those homeomorphisms which also send P_m to P_m, then it is possible to construct all isotopies used in the proof of Theorem 6 of [9] so that P_m is sent to P_m by these isotopies. Thus by a slight modification of the proof of Theorem 6 of [9] we have that π is an isomorphism from $H(Y_n, P_m)$ to $H(X_{m,n})$.

Remark. If m and n are such that $m+n = 3$, then the presentation obtained for $H(X_{m,n})$ yields the group $S_m \times S_n \times Z_2$. This follows since when F consists of three elements, the twist homeomorphisms a_{13} and a_{23} are isotopic (rel F) to the identity and each dial homeomorphism is its own inverse.

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Hyperspaces where convergence to a calm limit implies eventual shape equivalence

by

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Abstract. We introduce the *calm fundamental metric* as a means of topologizing the collection 2^X of nonempty subcompacta of a compactum X. The calm fundamental metric d_{CF} induces a topology stronger than that of Borsuk's fundamental metric and has the following property: if A_0 is calm and $\lim_{n\to\infty} d_{CF}(A_n, A_0) = 0$, then $\mathrm{Sh}(A_n) = \mathrm{Sh}(A_0)$ for almost all n. The relation between d_{CF} and other hyperspace metrics is explored for certain subsets of 2^X.

§ **1. Introduction.** For a metric space X, let 2^X denote the collection of nonempty compact subsets of X. There have been several methods developed for imposing a metric topology on 2^X. The best-known is by use of the *Hausdorff metric* d_H. The Hausdorff metric has interesting properties, but is displeasing from the following standpoint: for fixed $A \in 2^X$, we may have $\lim_{n\to\infty} d_H(A_n, A) = 0$ and yet for all n, A_n and A may be very different topologically. For example, every member of 2^X is a limit of finite sets in the topology of d_H.

Metrics for 2^X that induce stronger topologies than that induced by d_H were introduced by Borsuk in [B1] and [B2]. The *fundamental metric* d_F defined in the latter paper was shown in [Ce-So] to have the following property: if $\lim_{n\to\infty} d_F(A_n, A) = 0$ and A is a calm compactum (see § 3 for the definition of calm) then $\mathrm{Sh}(A_n) \geq \mathrm{Sh}(A)$ for almost all n.

In this paper, we assume that X is a nonempty compactum. Our main results include the introduction of the *calm fundamental metric* d_{CF}, which induces on 2^X a topology stronger than that of d_F and has the following property: if $\lim_{n\to\infty} d_{CF}(A_n, A) = 0$ and A is calm, then $\mathrm{Sh}(A_n) = \mathrm{Sh}(A)$ for almost all n.

After submitting the first draft of this paper, the author received a preprint of [Ce2]. We show the notion of calmly regular convergence introduced there is essentially equivalent to convergence in the topology of d_{CF} and we answer a question raised in [Ce2].

We assume the reader is familiar with shape theory [B3] and the topology of the Hilbert cube [Ch].

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§ 2. Preliminaries. We let Q denote the Hilbert cube. For $A \in 2^Q$, $\varepsilon > 0$, we let $N_\varepsilon(A) = \{x \in Q \mid d(x, A) < \varepsilon\}$.

By *map* we will always mean a continuous function. An *ε-map* is a map f whose domain and range lie in a metric space (Y, d) and that satisfies $d(y, f(y)) < \varepsilon$ for all y in the domain of f.

If A and B are compact subsets of an AR-space M, we say a fundamental sequence $\underline{f} = \{f_k, A, B\}_{M,M}$ is an ε-fundamental sequence if it satisfies: for some neighborhood U of A in M, there is a k_0 such that $k \geqslant k_0$ implies $f_k|U$ is an ε-map.

The *metric of continuity* d_C is defined [B1] as follows: for $A, B \in 2^Y$, $d_C(A, B) = \inf\{\varepsilon > 0 \mid$ there are ε-maps $f\colon A \to B$ and $g\colon B \to A\}$. The space obtained by topologizing 2^Y by d_C is denoted 2^Y_C.

