multi-map of $X$ into itself with the property that $\phi(x_0)$ for every $x \in X$, then we write $\phi = m$-$\text{id}_X$. If $\phi: (X, x_0) \to (Y, y_0)$ is a multi-map, then $\phi \equiv \theta$ will mean that $\phi$ is multi-homotopic to the map $\theta$ of $X$ into $Y$ such that $\theta(x) = y_0$ for every $x \in X$. A set $(X, x_0)$ is said to be multi-contractible, if $\text{id}_X \equiv \theta$.

3.3. If $\phi = m$-$\text{id}_X$, then $\phi \equiv \text{id}_X$.

Indeed, the conditions $\chi(x, 0) = \{x\}, \chi(x, 1) = \{x\}$ define the multi-homotopy connecting $\phi$ with $\text{id}_X$.

3.4. If $\phi: (X, x_0) \to (Y, y_0)$ is a multi-map of $X$ into $Y$ such that $\phi(x) = \phi(x_0)$ for every $x \in X$, then $\phi \equiv \theta$.

3.5. If $f$ is a map of $(X, x_0)$ into a multi-contractible space $(Y, y_0)$, then $f \equiv \theta$.

In fact, if $\chi: X \times I \to Y$ realizes the relation $\chi(x, 0) \equiv \theta$, then $\chi(x, t) \equiv \theta$ for every $x \in X$. Then $\phi \equiv \theta$.

3.6. Let $\phi: (X, x_0) \to (Y, y_0)$ be a multi-map of $X$ into $Y \in \text{FAR}$. Then $\phi \equiv \theta$.

To show this, it suffices to define the multi-homotopy $\chi: X \times I \to Y$ by the following conditions: $\chi(x, t) = \phi(x)$ for $t \in (0, 1)$ and $\chi(x, 1) = \chi(x, 0) = \{y\}$.

3.7. Let $\phi: (S^n, x_0) \to (Y, y_0)$ be a multi-map of the $n$-dimensional sphere $S^n$ into $Y$. Then the following conditions are equivalent

(a) $\phi$ has an extension onto the ball $B_0^n$.

(b) $\phi \equiv \theta$. 

— Fundamenta Mathematicae
Proof. Let \( \varphi' \) be the extension of \( \varphi \) onto \( B^{n}\). We define \( \chi: S^n \times I \to Y \) by the formula

\[
\chi(x,t) = \varphi'(1-t)x + t \varphi_0.
\]

Since \( \chi(x,0) = \varphi'(x) \), \( \chi(x,1) = \varphi'(x) = \varphi(x_0) \) and \( \chi(x,t) = \varphi(x_0) \) for every \( t \in I \) and \( x \in S^n \), then \( \chi \) is a multi-homotopy connecting \( \varphi \) with \( \chi|S^n \times \{1\} \). So, it suffices to apply 3.4. To prove the implication \( (b) \Rightarrow (a) \), we define the extension \( \varphi' \) of \( \varphi \) by setting \( \varphi'(x) = \{ y_0 \} \) for \( 0 < |x| \leq 1 \) and \( \varphi'(x) = \chi(x)/|x|, 2-2|x|/3 \) for \( 1 < |x| \leq 1 \), where \( \chi \) realizes the relation \( \varphi \cong \theta \).

3.5. Every \( m \)-ANR-set is multi-contraction.

Proof. Let \( X \in \mathcal{Q} \) be an \( m \)-ANR-set. Let us fix a point \( x_0 \in X \). Then there exists a multi-retraction \( \sigma: X \to x_0 \) such that \( x_\sigma(x) \) for every \( x \in X \). We define the multi-homotopy \( \chi: X \times I \to X \) connecting \( \sigma|X = m \cdot \sigma(x) \) with the multi-map \( \psi \), which assigns to every \( x \in X \) a set \( \sigma(x_0): \)

\[
\chi(x,t) = \sigma((1-t)x + t \sigma(x_0)).
\]

By 3.3 and 3.4 we get

\[
\text{id}_X \cong \sigma|X \cong \psi \cong \theta.
\]

This implies the following

3.9. FAR-sets are multi-contractions.

There exist multi-contraction spaces not having the shape of a point. For instance the non-movable compactum \( Y^* \) (see 2.11) is an \( m \)-ANR-set and, by 3.8, is multi-contraction.

