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There is a positive integer $n_3$ such that $n \geq n_3$ implies $(2n)$-close maps into $P$ are homotopic. Let $r = \max\{n_1, n_2, n_3\}$ be fixed. Let $r : P \to A_r$ be the retraction defined by

$$r(x, g_{n_k+1}, g_{n_k+2}, \ldots) = r_d(x), \quad \text{for} \quad x \in U.$$  

Since $g_m \circ f_0 : A_r \to A_r \subset P$ is a $(2n)$-map, our choice of $n_3$ implies there is a homotopy $F : A_r \times I \to P$ with

$$F(x, 0) = g_m \circ f_0(x) \quad \text{and} \quad F(x, 1) = x \quad \text{for all} \quad x \in A_r.$$  

Thus $r \circ F : A_r \times I \to A_r$ is a homotopy with

$$r \circ F(x, 0) = g_m \circ f_0(x) \quad \text{and} \quad r \circ F(x, 1) = x.$$  

This completes the proof.

The converse of (5.5) is not true: Let $\{A_0\}_{\alpha=0}^\infty$ be the sequence of [Bx. (4.3)], in which it was shown that $A_0 \neq \lim_{\alpha \to \infty} A_\alpha$ in the topology of $d_U$. However, $A_0 = \lim_{\alpha \to \infty} A_\alpha$ in the topology of $d_U$, hence in the topology of $d_U$, hence (by (5.2)) in the topology of $d_U$.

Thus $d_U$ induces a stronger topology on ANR$^*$ than does $d_{U'}$.

References


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On non compact FANR's and MANR's
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Abstract. It is proved that a finite dimensional metrizable space $X$ is a FANR if and only if $X$ is a MANR and the set of points at which $X$ is not locally contractible has the compact closure. As an application, for finite dimensional metrizable spaces $X$ and $Y$, a necessary and sufficient condition under which $X \times Y$ be a FANR is obtained in terms of $X$ and $Y$.

1. Introduction. The notion of FANR is introduced by K. Borsuk [2]. According to [2, p. 94] a metrizable space $X$ is a FANR if for every metrizable space $X'$ containing $X$ as a closed subset, $X$ is a fundamental neighborhood retract of $X'$. S. Godlewski [4] has introduced the concept of MANR. From the definition it is obvious that every FANR is a MANR. By [4] and [5] the properties "to be a FANR" are not generally shape invariants in the sense of Fox [3]. In this paper we shall show that a finite dimensional metrizable space $X$ is a FANR if and only if $X$ is a MANR and the set of points at which $X$ is not locally contractible has the compact closure. Obviously the second condition is not a shape invariant.

All spaces under considerations are metrizable and are continuous. AR and ANR mean those for metrizable spaces.

2. Theorems. Let $X$ be a space and let $x \in X$. If for every neighborhood $U$ of $x$ in $X$ there exists a neighborhood $V$ of $x$ such that $V$ is contractible in $U$, then $X$ is said to be locally contractible at $x$. Put $L(X) = \{x : x \in X \text{ and } X \text{ is locally contractible at } x\}$ and $L(X) = C(X \setminus L(X))$, where $C$ means the closure in $X$.

Theorem 1. A finite dimensional space $X$ is a FANR if and only if $X$ is a MANR and $L(X)$ is compact.

Proof. If "part". Let $M$ be an AR containing $X$ as a closed set. It is assumed by [7] that $M$ is finite dimensional and $X$ is unstable in $M$ in the sense of Sher [9, p. 346]. Since $X$ is a MANR, there is a closed neighborhood $W$ of $X$ in $M$ and a mutational retraction $r : U(W, M) \to U(X, M)$. Here $U(A, M)$ means the family of all open neighborhoods of $A$ in $M$. (See [3] and [5] for notations and definitions.) Let $d$ be a metric in $M$. Choose an open cover $\mathcal{U}$ of the set $M \setminus L(X)$ such that if $d(x_i, L(X)) = 0$ $(i \to \infty)$ for $x_i \in M \setminus L(X)$ then diameter $d(x_i, \mathcal{U}) = 0$ $(i \to \infty)$. Since $X$ is locally contractible at each point of the set $L(X) \setminus L(X)$ and $M$ is finite dimensional, by [1, Theorem (9.1), p. 80]
there exist a subset $H$ of $W - L(X)$ and a map $r': H \to X - L(X)$ satisfying the following conditions.