The fundamental metric is defined [B2] as follows: for $A, B \in 2^Y$, $d_F(A, B) = \inf\{\varepsilon > 0 \mid$ there are ε-fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,M} \quad \text{and} \quad \underline{g} = \{g_k, B, A\}_{M,M}\},$$

where M is an AR-space containing Y. Borsuk has shown that the choice of the AR-space M is irrelevant and that if $h\colon Y \to Y'$ is a homeomorphism then $\lim_{n\to\infty} A_n = A$ in 2^Y_C (in 2^Y_F) if and only if $\lim_{n\to\infty} h(A_n) = h(A)$ in $2^{Y'}_C$ (in $2^{Y'}_F$). Further, for all $A, B \in 2^Y$, $d_H(A, B) \leqslant d_F(A, B) \leqslant d_C(A, B)$.

We denote by Z^Q the set $\{Y \in 2^Q \mid Y$ is a Z-set in $Q\}$. We write $f \simeq g$ to indicate that f and g are homotopic maps.

§ 3. Calmness and d_F. Calm compacta were introduced by Čerin in [Ce1]. We recall some of Čerin's terminology.

Let $\mathscr{C}$ be a class of topological spaces. Let $A \in 2^Q$. Let V be a neighborhood of A in Q. We say h-$\mathrm{Comp}_{\mathscr{C}}(V, A)$ if the following homotopy compression property is satisfied:

for every neighborhood U of A in Q, there is a neighborhood W of A in Q such that for every $Y \in \mathscr{C}$, if $f, g\colon Y \to W$ are maps with $f \simeq g$ in V, then $f \simeq g$ in U.

If $\mathscr{C}$ is the class of all topological spaces, we abbreviate the above by h-$\mathrm{Comp}(V, A)$.

For $A \in Z^Q$, we say A is *$\mathscr{C}$-calm* if for every neighborhood U of A in Q there is a neighborhood V of A in U such that h-$\mathrm{Comp}_{\mathscr{C}}(V, A)$. We say A is *calm* if A is $\mathscr{C}$-calm when $\mathscr{C}$ is the class of all topological spaces.

In [Ce1] it is shown that $\mathscr{C}$-calmness is a hereditary shape property. The relation of calmness to more familiar shape properties is illustrated by the following facts: Solenoids are calm [Ce1]. If $Y \in Z^Q$, then $Y \in \mathrm{FANR}$ if and only if Y is calm and movable [Ce-So].

The next three results have easy proofs that are left to the reader:

(3.1) LEMMA. *Suppose $A \in 2^Q$, $\mathscr{C}$ is a class of topological spaces, and V is a neighborhood of A in Q such that h-$\mathrm{Comp}_{\mathscr{C}}(V, A)$. Let V' be a neighborhood of A in V. Then h-$\mathrm{Comp}_{\mathscr{C}}(V', A)$.*

(3.2) COROLLARY. *Let $A \in Z^Q$. Then A is $\mathscr{C}$-calm if and only if there is a neighborhood V of A in Q such that h-$\mathrm{Comp}_{\mathscr{C}}(V, A)$.*

(3.3) LEMMA. *Let $A \in 2^Q$ and let $f\colon Q \to Q$ be a homeomorphism. Let $B = f(A)$. If V is a neighborhood of A in Q such that h-$\mathrm{Comp}_{\mathscr{C}}(V, A)$ for a class $\mathscr{C}$ of topological spaces, then h-$\mathrm{Comp}_{\mathscr{C}}(f(V), B)$.*

We define for each $A \in Z^Q$ an *index of calmness* $i(A)$ as follows:

$$i(A) = \sup(\{0\} \cup \{\varepsilon > 0 \mid N_\varepsilon(A) \neq Q \text{ and } h\text{-}\mathrm{Comp}(N_\varepsilon(A), A)\}).$$

Observe that $i(A) \geqslant 0$, and by (3.2) we have $i(A) > 0$ if and only if A is calm.

According to [Ce-So], if A and B are compacta lying in AR-spaces M and N, respectively, then fundamental sequences $\underline{f} = \{f_k, A, B\}_{M,N}$ and $\underline{g} = \{g_k, A, B\}_{M,N}$ are ε-close if there is a neighborhood U of A in M such that for some integer m, $k \geqslant m$ implies $d(f_k(x), g_k(x)) < \varepsilon$ for all $x \in U$.