We now prove the following theorem on the extension of a multi-homotopy:

3.10. Theorem. Let \( Y: X \times I \to Y \) be an ANR, where \( X \in M \) and \( \text{dim}(\mathbb{M} \times X) < \infty \) or \( M \) is countable-dimensional. If \( Y|X \times \{0\}: X \to Y \) has an extension \( \chi_0: M \times I \to Y \), then there exists a multi-map \( \chi: M \times I \to Y \), such that \( Y|X \times I \cong \chi \) and \( Y|M \times \{0\} \cong \chi_0 \).

Proof. Put \( P = M \times I \cup X \times I \). By Theorems 1.1 and 1.2, there is a neighborhood \( U \) of \( P \) in \( M \times I \) such that the multi-map \( \psi: P \to Y \) defined by the formulae

\[
\psi(x,0) = \chi_0(x) \quad \text{for} \quad x \in M,
\]

\[
\psi(x,t) = \chi(x,t) \quad \text{for} \quad x \in X \quad \text{and} \quad t \in I,
\]

has an extension \( \psi' \) onto \( U \). There exists a map \( f: M \times I \to U \) which is the identity onto \( P \) (see \[4\], (8.2), p. 94). Then it suffices to define \( \tilde{\psi}: M \times I \to Y \) as the composition

\[
\tilde{\psi} = \psi \circ f.
\]

For \( (x,t) \in X \times I \) we get

\[
\tilde{\psi}(x,t) = \psi'(f(x,t)) = \chi(x,t).
\]

For \( x \in M \)

\[
\tilde{\psi}(x,0) = \psi'(f(x,0)) = \psi(x,0) = \chi_0(x).
\]

Thus \( \tilde{\psi} \) is the required multi-map.

One can easily see that the classical homotopy extension theorem cannot be directly transferred to the case of multi-maps into \( m \)-ANR-sets. Instead, the supposition \( Y \in m \)-ANR leads to the following

3.11. Theorem. Let \( h \) be a map of \( P = M \times \{0\} \cup X \times I \) into \( Y \) \( m \)-ANR, where \( X \) is a subset of \( M \). Then there exists a multi-map \( \psi: M \times I \to Y \), such that for every \( (x,t) \in P \), \( h(x,t) \in \psi(x,t) \).

Proof. Let \( Y \) be a subset of the Hilbert cube \( Q \). Then there are a neighborhood \( V \) of \( Y \) in \( Q \) and a multi-retraction \( \sigma: V \to Y \). Let \( h': M \times I \to Q \) be a map satisfying the condition \( h'(x,t) = h(x,t) \) for every \( (x,t) \in P \). Put \( U = h'^{-1}(V) \). There exists (see \[4\], (8.2), p. 94) a map \( f: M \times I \to U \) which is the identity onto \( P \). We define \( \phi: M \times I \to Y \) as the composition \( \phi = \sigma \circ h' \). For \((x,t) \in P \) we get

\[
\psi(x,t) = \sigma(h'(f(x,t))) = \sigma(h(x,t)) = \sigma(h(x,t)).
\]

Since \( \psi \in \mathcal{G} \) for every \( \psi \in \mathcal{Y} \), we have \( h(x,t) \in \sigma(h(x,t)) \). This completes the proof.

3.12. Every multi-contraction \( m \)-ANR-space is an \( m \)-ANR-space.

Proof. Let \((X,x_0)\) be a multi-contraction \( m \)-ANR-set contained in \( M \in \mathcal{E} \). Let us denote by \( \chi: X \times I \to X \) the multi-homotopy connecting \( \text{id}_X \) with the map \( \chi|X \times \{0\} \) of \( X \) into \( x_0 \). Setting

\[
\psi(x,0) = \{ x_0 \} \quad \text{for} \quad x \in M, \quad \psi(x,t) = \chi(x,t) \quad \text{for} \quad x \in X \quad \text{and} \quad t \in (0,1),
\]

we get the multi-map \( \psi \) of \( P = M \times \{0\} \cup X \times I \) into \( X \). Since \( P \) is an \( m \)-ANR, there exists a neighborhood \( U \) of \( P \) in \( M \times I \) and a retraction \( r: U \to P \). There exists a map \( f: M \times I \to U \) which is the identity onto \( P \) (see \[4\], (8.2), p. 94). Therefore

\[
\psi = \sigma \circ r \circ f|\{x\} \quad M \times \{0\} \to X \quad \text{is a multi-map such that} \quad \psi(x) = \sigma(r(x)) = \sigma(h(x)) = \sigma(h(x,t)) = \sigma(h(x,0)) = \chi(x,0) \quad \text{for} \quad x \in x \times \{0\}. \quad \text{Thus} \quad \psi \quad \text{is a multi-retraction of} \quad M \quad \text{to} \quad X \quad \text{and we conclude that} \quad X \in m \ ANR.
\]

