There is a positive integer n_3 such that $n \geqslant n_3$ implies $(2\varepsilon_n)$-close maps into P are homotopic. Let $n \geqslant \max\{n_1, n_2, n_3\}$ be fixed. Let $r\colon P \to A_n$ be the retraction defined by

$$r(x, q_{m+1}, q_{m+2}, \ldots) = r_n(x), \quad \text{for} \quad x \in U.$$

Since $g_n \circ f_n\colon A_n \to A_n \subset P$ is a $(2\varepsilon_n)$-map, our choice of n_3 implies there is a homotopy $F\colon A_n \times I \to P$ with

$$F(x, 0) = g_n \circ f_n(x) \quad \text{and} \quad F(x, 1) = x \quad \text{for all } x \in A_n.$$

Thus $r \circ F\colon A_n \times I \to A_n$ is a homotopy with

$$r \circ F(x, 0) = g_n \circ f_n(x) \quad \text{and} \quad r \circ F(x, 1) = x.$$

This completes the proof.

The converse of (5.5) is not true: Let $\{A_n\}_{n=0}^{\infty}$ be the sequence of [Bx, (4.9)], in which it was shown that $A_0 \neq \lim\limits_{n\to\infty} A_n$ in the topology of d_h. However, $A_0 = \lim\limits_{n\to\infty} A_n$ in the topology of d_C, hence in the topology of d_F, hence (by (5.2)) in the topology of d_{CF}.

Thus d_h induces a stronger topology on ANR^X than does d_{CF}.

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DEPARTMENT OF MATHEMATICS
NIAGARA UNIVERSITY
Niagara University, New York 14109

Accepté par la Rédaction le 28. 4. 1980

On non compact FANR's and MANR's

by

Yukihiro Kodama (Ibaraki)

Abstract. It is proved that a finite dimensional metrizable space X is a FANR if and only if X is a MANR and the set of points at which X is not locally contractible has the compact closure. As an application, for finite dimensional metrizable spaces X and Y, a necessary and sufficient condition under which $X \times Y$ be a FANR is obtained in terms of X and Y.

1. Introduction. The notion of FANR is introduced by K. Borsuk [2]. According to [2, p. 94] a metrizable space X is a FANR if for every metrizable space X' containing X as a closed subset, X is a fundamental neighborhood retract of X'. S. Godlewski [4] has introduced the concept of MANR. From the definition it is obvious that every FANR is a MANR. By [4] and [6] the properties "to be a FANR" are not generally shape invariants in the sense of Fox [3]. In this paper we shall show that a finite dimensional metrizable space X is a FANR if and only if X is a MANR and the set of points at which X are not locally contractible has the compact closure. Obviously the second condition is not a shape invariant.

All spaces under considerations are metrizable and maps are continuous. AR and ANR mean those for metrizable spaces.

2. Theorems. Let X be a space and let $x \in X$. If for every neighborhood U of x in X there exists a neighborhood V of x such that V is contractible in U, then X is said to be *locally contractible at* x. Put $L'(X) = \{x\colon x \in X \text{ and } X \text{ is locally contractible at } x\}$ and $L(X) = \mathrm{Cl}(X - L'(X))$, where Cl means the closure in X.

THEOREM 1. *A finite dimensional space X is a* FANR *if and only if X is a* MANR *and $L(X)$ is compact.*

Proof. "If part". Let M be an AR containing X as a closed set. It is assumed by [7] that M is finite dimensional and X is unstable in M in the sense of Sher [9, p. 346]. Since X is a MANR, there is a closed neighborhood W of X in M and a mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}(W, M) \to \boldsymbol{U}(X, M)$. Here $\boldsymbol{U}(A, M)$ means the family of all open neighborhoods of A in M. (See [3] and [5] for notations and definitions.) Let d be a metric in M. Choose an open cover $\mathscr{U}$ of the set $M - L(X)$ such that if $d(x_i, L(X)) \to 0$ $(i \to \infty)$ for $x_i \in M - L(X)$ then diameter $\mathrm{St}(x_i, \mathscr{U}) \to 0$ $(i \to \infty)$, where $\mathrm{St}(x, \mathscr{U}) = \bigcup\{U\colon x \in U \text{ and } U \in \mathscr{U}\}$. Since X is locally contractible at each point of the set $X - L(X)$ and M is finite dimensional, by [1, Theorem (9.1), p. 80] there exist a subset H of $W-L(X)$ and a map $r'\colon H \to X-L(X)$ satisfying the following conditions.