(3.4) THEOREM [Ce-So, (4.1)]. *If $Y \in 2^Q$ is topologically calm, then there is an $\varepsilon > 0$ such that for every compactum X lying in an AR-space M, every pair of ε-close fundamental sequences $\underline{f} = \{f_k, X, Y\}_{M,Q}$ and $\underline{g} = \{g_k, X, Y\}_{M,Q}$ are homotopic.*

(3.5) LEMMA. *Suppose $\{A_n\}_{n=0}^{\infty} \subset Z^Q$, $\lim_{n\to\infty} d_F(A_n, A_0) = 0$, and A_0 is calm. If $\delta > i(A_0)$ then for almost all n, $\delta > i(A_n)$.*

Proof. Otherwise there would be a sequence $\{A_n\}_{n=0}^{\infty} \subset Z^Q$ and a $\delta > 0$ such that $\lim_{n\to\infty} d_F(A_n, A_0) = 0$ and $i(A_0) < \delta < i(A_n)$ for $n = 1, 2, \ldots$

Let $0 < i(A_0) < s < \delta$. Since convergence in the fundamental metric implies convergence in the Hausdorff metric, there is an integer m_1 such that $n \geqslant m_1$ implies $A_n \subset N_s(A_0) \subset N_\delta(A_n)$.

Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers converging to 0 such that there are ε_n-fundamental sequences $\underline{f}^n = \{f^n_k, A_n, A_0\}_{Q,Q}$ and $\underline{g}^n = \{g^n_k, A_0, A_n\}_{Q,Q}$. By (3.4), there is a positive integer m_2 such that $n \geqslant m_2$ implies $(2\varepsilon_n)$-close fundamental sequences to A_0 are homotopic.

There is a compact ANR neighborhood A of A_0 in Q such that $A \subset N_s(A_0)$. There is a positive integer m_3 such that $n \geqslant m_3$ implies ε_n-close maps into A are homotopic. There is a positive integer m_4 such that $n \geqslant m_4$ implies $A_n \subset A$.

Let $n \geqslant \max\{m_1, m_2, m_3, m_4\}$ be fixed. Let U be a neighborhood of A_0 in Q. There is a neighborhood V of A_n in Q and a positive integer $k_1(n)$ such that $k \geqslant k_1(n)$ implies $f^n_k(V) \subset U$ and $f^n_k|V$ is an ε_n-map.

Since $\delta < i(A_n)$, it follows from (3.1) that h-$\mathrm{Comp}(N_\delta(A_n), A_n)$. Thus there is a neighborhood W of A_n with (by choice of m_4) $W \subset A$ such that if f and g are maps of a topological space into W with $f \simeq g$ in $N_\delta(A_n)$, then $f \simeq g$ in V.

There is a neighborhood T of A_0 in Q and a positive integer $k_2(n)$ such that $k \geqslant k_2(n)$ implies

$$g_k^n|T \text{ is an } \varepsilon_n\text{-map},$$
$$g_k^n(T) \subset W,$$

and (by choice of m_2, since $\underline{f}^n \circ \underline{g}^n$ is $(2\varepsilon_n)$-close to $\underline{1}_{A_0}$)

$$f_k^n \circ g_k^n|T \simeq 1_T \text{ in } U.$$

Let P be any topological space. Let $f, g\colon P \to T$ with $f \simeq g$ in $N_s(A_0)$. Fix $k \geqslant \max\{k_1(n), k_2(n)\}$. Then $g_k^n \circ f$ and $g_k^n \circ g$ are maps from P into $W \subset A \subset N_s(A_0)$, with $g_k^n \circ f \simeq f$ in A and $g \simeq g_k^n \circ g$ in A by choice of m_3. By choice of A, f, and g, $g_k^n \circ f \simeq g_k^n \circ g$ in $N_s(A_0)$. By choice of m_1, $g_k^n \circ f \simeq g_k^n \circ g$ in $N_\delta(A_n)$. By choice of W, $g_k^n \circ f \simeq g_k^n \circ g$ in V.