From this it follows the following corollary

3.13. The sphere \( S^n \) is not multi-contraction.

4. Multi-homotopy groups. Let \( S^n \) and \( S^m \) denote disjoint sets, homeomorphic to the closed half-spheres in the \( n \)-dimensional sphere \( S^n \), \( n \in \mathbb{N} \). By \( S^n \) we denote the set obtained from the disjoint union \( S^n \cup S^m \cup (S^{n-1} \times I) \) by identifying \( S^{n-1} \times \{0\} \) with the boundary \( S^n = (S^n \cap S^m) \times \{0\} \) and \( S^{n-1} \times \{1\} \) with the boundary \( S^m = (S^n \cap S^m) \times \{1\} \). Clearly \( S^n \cong S^m \). Let \( x_0 \) be a selected point of \( S^n \) belonging to \( S^{n-1} \times \{0\} \). If \( Y \) is a space with the basic point \( y_0 \), then the multi-homotopy class (see 3.1) containing the multi-map \( \phi: (S,x_0) \to (Y,y_0) \) we denote by \([\phi] \). In the set of the multi-homotopy classes we introduce the group operation.

To this end we first define the operation assigning to every two multi-maps \( \phi, \psi: (S,x_0) \to (Y,y_0) \) a multi-map \( \psi \psi: (S,x_0) \to (Y,y_0) \). It is known that one can define isotopies \( h_i: (S \times \{x_0\}) \to (S \times \{x_0\}), i = 1, 2, \)
such that $h_i((S \setminus \{x_0\}) \times \{1\}) = \{y_0\}$ for $i = 1, 2$, $h_1((S \setminus \{x_0\}) \times \{1\}) = S^+ \setminus S^+$, and $h_2((S \setminus \{x_0\}) \times \{1\}) = S^+ \setminus S^-$. If we set

$$\varphi \left( (S \setminus \{x_0\}, 1) \right) = \begin{cases} \varphi \left( (S \setminus \{x_0\}, 1) \right) & x \in S^+ \setminus S^+ \\ \varphi \left( (S \setminus \{x_0\}, 1) \right) & x \in S^+ \\
\varphi \left( (S \setminus \{x_0\}, 1) \right) & x \in S\{x_0\} \times (0, 1), \\
\psi \left( (S \setminus \{x_0\}, 1) \right) & x \in S^+ \setminus S^- \\
\psi \left( (S \setminus \{x_0\}, 1) \right) & x \in S^+ \setminus S^- 
\end{cases}$$

then we get the multi-map $\varphi \nabla \psi : (S, x_0) \to (Y, y_0)$. In particular, $(\varphi \nabla \psi)(x_0) = (y_0)$. For classes $[\varphi]$ and $[\psi]$ we define their group operation as follows

(i) $[\varphi][\psi] = [\varphi \nabla \psi]$. Let us verify that the multi-homotopy class $[\varphi \nabla \psi]$ does not depend on the choice of the multi-maps $\varphi$ and $\psi$, i.e., that if $\varphi', \psi' : (S, x_0) \to (Y, y_0)$ are multi-maps and $\varphi \cong \varphi'$, $\psi \cong \psi'$, then $[\varphi \nabla \psi] = [\varphi' \nabla \psi']$. Let $\chi_1, \chi_2 : S \times I \to Y$ denote multi-homotopies connecting $\varphi$ with $\varphi'$ and $\psi$ with $\psi'$, respectively. Then $\chi : S \times I \to Y$ defined by the formula

$$\chi(x, t) = \begin{cases} \chi_1(x, 1, t) & x \in S^+ \setminus S^+ \\ \chi_2(x, 0, t) & x \in S^+ \\
\chi_2(x, t) & x \in S\{x_0\} \times (0, 1), \\
\chi_2(x, 0, t) & x \in S^- \\
\chi_2(x, t) & x \in S^+ \setminus S^- 
\end{cases}$$

for every $t \in I$, is a multi-homotopy connecting $\varphi \nabla \psi$ with $\varphi' \nabla \psi'$. Therefore (ii) is satisfied.