(1) H is a closed neighborhood of $X-L(X)$ in $W-L(X)$,

(2) r' is a retraction, that is, $r'(x) = x$ for $x \in X-L(X)$,

(3) there is a deformation retraction $\xi'\colon H\times I \to W-L(X)$ such that $\xi'(x, 0) = x$ and $\xi'(x, 1) = r'(x)$ for $x \in H$, and $\xi'(x, t) = x$ for $x \in X-L(X)$, and ξ' is $\mathcal{U}$-limited on some neighborhood H' of $X-L(X)$ in $M-L(X)$, that is, if $x \in H' \cap H$ $\xi'(x\times I) \subset U$ for some $U \in \mathcal{U}$.

Consider the subset $T = (X \cup H)\times I \cup W\times\{0\}$ of $M\times I$. Let us define a map $\xi\colon T \to M$ by $\xi(x, t) = (x, t)$ for $(x, t) \in X\times I \cup W\times\{0\}$ and $\xi|H\times I = \xi'$. From (3) ξ is continuous. By Borsuk's homotopy extension theorem ξ has an extension over $W\times I$ which we denote by ξ again. Define $r\colon W \to M$ by $r(x) = \xi(x, 1)$ for $x \in W$. Then we have

(4) $$r(x) = x \quad \text{for} \quad x \in X \text{ and } r(H) \subset X.$$

Let $\{U_i\colon i = 1, 2, \ldots\}$ be a decreasing countable neighborhood basis of $L(X)$ in M. Such a basis exists by the compactness of $L(X)$. Each set $U_i \cup H$ is a neighborhood of X in M. From (4) and the continuity of r there exist a neighborhood W_i of $L(X)$ in M such that

(5) $$W_{i+1} \subset W_i \subset W \quad \text{and} \quad r(W_i) \subset U_i \quad \text{for} \quad i = 1, 2, \ldots$$

Now consider a mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}(W, M) \to \boldsymbol{U}(X, M)$. For each i, $W_i \cup H$ is a neighborhood of X in M. Choose $r_i \in \boldsymbol{r}$ such that

(6) $$r_i(W) \subset W_i \cup H.$$

Let us define

(7) $$f_i\colon W \to M \quad \text{by} \quad f_i(x) = rr_i(x) \quad \text{for} \quad x \in W.$$

Since M is an AR, f_i is extendable over M. We denote by f_i its extension again. Put $f = \{f_i\colon i = 1, 2, \ldots\}$. We shall prove that f forms a fundamental retraction from W into X. To do it we have to show that

(8) $$f_i(x) = x \quad \text{for} \quad x \in X,\ i = 1, 2, \ldots,$$

(9) for every neighborhood V of X in M there exists i_0 such that if $i \geqslant i_0$ then $f_i(W) \subset V$ and $f_i|W \simeq f_{i_0}|W$ in V.

Since $r_i(x) = x$ for $x \in X$, (8) is obvious by (4) and (7). Let V be a neighborhood of X in M. Since $\{U_i\}$ is a neighborhood basis of $L(X)$ in M, there is i_0 such that $U_i \subset V$ for $i \geqslant i_0$. By (4), (5), (6) and (7) we have $f_i(W) \subset U_i \cup X \subset V$ for each $i \geqslant i_0$. Since $r_i(W) \cup r_{i_0}(W) \subset W_{i_0} \cup H$ by (6) and $W_{i_0} \cup H$ is a neighborhood of X, by the definition of a mutation (cf. [5, p. 49]) there exists a homotopy $\eta\colon W\times I \to W_{i_0} \cup H$ such that

(10) $$\eta(x, 0) = r_i(x) \quad \text{and} \quad \eta(x, 1) = r_{i_0}(x) \quad \text{for} \quad x \in W.$$

Here r_i and r_{i_0} are the members of $\boldsymbol{r}$ used to define f_i and f_{i_0} respectively (cf. (7)). Define $\mu\colon W\times I \to M$ by $\mu = r\eta$. Then by (10), (7) and (4) we have $\mu(x, 0) = f_i(x)$ and $\mu(x, 1) = f_{i_0}(x)$ for $x \in W$, and $\mu(W\times I) = r\eta(W\times I) \subset r(W_{i_0} \cup H) \subset U_{i_0} \cup \cup X \subset V$. Thus $f_i|W \simeq f_{i_0}|W$ in V.