Our choice of V implies $f_k^n \circ g_k^n \circ f \simeq f_k^n \circ g_k^n \circ g$ in U. The last condition on our choice of T implies

$$f \simeq f_k^n \circ g_k^n \circ f \text{ in } U \quad \text{and} \quad f_k^n \circ g_k^n \circ g \simeq g \text{ in } U.$$

Thus $f \simeq g$ in U. It follows that $h\text{-Comp}(N_s(A_0), A_0)$. This is impossible, since $s > i(A_0)$. The assertion follows.

(3.6) Lemma. *Suppose* $\{A_n\}_{n=0}^\infty \subset Z^Q$, $\lim_{n\to\infty} d_F(A_n, A_0) = 0$, $\lim_{n\to\infty} i(A_n) = i(A_0) > 0$, *and V is a neighborhood of A_0 in Q such that* $h\text{-Comp}(V, A_0)$. *Then for almost all n,* $h\text{-Comp}(V, A_n)$.

Proof. Let $\varepsilon = i(A_0)/2$. Let P be a compact ANR neighborhood of A_0 in Q such that $P \subset V \cap N_\varepsilon(A_0)$. There is an integer n_1 such that $n \geqslant n_1$ implies $A_n \subset P$. There is a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$ whose limit is 0 and there are ε_n-fundamental sequences $\underline{f}^n = \{f_k^n, A_n, A_0\}_{Q,Q}$ and $\underline{g}^n = \{g_k^n, A_0, A_n\}_{Q,Q}$. Since $P \in \text{ANR}$, there is a positive integer n_2 such that $n \geqslant n_2$ implies maps into P that are $(2\varepsilon_n)$-close are homotopic.

Let δ satisfy $i(A_0)/2 < \delta < i(A_0)$. Since convergence in d_F implies convergence in d_H, our choice of P implies $P \subset N_\delta(A_n)$ for almost all n. Since $\lim_{n\to\infty} i(A_n) = i(A_0)$, our choice of δ and (3.1) imply $h\text{-Comp}(N_\delta(A_n), A_n)$ for almost all n. Thus there is, by (3.1), a positive integer n_3 such that $n \geqslant n_3$ implies $h\text{-Comp}(P, A_n)$.

Let $n \geqslant \max\{n_1, n_2, n_3\}$ be fixed. Let U be a neighborhood of A_n in Q. There is a neighborhood W of A_0 in Q such that for some positive integer $k_1(n)$, $k \geqslant k_1(n)$ implies

$$g_k^n(W) \subset P \quad \text{and} \quad g_k^n|W \text{ is an } \varepsilon_n\text{-map}.$$

There is a neighborhood S of A_0 in Q such that $S \subset P \cap W$ and maps into S that are homotopic in V are homotopic in W. There is a neighborhood T of A_n in P and a positive integer $k_2(n)$ such that $k \geqslant k_2(n)$ implies

$f_k^n(T) \subset S$,
$f_k^n|T$ is an ε_n-map,

and (by choice of n_3)

maps into T that are homotopic in P are homotopic in U.

Let $k \geqslant \max\{k_1(n), k_2(n)\}$ be fixed. Let Y be a topological space and let $f, g\colon Y \to T$ be maps such that $f \simeq g$ in V.

Then $f_k^n \circ f, f_k^n \circ g\colon Y \to S \subset P \subset V$, with $f_k^n \circ f \simeq f$ in P and $g \simeq f_k^n \circ g$ in P by choice of n_2. Hence $f_k^n \circ f \simeq f_k^n \circ g$ in V. By choice of S, $f_k^n \circ f \simeq f_k^n \circ g$ in W. By choice of W, $g_k^n \circ f_k^n \circ f \simeq g_k^n \circ f_k^n \circ g$ in P. Our choice of n_2 implies $f \simeq g_k^n \circ f_k^n \circ f$ in P and $g_k^n \circ f_k^n \circ g \simeq g$ in P. Hence $f \simeq g$ in P.

By choice of T we have $f \simeq g$ in U. Therefore $h\text{-Comp}(V, A_n)$.

(3.7) Lemma. *Let* $\{A_n\}_{n=0}^\infty \subset Z^Q$, *let* $f\colon Q \to Q$ *be a homeomorphism, and let* $B_n = f(A_n)$ *for all n. Suppose* $\lim_{n\to\infty} d_F(A_n, A_0) = 0$. *Then* $\lim_{n\to\infty} i(A_n) = i(A_0)$ *if and only if* $\lim_{n\to\infty} i(B_n) = i(B_0)$.