It is not hard to prove (cf. the definition of homotopy groups) that the set of all multi-homotopy classes of multi-maps $(S, x_0) \to (Y, y_0)$ with the operation defined by (i) constitutes a group, abelian for $n \geq 3$. The neutral element of this group is the multi-map $\varphi \nabla \psi$ containing a map $\theta : (S, x_0) \to (Y, y_0)$ such that $\theta(x) = y_0$ for every $x \in S$. Moreover, the group does not depend on the choice of the point $x_0 \in S$.

Let us call that group $m$-th multi-homotopy group of $Y$ with base point $y_0$. We shall denote this group by $m\pi_n(Y, y_0)$. Indeed, it suffices to remark that if $F \in \text{Far}$ contains $y_0$ and $y_0$, then for every multi-map $\varphi : (S, x_0) \to (Y, y_0)$ there exists a multi-map $\psi : (S, x_0) \to (Y, y_0)$, multi-homotopic to $\varphi$ (relative $y_0$), namely $\varphi \cong \varphi \nabla \theta \cong \psi$.

where $[a]$ is the neutral element of $m\pi_n(Y, y_0)$ and $\psi$ is defined as follows:

$$\psi|_{(S \setminus \{x_0\}) \times \{1\}} = \varphi \nabla \theta, \quad \psi(x_0) = F.$$ 

The isomorphism of the groups $m\pi_n(Y, y_0)$ and $m\pi_n(Y, y_0)$ is then established by the assignment

$$[\varphi] \mapsto [\psi].$$

If $h$ is a homeomorphism of spaces $(Y, y_0)$ and $(Y', y_0')$, then assigning to every element $[\varphi] \in m\pi_n(Y, y_0)$ the element $[h \circ \varphi] \in m\pi_n(Y', y_0')$ we define a homomorphism of the multi-homotopy groups of $(Y, y_0)$ and $(Y', y_0')$. Moreover, this homomorphism is an isomorphism; hence

4.2. The groups $m\pi_n(Y, y_0)$ are topological invariants of the space $(Y, y_0)$. It follows from 3.6 that

4.3. All the multi-homotopy groups of an FAR-set are trivial.

We now define a homomorphism of the group $m\pi_n(Y, y_0)$ into the $n$th fundamental group $\pi_n(Y, y_0)$ of a space $(Y, y_0)$ (see [3], p. 132 for the definition of fundamental groups).

Let $\varphi : (S, x_0) \to (Y, y_0)$ be a multi-map. We may assume (taking, if necessary, the multi-map $\varphi \nabla \psi$ instead of $\varphi$) that $\varphi(x_0) = (y_0)$. Assume that $S$ is a $Z$-set in $Y \subseteq Q$ and that $Y \subseteq Q \subseteq Q$. Let us take the upper semi-continuous decomposition $\mathcal{B}$ of the space $Q \times Q$, whose elements are the sets $\{x\} \times \mathcal{B}(x)$ for $x \in S$ and the remaining one-point sets. Denote by $r$ the projection of $Q \times Q$ onto the decomposition space $\mathcal{M} = Q \times Q / \mathcal{B}$ and by $\Phi$ the graph of $\varphi$. It follows by Theorem 1.1 that $\mathcal{M} \in \mathcal{AR}$. Since $r$ is a cellular map of $Q$ onto an AR-set and the set of non-degenerated point-images of $r$ is a $Z$-set in $Q$, by (17), Theorem 3, we get $\mathcal{M} \subseteq Q$. From (17), also [12], Lemma 2.3 we infer that $r(\Phi)$ is a $Z$-set in $\mathcal{M}$.

Let us denote by $q$ and $p$ the projections of $Q \times Q$ onto the first and second factor, respectively. Setting

$$h : x \mapsto r((\Phi)^{-1}(x)) \quad \text{for} \quad x \in S,$$

we get a homeomorphism of $S$ onto $r(\Phi)$. Denote by $Q_0$ the space formed from $Q_0 \times I$ by contracting $\{x_0\} \times I$ to a point. Let $t_1 : Q_0 \times I \to Q_0$ be the projection. Then $Q_0 \subseteq Q$ and by (11), Theorem 10.1 the homeomorphism $h$ between $Z$-sets $S \subseteq Q_0$ and $r(\Phi) \subseteq \mathcal{M}$ extends to a homeomorphism $h : Q_0 \to \mathcal{M}$.