1. $H$ is a closed neighborhood of $X - L(X)$ in $W - L(X)$,
2. $r'$ is a retraction, that is, $r'(x) = x$ for $x \in X - L(X)$,
3. there is a deformation retraction $\xi': H \times I \to W - L(X)$ such that $\xi'(x, 0) = x$ and $\xi'(x, 1) = r'(x)$ for $x \in H$, and $\xi'(x, 1) = x$ for $x \in X - L(X)$, and $\xi'$ is $\mathcal{H}$-limited on some neighborhood $H'$ of $X - L(X)$ in $H \times I$.

Consider the subset $T = (X \cup H) \times I \cup W \times \{0\}$ of $M \times I$. Let us define a map $\xi: T \to M$ by $\xi(x, i) = (x, i)$ for $(x, i) \in X \times I \cup W \times \{0\}$ and $\xi(H \times I) = \xi'$. From (3) $\xi$ is continuous. By Borsuk's homotopy extension theorem $\xi$ has an extension over $W \times I$ which we denote by $\xi$. Define $r: W \to M$ by $r(x) = \xi(x, 1)$ for $x \in W$. Then we have

$$r(x) = x \quad \text{for} \quad x \in X \quad \text{and} \quad r(H) = X.$$

Let $\{U_i: i = 1, 2, \ldots\}$ be a decreasing countable neighborhood basis of $L(X)$ in $M$. Such a basis exists by the compactness of $L(X)$. Each set $U_i \cup H$ is a neighborhood of $X$ in $M$. From (4) and the continuity of $r$ there exist a neighborhood $W_i$ of $L(X)$ in $M$ such that

$$W_{i+1} \subseteq W_i \subseteq W \quad \text{and} \quad r(W_i) \subseteq U_i \quad \text{for} \quad i = 1, 2, \ldots.$$

Now consider a mutually continuous retraction $r_0: W \to U \cup H$, (7). For each $i$, $W_i \cup H$ is a neighborhood of $X$ in $M$. Choose $r_i \in r_0$ such that

$$r_i(W) \subseteq W_i \cup H.$$

Let us define

$$f_i: W \to M \quad \text{by} \quad f_i(x) = r(x) \quad \text{for} \quad x \in W.$$

Since $M$ is an AR, $f_i$ is extendable over $M$. We denote by $f_i$ its extension again. Put $f = \{f_i: i = 1, 2, \ldots\}$. We shall prove that $f$ forms a fundamental retraction from $W$ into $X$. To do it we have to show that

$$f_i(x) = x \quad \text{for} \quad x \in W_i \quad \text{and} \quad i = 1, 2, \ldots.$$  

Since $r_i(W) \subseteq W_i \cup H$, (8) is obvious by (4) and (7). Let $V$ be a neighborhood of $X$ in $M$. Since $\{U_i\}$ is a neighborhood basis of $L(X)$ in $M$, there is $i_0$ such that $U_{i_0} \cap V \neq \emptyset$. By (4), (5), (6) and (7) we have $f_i(W) \subseteq U_i \cup X \subseteq V$ for each $i \geq 2$. Since $r_i(W) \subseteq W_i \cup H$ by (6) and $W_i \cup H$ is a neighborhood of $X$, by the definition of a mutation (cf. [5, p. 49]) there exists a homotopy $\eta: W \times I \to W_i \cup H$ such that

$$\eta(x, 0) = r(x) \quad \text{and} \quad \eta(x, 1) = r_i(x) \quad \text{for} \quad x \in W.$$

Here $r_i$, $r_i$, and $r_i$ are the members of $r$ used to define $f_i$ and $f_i$ respectively (cf. (7)). Define $\mu: W \times I \to M$ by $\mu = r_i$. Then by (10), (7) and (4) we have $\mu(x, 0) = f_i(x)$ and $\mu(x, 1) = f_i(x)$ for $x \in W$, and $\mu(W \times I) = r_i(W \times I) \subseteq U_i \cup X$. Thus $f_i(W) = f_i(W) \subseteq W$.