Proof. By [B2, (6.1)], it suffices to show that $\lim_{n\to\infty} i(A_n) = i(A_0)$ implies $\lim_{n\to\infty} i(B_n) = i(B_0)$.

Let $\varepsilon > 0$. By (3.5) we have $i(B_n) < i(B_0) + \varepsilon$ for almost all n. Since ε is arbitrary, we are done if we can show $i(B_0) - \varepsilon < i(B_n)$ for almost all n. Clearly we may assume $0 < i(B_0) - \varepsilon$.

Let δ satisfy $i(B_0) - \varepsilon < \delta < i(B_0) - \varepsilon/2$. Let $V = N_{i(B_0)-\varepsilon/2}(B_0)$. Since (3.1) implies $h\text{-Comp}(V, B_0)$, we have $h\text{-Comp}(f^{-1}(V), A_0)$ by (3.3). It follows from (3.6), since we assume $\lim_{n\to\infty} i(A_n) = i(A_0)$, that there is an integer n_1 such that $n \geqslant n_1$ implies $h\text{-Comp}(f^{-1}(V), A_n)$. Hence, by (3.3), $h\text{-Comp}(V, B_n)$ for $n \geqslant n_1$.

Our choice of δ implies there is an integer n_2 such that $n \geqslant n_2$ implies $N_\delta(B_n) \subset V$. It follows from (3.1) that for $n \geqslant \max\{n_1, n_2\}$, $h\text{-Comp}(N_\delta(B_n), B_n)$. Hence for $n \geqslant \max\{n_1, n_2\}$, $i(B_n) \geqslant \delta > i(B_0) - \varepsilon$, and the proof is done.

§ 4. **The calm fundamental metric.** Let X be a compactum. The index of calmness allows us to compare quantitatively members of 2^X as follows: let $h\colon X \to Q$ be an embedding such that $h(X)$ is a Z-set in Q. Since closed subsets of Z-sets in Q are also Z-sets in Q, we define for all $A, B \in 2^X$, $\lambda_h(A, B) = |i(h(A)) - i(h(B))|$.

We define the *calm fundamental metric* on 2^X (for the embedding h) by $d_{\text{CF}}^h(A, B) = d_F(A, B) + \lambda_h(A, B)$ for all $A, B \in 2^X$. It is easily seen that this formula defines a metric, since λ_h is symmetric in A and B, nonnegative, and satisfies the triangle inequality. Let us denote by $2^X_{\text{CF},h}$ the space obtained by topologizing 2^X by d_{CF}^h.

An embedding of a compactum as a Z-set of Q will be called a Z-embedding. The following shows that $2^X_{\text{CF},h}$ is a topological invariant of X and is topologically independent of the Z-embedding h chosen:

(4.1) Theorem. *Let* $g\colon X \to X'$ *be a homeomorphism. Let* $h\colon X \to Q$ *and* $h'\colon X' \to Q$ *be Z-embeddings. Let* $G\colon 2^X_{\text{CF},h} \to 2^{X'}_{\text{CF},h'}$ *be the function defined by* $G(A) = g(A)$ *for all* $A \in 2^X$. *Then G is a homeomorphism.*

Proof. By [Ch, 11.1, p. 14] there is a homeomorphism H of Q extending $h' \circ g \circ h^{-1}$: $h(X) \to h'(X')$. (We remark that this is the reason we have insisted on working with Z-sets.)

It is clear that G is a bijection. It follows from [B2] that $\lim_{n\to\infty} d_F(A_n, A_0) = 0$ in 2^X if and only if

$$\lim_{n\to\infty} d_F(g(A_n), g(A_0)) = \lim_{n\to\infty} d_F(G(A_n), G(A_0)) = 0 \quad \text{in } 2^{X'}.$$

Further, it follows from (3.7) that $\lim_{n\to\infty} \lambda_h(A_n, A_0) = 0$ if and only if $0 = \lim_{n\to\infty} \lambda_{h'}(G(A_n), G(A_0))$, since for $n = 0, 1, 2, \ldots$, we have $h'(G(A_n)) = h'(g(A_n)) = H(h(A_n))$. Thus G and G^{-1} are continuous, and the proof is complete.