We define $\chi^0 : S \times I \to Q_0 \times Q$ by the formula

$$\chi^0(x, t) = r^{-1}(t) \quad \text{for} \quad x \in S,$$

Then $\chi^0$ is a multi-map and $\chi^0(x_0, t) = \{(x_0, r(t))\}$ for every $t \in I$. Hence $\chi^0$ is a multi-homotopy connecting $\chi^0|_{S \times \{1\}}$ with the multi-map $\chi^0|_{S \times \{0\}}$. Moreover, $\chi^0(x, 0) = \{x\} \times \varphi(x)$ for every $x \in S$ and $\chi^0|_{S \times \{t\}}$ is a homeomorphism onto $\chi^0(S \times \{t\})$ for each $t \in (0, 1)$. Therefore, defining the sequence of maps

$$\theta^0 = \{t : (S, x_0) \to (Q_0, y_0)\}$$
by the formula
\[
d'_{q_k}(x) = q(x, 1/k) \quad \text{for} \quad k = 1, 2, \ldots,
\]
we obtain an approximative map from \((S, x_0)\) towards \((Y, y_0)\) (for the definition of approximative maps see [3], p. 128).

4.4. Theorem. Assigning to every element of the group \(m\pi(S, y_0)\) with a representative \(\varphi: (S, x_0) \to (Y, y_0)\) the approximative class with
\[
\varphi^a = (\varphi^a: (S, x_0) \to (Q_2, y_0))
\]
as a representative, we get a homeomorphism \(\mathfrak{Z}\) of the multi-homotopy group \(m\pi(S, y_0)\) into the fundamental group \(\pi_1(Y, y_0)\). Moreover, if \(Y \in \text{ANR}\), then \(\mathfrak{Z}\) is an isomorphism.

Proof. First we show that
\[
\mathfrak{Z}([\varphi], [\psi]) = [\varphi^{a\psi}] = [\varphi^a],
\]
where \(\varphi, \psi: (S, x_0) \to (Y, y_0)\) are multi-maps. With this aim we remark that a representative of the class \([\varphi^a]\) is to be found in the approximative class \(b = (b_i; (S, x_0) \to (Y, y_0))\), defined by the formula
\[
b_i(x) = \begin{cases} 
q_0(x) & \text{for } x \in S^+, \\
q_0(x) & \text{for } x \in S^- \cup (S^+ \cup S^-).
\end{cases}
\]

Thus, to prove (iii) it suffices to show that the approximative map
\[
\{d'_{q_k}(S^+, S^-) \to (Y, y_0)\}_0
\]
is homotopic to the approximative map
\[
\{d'_{q_k}(S^+, S^-) \to (Y, y_0)\}_0 \quad \text{homotopic to the approximative map} \quad \{d'_{q_k}(S^+, S^-) \to (Y, y_0)\}_0
\]
under the map \(\varphi\) (where \(S^+ = \text{reg} \) and \(S^- = \text{reg} \) are regarded here as points). We only show the former condition. The proof of the latter is similar.

Let us observe that \(\varphi^a\) and \(\varphi^{a\psi}\), restricted to the set \(S^+ \times I\), are multi-homotopic to the unit map \(x \mapsto (x, 0)\), \(x \in S^+ \) and \(x \in S^- \). Hence \(\varphi^{a\psi}\) is homotopic to \(\varphi^a\) in \(\mathfrak{Z}^a\). Since \(\mathfrak{Z}^a\) is an ANR, there is a \(e > 0\) such that every \(e\)-neighborhood of \(\varphi^a\) in \(\mathfrak{Z}^a\) is homotopic in \(\mathfrak{Z}^a\). Choose \(k_0 \in N\) so large that for every \(k > k_0\) and \(x \in S^+ \times (1/k)\)
\[
\varphi\left(t \cdot \varphi^a(x, 1/k)\right) + \varphi\left(t \cdot \varphi (x, 1/k)\right) < e.
\]
Then for every \(k > k_0\) the maps \( r \to \varphi^{a\psi}(S^+ \times (1/k)) \) and \( r \to \varphi^a(S^+ \times (1/k)) \) of the set \(S^+ \times (1/k)\) into \(\mathfrak{Z}^a\) are homotopic in \(\mathfrak{Z}^a\). Let \(g_k\) be a homotopy joining these maps. Setting
\[
g_k(x, t) = (g \circ r^{-1} \circ g_k)(x, 1/k, t) \quad \text{for} \quad (x, t) \in S^+ \times I,
\]
we get a homotopy \(g_k: S^+ \times I \to V_2\) joining \(V_2\) to \(V_2\) with the map \(d'_{q_k}\). Thus (iii) is satisfied and hence \(\mathfrak{Z}\) is a homomorphism. We now show that, in the case of \(Y \in \text{ANR}\), \(\mathfrak{Z}\) is an isomorphism.