"Only if part". Let X be a MANR such that $L(X)$ is not compact. We shall show that X is not a FANR. For the proof the same argument as in [6, (11)] is used. Let M be a finite dimensional ANR containing X as a closed set. Since $L(X)$ is not compact, there is a discrete sequence $\{x_i\colon i = 1, 2, \ldots\}$ such that X is not locally contractible at each x_i. Suppose that X is a FANR. Then there exist a closed neighborhood W of X in M and a fundamental retraction $r = \{r_i\colon i = 1, 2, \ldots\}$ from W into X. For each i, choose a neighborhood U_i' of x_i in W such that $\{U_i'\colon i = 1, 2, \ldots\}$ forms a discrete family in W. Since X is not locally contractible at x_i, there exists neighborhoods U_i and V_i of x_i, and a map f_i from an n_i-sphere S^{n_i} to $V_i \cap X$ satisfying the following conditions.

(11) $$U_i \subset U_i' \text{ and } V_i \text{ is contractible in } r_i^{-1}(U_i),$$

(12) f_i has not any extension from E^{n_i+1} to $U_i \cap X$, where E^{n_i+1} is an (n_i+1)-cell whose boundary is S^{n_i}.

By (11) there is an extension $g_i\colon E^{n_i+1} \to r_i^{-1}(U_i)$ of f_i. Then, by (12), $\emptyset \neq r_i g_i(E^{n_i+1}) \setminus X \subset U_i$ for each i. Choose a point $x_i \in r_i g_i(E^{n_i+1}) \setminus X$ for $i = 1, 2, \ldots$ and put $F = \{x_i\colon i = 1, 2, \ldots\}$. Note that

(13) $$r_i(W) \cap F \neq \emptyset \quad \text{for} \quad i = 1, 2, \ldots$$

Since $\{U_i\}$ is a discrete family in W, F is closed in W and $F \cap X = \emptyset$. Hence $W-F$ is a neighborhood of X in M. Since r is a fundamental retraction, $r_k(W) \subset W-F$ for some $r_k \in r$. This contradicts (13). The proof is completed.

Let Y be a metrizable space. For a closed set A of Y, denote by $\chi(A, Y)$ the *character* of A in Y, that is, the smallest cardinal number of neighborhood bases of A in Y. For a metrizable space X, put $\chi(X) = \chi(X\times\{0\}, X\times I)$. If X is empty, we put $\chi(X) = 1$. The following is proved.

(14) For a metrizable space X, $\chi(X) = \sup\{\chi(X, Y)\colon Y$ is a metrizable space containing X as a closed set$\}$.

The inequality $\chi(X) \leqslant \sup\{\chi(X, Y)\colon Y$ is a metrizable space containing X as a closed set$\}$ is obvious. To prove the converse inequality, let Y be a metrizable space containing X as a closed set. Given a neighborhood U of X in Y, there is a continuous function $f\colon X \to (0, 1]$ such that $\bigcup_{x\in X} S(x, f(x)) \subset U$, where $S(x, r)$ is a spherical neighborhood of x in Y with radius r. Since $\chi(X) = \chi(X\times\{0\}, X\times I)$, there is a family $M = \{f_\alpha\colon \alpha \in A\}$ of continuous functions $f_\alpha\colon X \to (0, 1]$, where A is the set of indices with cardinality $\chi(X)$, having the following property: If $f\colon X \to (0, 1]$ is continuous, then there is $f_\alpha \in M$ such that $f_\alpha \leqslant f$. For each $\alpha \in A$, let $U_\alpha = \bigcup_{x\in X} S(x, f_\alpha(x))$. Then $\{U_\alpha\colon \alpha\in\Lambda\}$ forms a neighborhood basis of X in Y. Thus $\chi(X, Y)\leqslant\chi(X)$.