In view of (4.1), we will drop "h" from the notation, writing 2^X_{CF} for the hyperspace of X metrized by the calm fundamental metric d_{CF}, for any (fixed) Z-embedding of X into Q. Alternately, when it suits our purpose, we may simply consider X as a Z-set of Q, using the inclusion map for the Z-embedding.

Let us say that if $A, B \in 2^X$ and there are ε-fundamental sequences $\underline{f} = \{f_k, A, B\}_{Q,Q}$ and $\underline{g} = \{g_k, B, A\}_{Q,Q}$ (where $X \in Z^Q$) such that $\underline{f} \circ \underline{g} \simeq \underline{1}_B$, the identity fundamental sequence on B, then A *ε-dominates* B. If we also have $\underline{g} \circ \underline{f} \simeq \underline{1}_A$, we say A and B are *ε-shape equivalent*.

The *metric of calmly regular convergence* d_{ca} for the collection $\mathrm{ca}(X)$ of members of 2^X (X not necessarily compact) satisfies, for M an ANR containing X:

(4.2) THEOREM [Ce2 (2.2) and (4.6)]. *Let* $\{A_n\}_{n=0}^{\infty} \subset \mathrm{ca}(X)$. *Then*

$$\lim_{n\to\infty} d_{\mathrm{ca}}(A_n, A_0) = 0$$

if and only if

a) $\lim_{n\to\infty} d_F(A_n, A_0) = 0$ *and*

b) *there is a neighborhood* V *of* A_0 *in* M *such that* h-$\mathrm{Comp}(V, A_n)$ *for almost all* n.

In the following theorem, the requirement that X be compact in using d_{CF} is avoided by observing that $\lim_{n\to\infty} d_F(A_n, A_0) = 0$ implies that $A = \bigcup_{n=0}^{\infty} A_n$ is compact; then we consider d_{CF} on the hyperspace of A. We remark that the equivalence of a) and c) below motivated this paper, while the equivalence of b) and c) improves [Ce2, (4.9)].

(4.3) THEOREM. *Let* $\{A_n\}_{n=0}^{\infty} \subset 2^X$, $A_0 \in \mathrm{ca}(X)$. *The following are equivalent*

a) $\lim_{n\to\infty} d_{\mathrm{CF}}(A_n, A_0) = 0$.

b) $\lim_{n\to\infty} d_{\mathrm{ca}}(A_n, A_0) = 0$.

c) *Given* $\varepsilon > 0$, *there is an integer* m *such that* $n \geqslant m$ *implies* A_n *and* A_0 *are ε-shape equivalent.*

Proof. That a) implies b) follows from (3.6) and (4.2).

To show b) implies c): Assume b). Then there is a compact ANR neighborhood P of A_0 in Q such that h-$\mathrm{Comp}(P, A_n)$ for $n = 0$ and $n \geqslant n_1$, where n_1 is some positive integer. Let $\varepsilon > 0$ be such that any two 2ε-close maps into P are homotopic. There is a positive integer n_2 such that if $n \geqslant n_2$, $A_n \subset P$ and $d_F(A_n, A_0) < \varepsilon$. Fix $n \geqslant \max\{n_1, n_2\}$.

By choice of n_2, there are ε-fundamental sequences $\underline{f} = \{f_k, A_n, A_0\}_{Q,Q}$ and $\underline{g} = \{g_k, A_0, A_n\}_{Q,Q}$. By the proof of [Ce–So, (4.2)] we have $\underline{f} \circ \underline{g} \simeq \underline{1}_{A_0}$.

Let W be a neighborhood of A_n in Q. By choice of P, there is a neighborhood T of A_n in Q, $T \subset P$, such that maps into T that are homotopic in P are homotopic in W. There is a neighborhood Y of A_n in Q, $Y \subset T$, and a positive integer m such that $k > m$ implies $d(g_k \circ f_k(x), x) < 2\varepsilon$ for all $x \in T$, and $g_k \circ f_k(Y) \subset T$.

Our choice of ε implies $g_k \circ f_k | Y \simeq 1_Y$ in P, hence in W by choice of T. Therefore $\underline{g} \circ \underline{f} \simeq \underline{1}_{A_n}$, and the assertion follows.