By (4.3), p. 129 if \( \varphi = (q_0, S \to (Y, y_0)) \) is an approximative map and \(Y \in \text{ANR}\), then the class \([\varphi]\) is generated by a map \(c: (S, x_0) \to (Y, y_0)\). But every map is a multi-map; therefore we may obtain the approximative map \(\varphi^a = (d'_{q_k}, S \to (Y, y_0)) \) for the map \(c\). Moreover, \(\varphi^a\) is then a homotopy, and for every neighborhood \(W\) of \(Y\) in \(Q_2\) there is a \(k_0 \in N\) such that, for \(k > k_0\), \(\varphi^a(S \times (0, 1/k))\) is a homotopy joining in \(W\) the maps \(d'_{q_k}\) and \(c\). Hence \(\varphi^a\) and \(\varphi\) are homotopic and \(\mathfrak{Z}\) is an epimorphism. In order to prove that \(\mathfrak{Z}\) is a monomorphism, let \(\varphi: (S, x_0) \to (Y, y_0)\) be a multi-map and assume that an approximative map \(\varphi^a\) is null-homotopic. The assumption \(Y \in \text{ANR}\) implies the existence of a neighborhood \(V'_2\) of \(Y\) in \(Q_2\) and a retraction \(b: V'_2 \to Y\). Besides there is a \(k_1 \in N\) such that
\[
\chi^a(S \times (0, 1/k)) \subset Q_1 \times V'_2.
\]
Since \(\varphi^a\) is null-homotopic, there exists a homotopy \(g: S \times I \to V'_2\) joining \(V'_2\) to \(V'_2\) with the map \(d'_{q_1}\), which is the constant map \(c\) into \(y_0\). Then \(\mathfrak{Z}\) defined by the formula
\[
\mathfrak{Z}(x, t) = \begin{cases} 
b_i(x, t) & \text{for } x \in S \text{ and } t \in (0, 1/k_1), \\
b_i\left(x, \frac{k_1 - 1}{k_1 - 1}\right) & \text{for } x \in S \text{ and } t \in (1/k_1, 1)
\end{cases}
\]
is a multi-homotopy connecting \(Y\) the multi-map \(\varphi\) with the constant map \(S\) into \(y_0\). Hence \(\mathfrak{Z}\) is a homomorphism and the proof of 4.4 is finished.

By Theorem 4.4 and (4.1), p. 133 we have the following corollary:

4.5. If \(y_0 \in Y \in \text{ANR}\), then the group \(m\pi(S, y_0)\) is isomorphic to the \(n\)-th homotopy group \(\pi_n(Y, y_0)\).

Let us remark that if the group \(\pi_n(Y, y_0)\) of a space \((Y, y_0)\) is isomorphic to the group \(\pi_n(Y, y_0)\), e.g., when \(Y\) is locally \(n\)-connected (see [11], also [10]), then by an analogous argument as in the proof of 4.4 we infer that \(\mathfrak{Z}\) is an epimorphism.

Notice that a certain kind of classes of multi-homotopy was investigated in [5], namely multi-functions from finite-dimensional compacta to \(S^\infty\) with values which are cellular subsets of \(S^\infty\). Our Theorem 4.4 is a certain analogue of Theorems 2 and 3 in [5].