For metrizable spaces X and Y, let $\boldsymbol{r}\colon \boldsymbol{U}(X, M)\to\boldsymbol{U}(Y, N)$ be a mutation, where M and N are ANR's containing X and Y as closed sets respectively. A subfamily $\boldsymbol{r}'$ of $\boldsymbol{r}$ is said to *generate* $\boldsymbol{r}$ if for any $V\in\boldsymbol{U}(Y, N)$ there is $r'\in\boldsymbol{r}'$ whose range is contained in V. By the *character* $\mu(\boldsymbol{r})$ of a mutation $\boldsymbol{r}$ we mean the smallest cardinal number of subfamilies generating $\boldsymbol{r}$. For example, if $\boldsymbol{r}$ is a mutation into an ANR, that is, the range of $\boldsymbol{r}$ is an ANR, then $\mu(\boldsymbol{r}) = 1$, because $\boldsymbol{r}$ is generated by one continuous map. We have the following theorem.

THEOREM 2. *Let Y be a finite dimensional metrizable space. Then, for every metrizable space X and for every mutation $\boldsymbol{r}\colon \boldsymbol{U}(X, M)\to\boldsymbol{U}(Y, N)$ the relation*

$$\mu(\boldsymbol{r})\leqslant\chi(L(Y)) \tag{15}$$

holds. There is a mutation $\boldsymbol{r}$ for which the equality holds in (15).

The proof is given under consideration of (14) by the same way as in Theorem 1. We omit it.

For a given infinite cardinal number τ, let X be a topological sum of τ copies of the continuum constructed by Borsuk [1, p. 125]. Since X is locally contractible, $L(X) = \emptyset$ and hence $\chi(L(X)) = 1$. However there exists a mutation $\boldsymbol{r}$ such that the range of $\boldsymbol{r}$ is X and $\mu(\boldsymbol{r}) = \tau^{\aleph_0}$. Thus the finite dimensionality of X in Theorem 2 cannot be omitted.

Finally, we have the following corollaries.

COROLLARY 1. *A finite dimensional metrizable space X is a* FAR *if and only if X is a* MAR *and $L(X)$ is compact.*

COROLLARY 2. *A finite dimensional contractible metrizable space X is a* FAR *if and only if $L(X)$ is compact.*

COROLLARY 3. *Let Y be a finite dimensional metrizable space. The following are equivalent.*

(i) *$L(Y)$ is compact.*

(ii) *For a metrizable space X, every mutation $\boldsymbol{r}\colon \boldsymbol{U}(X, M)\to\boldsymbol{U}(Y, N)$ is generated by a fundamental sequence.*

COROLLARY 4. *Let X and Y be finite dimensional metrizable spaces. Then $X\times Y$ is a* FANR (*resp.* FAR) *if and only if either*

(i) *X is a* FANR (*resp.* FAR) *and Y is a compact* ANR (*resp.* AR), *or*

(ii) *X is a compact* ANR (*resp.* AR) *and Y is a* FANR (*resp.* FAR), *or*

(iii) *X and Y are both compact* FANR*'s* (*resp.* FAR*'s*), *or*

(iv) *X and Y are both* ANR*'s* (*resp.* AR*'s*).

Corollaries 1, 2 and 3 are immediate consequences of Theorems 1 and 2. We shall prove Corollary 4 in the case for FANR.

Suppose that $X\times Y$ is a FANR. Obviously X and Y are both FANR's. Let $L(X)\neq\emptyset$ and $L(Y)\neq\emptyset$. Then X and Y are both compact by Theorem 1, that is, X and Y are compact FANR's. Let $L(X)=\emptyset$ and $L(Y)\neq\emptyset$. Theorem 1 implies that X is compact. Since X is locally contractible at each point, it is an ANR. Thus (ii) holds. Similarly, if $L(X)\neq\emptyset$ and $L(Y)=\emptyset$ then (i) holds. If $L(X)$ and $L(Y)$ are both empty, then X and Y are ANR's.

Conversely, assume that (i) holds. It follows from [8, Theorem 3.8] that $X\times Y$ is a finite dimensional MANR. Since $L(X)$ is compact by Theorem 1 and Y is a compact ANR, $L(X\times Y)$ is compact. By applying Theorem 1 again, it is seen that $X\times Y$ is a FANR. The cases (iii) and (iv) are obvious. This completes the proof.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TSUKUBA

Accepté par la Rédaction le 28. 4. 1980