"Only if part.\" Let $X$ be a MANR such that $L(X)$ is not compact. We shall show that $X$ is not a MANR. For the proof the same argument as in [6, (11)] is used. Let $M$ be a finite dimensional ANR containing $X$ as a closed set. Since $L(X)$ is not compact, there is a discrete sequence $\{x_i: i = 1, 2, \ldots\}$ such that $X$ is not locally contractible at each $x_i$. Suppose that $X$ is a MANR. Then there exist a closed neighborhood $W$ of $X$ in $M$ and a fundamental retraction $r = \{r_i: i = 1, 2, \ldots\}$ from $W$ into $X$. For each $i$, choose a neighborhood $U_{i, i}$ of $x_i$ in $X$ such that $\{U_{i, i}: i = 1, 2, \ldots\}$ is a discrete family in $W$. Since $X$ is not locally contractible at each $x_i$, there exists neighborhoods $U_{i, i} \subseteq V_{i, i}$ $x_i$, and a map $f_i$ from an $n_i$-sphere $S_{i, i}^n$ to $V_{i, i}$ intersecting the following conditions.

$$U_{i} \subseteq U_{i, i} \quad \text{and} \quad V_{i} \subseteq V_{i, i} \quad \text{is contractible in} \quad r^{-1}(U_{i}).$$

(11) $f_i$ has no extension from $E_{i, i} = W \setminus E_i$ to $V_{i} \cap X$, where $E_{i, i}$ is an $(n_i+1)$-cell whose boundary is $S_{i, i}^n$.

By (11) there is an extension $g_i: E_{i, i} \to r^{-1}(U_{i})$ of $f_i$. Then, by (12), $\emptyset \neq r_i\partial(E_{i, i}) \cap X \neq \emptyset$ for each $i$. Choose a point $x_i \in \partial(E_{i, i}) \cap X$ for $i = 1, 2, \ldots$ and put $F = \{x_i: i = 1, 2, \ldots\}$. Note that

$$r(F) \subseteq F \neq \emptyset \quad \text{for} \quad i = 1, 2, \ldots.$$

Since $\{U_i\}$ is a discrete family in $W$, $F$ is closed in $W$ and $F \cap X = \emptyset$. Hence $W \setminus F$ is a neighborhood of $X$ in $M$. Since $F$ is a fundamental retraction, $r_i(W)$ is contractible for some $r_i \in r$. This contradicts (13). The proof is completed.

Let $Y$ be a metrizable space. For a closed set $A$ of $Y$, denote by $\chi(A, Y)$ the character of $A$ in $Y$, that is, the smallest cardinal number of neighborhood bases of $A$ in $Y$. For a metrizable space $X$, put $\chi(X) = \chi(X \setminus \{0\}, X \setminus I)$. If $X$ is empty, we put $\chi(X) = 1$. The following is proved.

(14) For a metrizable space $X$, $\chi(X) = \sup \{\chi(A, Y): Y$ is a metrizable space containing $X$ as a closed set$.}$

The inequality $\chi(X) \leq \sup \{\chi(A, Y)$ is obvious. To prove the converse inequality, let $Y$ be a metrizable space containing $X$ as a closed set. Given a neighborhood $U$ of $X$ in $Y$, there is a continuous function $f: X \to (0, 1]$ such that $\bigcup S(f(x)) = U$, where $S(x, r)$ is a spherical neighborhood of $x$ in $Y$ with radius $r$. Since $\chi(X) = \chi(X \setminus \{0\}, X \setminus I)$, there is a family $M = \{f_a: a \in A\}$ of continuous functions $f_a: X \to (0, 1]$, where $A$ is the set of indices with cardinality $\chi(X)$, having the following property: If $f: X \to (0, 1]$ is continuous, then there is $f_a \in M$ such that $f_a < f$.
let $U_s = \bigcup_{x \in X} S(x, f'_s(x))$. Then $(U_s : s \in A)$ forms a neighborhood basis of $X$ in $Y$. Thus $\mu(X, Y) \leq \varepsilon(X)$.