4.6. Theorem. Let \(f: (Y, y_0) \to (Y, y_0)\) be a cellular map of \(Y\) onto \(Y \in \text{ANR}\). Then, assigning
\[
[g] \mapsto [f^{-1} \circ g]
\]
to every map \(g: (S^\infty, x_0) \to (Y, y_0)\), we get a monomorphism of the group \(\pi_n(Y, y_0)\) into the group \(m\pi_n(Y, y_0)\)
Proof. We show that the function defined in the theorem is a homomorphism. Let \( g_1, g_2 : (S, x_0) \to (Y, y_0) \) be maps (\( S = S^1 \)). Then
\[
[g_1; g_2] = [g_1 \lor g_2] = [(f^{-1} \circ g_1) \lor (f^{-1} \circ g_2)]
\]
But
\[
f^{-1} \circ (g_1 \lor g_2) = (f^{-1} \circ g_1) \lor (f^{-1} \circ g_2)
\]
under the multi-homotopy \( \gamma : S \times I \to Y' \) defined by the conditions
\[
\gamma(S) \times I = f^{-1} \circ g_1, \quad \gamma(S^c) \times I = f^{-1} \circ g_2
\]
and, for \( x \in S \setminus (S^+ \cup S^-) \),
\[
\gamma(x, 0) = f^{-1} \circ (y_0) \equiv y_0, \quad \gamma(x, t) = (y_t) \quad \text{for} \quad t \in (0, 1).
\]
Hence \( [f^{-1} \circ (g_1 \lor g_2)] = [f^{-1} \circ g_1] \lor [f^{-1} \circ g_2] \). We now prove that the homomorphism is a monomorphism. Let \( Y' \in CQ \). Since \( Y' \in CQ \), there exist a neighborhood \( U \in CQ \) of \( Y' \) in \( Q \) and a map \( f' : U \to Y \) such that \( f'|Y' = f \). The assumption \( f^{-1} \circ g \equiv \theta \) for a map \( g : (S, x_0) \to (Y, y_0) \) implies the existence of a multi-map \( \gamma : S \times I \to Y' \) such that \( \gamma(S) \times \{0\} = f^{-1} \circ g \) and \( \gamma(S) \times \{1\} \) is the constant map of \( S \) into \( y_0 \). It follows from Theorem 4.4 that \( \gamma \) is a multi-homotopy from \( U \) to a map \( g_1 : S \times I \to Y' \), and moreover the multi-homotopy \( \gamma_1 \) connecting \( \gamma_1 \) with \( g_1 \) is a map for \( 0 < t < 1 \). Hence we infer that \( h : S \times I \to Y \) defined as
\[
h(x, t) = \begin{cases} f_1(x, [x_0, 0, 2t]) & \text{for} \quad 0 \leq t \leq \frac{1}{2}, \\ f_2(x, [x_0, 2t - 1, 1]) & \text{for} \quad \frac{1}{2} < t \leq 1 \end{cases}
\]
is a homotopy joining the map \( g : (S, x_0) \to (Y, y_0) \) with the constant map of \( S \) into \( y_0 \). This completes the proof.

4.7. Corollary. If \( f : (Y', y_0) \to (S^1, x_0) \) is a cellular map of \( Y' \) onto \( S^1 \), then the group \( m_{\pi_1}(Y', y_0) \) is isomorphic to \( \pi_1(S^1) \).

4.8. Example. Let \( Y \) be the same set as in Example 2.27. Put \( a = (2, 1), b = (1, 1) \). Then \( Sh(Y, a) = Sh(Y, b) \) (see [3], (15.1), p. 240). If \( x \in S = S^1 \) and \( \phi \) is a multi-map of \( S \) into \( Y \), then
\[
\phi(x) \subseteq Y \setminus L \quad \text{or} \quad \phi(x) \subseteq L.
\]
Furthermore,
\[
\phi(S) \subseteq Y \setminus L \quad \text{or} \quad \phi(S) \subseteq L.
\]
Condition (1) is clear. We show that also condition (2) is satisfied. It follows from the upper semi-continuity of \( \phi \) and condition (1) that the inverse image of \( Y \setminus L \) under \( \phi \) is closed in \( S \). If (2) is not true, then there is a point \( x \in S \) such that in each of its neighborhoods in \( S \) there exists a point \( x' \in S \) with the property that \( \phi(x') \subseteq L \). Let \( x \in (Y \setminus L) \phi(x) \). Take a closed neighborhood \( G \) of \( \phi(x) \) in \( Y \), disjoint with the point \( x \). Then there is a neighborhood \( V \) of \( x \) in \( S \) such that \( \phi(Y) \subseteq G \) (we may assume \( V \) to be an arc).

Let \( x' \in V \) be the point described above. Then the image \( \phi(V) \subseteq G \) of the connected set \( V \) contains points lying in different components of the set \( G \) which contradicts the property of preserving connectedness under multi-maps. Thus, condition (2) is proved.

From (2) we infer that \( m_{\pi_1}(Y, a) = 0 \) and that \( m_{\pi_1}(Y, b) \) is a cyclic infinite group. Hence we infer that multi-homotopy groups are not the invariants of shape of pointed spaces.