For metrizable spaces $X$ and $Y$, let $r: U(X, M) \to U(Y, N)$ be a mutation, where $M$ and $N$ are ANR's containing $X$ and $Y$ as closed sets respectively. A sub-family $r'$ of $r$ is said to generate $r$ if for any $V \in U(Y, N)$ there is $r' \in r'$ whose range is contained in $V$. By the character $\mu(r)$ of a mutation $r$ we mean the smallest cardinal number of sub-families generating $r$. For example, if $r$ is a mutation into an ANR, that is, the range of $r$ is an ANR, then $\mu(r) = 0$, because $r$ is generated by one continuous map. We have the following theorem.

**Theorem 2.** Let $Y$ be a finite dimensional metrizable space. Then, for every metrizable space $X$ and for every mutation $r: U(X, M) \to U(Y, N)$ the relation

$$\mu(r) \leq \varepsilon(L(Y))$$

holds. There is a mutation $r$ for which the equality holds in (15).

The proof is given under consideration of (14) by the same way as in Theorem 1. We omit it.

For a given infinite cardinal number $\tau$, let $X$ be a topological sum of $\tau$ copies of the continuum constructed by Borsuk [1, p. 125]. Since $X$ is locally contractible, $L(X) = \emptyset$ and hence $\varepsilon(L(X)) = 1$. However there exists a mutation $r$ such that the range of $r$ is $X$ and $\mu(r) = \varepsilon^m$. Thus the finite dimensionality of $X$ in Theorem 2 cannot be omitted.

Finally, we have the following corollaries.

**Corollary 1.** A finite dimensional metrizable space $X$ is a FAR if and only if $X$ is a MAR and $L(X)$ is compact.

**Corollary 2.** A finite dimensional contractible metrizable space $X$ is a FAR if and only if $L(X)$ is compact.

**Corollary 3.** Let $Y$ be a finite dimensional metrizable space. The following are equivalent.

(i) $L(Y)$ is compact.

(ii) For a metrizable space $X$, every mutation $r: U(X, M) \to U(Y, N)$ is generated by a fundamental sequence.

**Corollary 4.** Let $X$ and $Y$ be finite dimensional metrizable spaces. Then $X \times Y$ is a FAR (resp. FAR) if and only if either

(i) $X$ is a FAR (resp. FAR) and $Y$ is a compact ANR (resp. AR), or

(ii) $X$ is a compact ANR (resp. AR) and $Y$ is a FAR (resp. FAR), or

(iii) $X$ and $Y$ are both compact FAR's (resp. FAR's), or

(iv) $X$ and $Y$ are both ANR's (resp. AR's).

Corollaries 1, 2 and 3 are immediate consequences of Theorems 1 and 2. We shall prove Corollary 4 in the case for FAR.

Suppose that $X \times Y$ is a FAR. Obviously $X$ and $Y$ are both FAR's. Let $L(X) \neq \emptyset$ and $L(Y) \neq \emptyset$. Then $X$ and $Y$ are both compact by Theorem 1, that is, $X$ and $Y$ are compact FAR's. Let $L(X) = \emptyset$ and $L(Y) = \emptyset$. Theorem 1 implies that $X$ is compact. Since $X$ is locally contractible at each point, it is an ANR. Thus (i) holds. Similarly, if $L(X) = \emptyset$ and $L(Y) = \emptyset$ then (i) holds. If $L(X) = \emptyset$ and $L(Y) = \emptyset$ then $X$ and $Y$ are ANR's.

Conversely, assume that (i) holds. It follows from [8, Theorem 3.8] that $X \times Y$ is a finite dimensional MANR. Since $L(X)$ is compact by Theorem 1 and $Y$ is a compact ANR, $L(X \times Y)$ is compact. By applying Theorem 1 again, it is seen that $X \times Y$ is a FAR. The cases (iii) and (iv) are obvious. This completes the proof.

References


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