4.9. Example. Let \( S \) be the union of the circles \( S_1 \) lying in the plane, \( i = 0, 1, 2, \ldots \) with centres 0 and radii \( r_i \), such that
\[
r_0 = 1, \quad r_1 = \frac{i - 1}{i + 1} \quad \text{for} \quad i = 1, 2, \ldots
\]
In the set \( S \) we identify each circle \( S_i \times \{1\} \) with the circle \( S_{i+1} \times \{0\}, i = 1, 2, \ldots \), the circle \( S_0 \times \{0\} \) with \( S_0 \times \{1\} \) and the circle \( S_1 \) with a point. Let \( e \in S_0 \). Denote by \( T \) the union of the resulting set and the disc whose boundary is the circle \( \{a\} \times I \) in \( T \). Then \( T \) is approximately 1-connected, hence it follows that the group \( \pi_1(T, a) \) is trivial. However, \( m_{\pi_1}(T, a) \neq 0 \), because there is an embedding of the sphere \( S^2 \) into \( T \) (for instance with values in \( S_0 \times \{0\} \) which is not multi-homotopic to the constant map of \( S^2 \) into \( a \) (the proof of this fact is similar to that given in Example 4.8). Moreover, one can prove that \( T \notin m\text{-ANR} \) (cf. Example 2.27).

Example 4.9 and Theorems 4.4 and 4.6 suggest a positive answer to the following problem:

4.10. Problem. Is it true that homomorphism \( \mathcal{I} \) defined in Theorem 4.4 is an isomorphism if \( Y \in m\text{-ANR} \)?

4.11. Problem. Is it true that the multi-homotopy groups are invariant under cellular maps?

4.12. Problem. Are the multi-homotopy groups of \( m\text{-ANR}-sets \) finitely generated?

References

On elementary cuts in models of arithmetic

by

Henryk Kotlarski (Warszawa)

Abstract. Let $M \models \text{Peano arithmetic}$. We put $Y := (N \subseteq M: N \nless M)$. This family, as each family of initial segments of $M$, is simply ordered by inclusion. The order type of $Y$ heavily depends on $M$; we shall compute this order type in the following cases: (a) $M$ is countable and recursively saturated and (b) $M$ is saturated. In both cases the proofs give fairly complete description of the situation.

We assume that the reader is familiar with saturated models (see Chang, Keisler [1]) and with recursively saturated models (see Schlipf [5]). We use standard model-theoretic terminology and notation.

§ 1. The recursively saturated case. Toward this section let $M \models P$ be recursively saturated. Our result is the following.

THEOREM 1. If $M$ is countable, then $Y$ is of the order type of the Cantor set $2^n$ with its lexicographical ordering:

$$b^{1} < b^{2} \equiv (\exists n \in \omega) (b^{1}_{n} = 0 \land b^{2}_{n} = 1 \land (\forall m < n) (b^{1}_{m} = b^{2}_{m})).$$

Before proving this we shall prove some lemmas. For $a \in M$ we shall denote by $M(a)$ the closure under the initial segment of the Skolem closure of $a$; formally

$$M(a) := \{ b \in M: \text{there exists a parameter-free term } t(a) \text{ such that } M \models b < t(a) \}.$$

By Gaifman [2, Theorem 4.1] $M(a) \nless M$.

LEMMA 2. $M(a)$ is not recursively saturated.

Proof. $M(a)$ omits the type $\{ t < t(a): t \text{ is a term} \}$. □

Let $Y_{1} = \{ N \in Y: N \text{ is not recursively saturated} \}$. Our first aim is to prove the converse of Lemma 2, it will be our Lemma 4.

LEMMA 3 (with W. Mark). If $D \subseteq Y$ has no greatest element, then $\bigcup D$ is recursively saturated.

Proof. Let $p(a)$ be a recursive type in parameters $b_{1}, ..., b_{9}$. There exists an $N \in D$ such that $b_{1}, ..., b_{9} \in N$ (in fact, $D$ is linearly ordered by inclusion), and so by the assumption there exists $N_{1} \in D$ such that $N_{1} < N$. Therefore pick $c \in N_{1}$ such that $\forall a \in N \, M \models a < c$. Now consider the type $p(a) \cup \{ t < c \}$. This is still a recursive type and consistent, and so it is realized in $M$. But any of its realizations is in $\bigcup D$. □