To show c) implies a): suppose there is a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ whose limit is 0 such that A_n and A_0 are ε_n-shape equivalent. Clearly we have $\lim_{n\to\infty} d_F(A_n, A_0) = 0$. Thus we must show $\lim_{n\to\infty} i(A_n) = i(A_0)$. By (3.5), it suffices to show that given τ such that $0 < \tau < i(A_0)$, we have $\tau \leqslant i(A_n)$ for almost all n.

Suppose δ satisfies $\tau < \delta < i(A_0)$. By (3.1), h-$\mathrm{Comp}(N_\delta(A_0), A_0)$. Let $P \subset N_\delta(A_0)$ be a compact ANR neighborhood of A_0 in Q. There is a positive integer n_1 such that $n \geqslant n_1$ implies $A_n \subset P$ (since convergence in d_F implies convergence in d_H) and ε_n-close maps into P are homotopic.

For each n there are ε_n-fundamental sequences $\underline{f}^n = \{f_k^n, A_n, A_0\}_{Q,Q}$ and $\underline{g}^n = \{g_k^n, A_0, A_n\}_{Q,Q}$ such that $\underline{f}^n \circ \underline{g}^n \simeq \underline{1}_{A_0}$ and $\underline{g}^n \circ \underline{f}^n \simeq \underline{1}_{A_n}$.

Let $n \geqslant n_1$ be fixed and let U be a neighborhood of A_n in Q. There is a neighborhood V of A_0 in Q and a positive integer $k_1(n)$ such that $k \geqslant k_1(n)$ implies $g_k^n(V) \subset U$. Let W be a neighborhood of A_0 in Q such that

$W \subset P$ and

maps into W that are homotopic in $N_\delta(A_0)$ are homotopic in V. Our first condition on the choice of n_1 implies there is a neighborhood T of A_n with $T \subset P$ and there is a positive integer $k_2(n)$ such that $k \geqslant k_2(n)$ implies

$$f_k^n(T) \subset W, \quad f_k^n|T \text{ is an } \varepsilon_n\text{-map}, \quad \text{and} \quad g_k^n \circ f_k^n|T \simeq 1_T \text{ in } U.$$

Let Y be a topological space and let $f, g\colon Y \to T$ be maps that are homotopic in $N_\delta(A_0)$. Let $k \geqslant \max\{k_1(n), k_2(n)\}$ be fixed. Then $f_k^n \circ f$, $f_k^n \circ g\colon Y \to W \subset P$, and our choices of n_1 and $k_2(n)$ imply $f_k^n \circ f \simeq f$ in P and $g \simeq f_k^n \circ g$ in P. By choice of P, $f_k^n \circ f \simeq f_k^n \circ g$ in $N_\delta(A_0)$. By choice of W, $f_k^n \circ f \simeq f_k^n \circ g$ in V. Our choice of V implies $g_k^n \circ f_k^n \circ f \simeq g_k^n \circ f_k^n \circ g$ in U. By choice of $k_2(n)$,

$$f \simeq g_k^n \circ f_k^n \circ f \text{ in } U \quad \text{and} \quad g_k^n \circ f_k^n \circ g \simeq g \text{ in } U.$$

Hence $f \simeq g$ in U. It follows that h-$\mathrm{Comp}(N_\delta(A_0), A_n)$.

Since convergence in d_F implies convergence in d_H and $\tau<\delta$, we have, for some positive integer n_2, $N_\tau(A_n)\subset N_\delta(A_0)$ for $n\geqslant n_2$. It follows from (3.1) that for $n\geqslant\max\{n_1, n_2\}$ we have h-Comp$(N_\tau(A_n), A_n)$, and thus $i(A_n)\geqslant\tau$. This completes the proof.

For $A\in\mathrm{ca}(X)$, let $X[A] = \{B\in 2^X \mid \mathrm{Sh}(B) = \mathrm{Sh}(A)\}$. An immediate consequence of (4.3) is:

(4.4) Corollary. a) *For X compact, $X[A]$ is open in 2^X_{CF}.*

b) *$X[A]$ is open and closed in $(\mathrm{ca}(X), d_{\mathrm{ca}})$.*

The following question is raised in [Ce2]: If X is separable, is $(\mathrm{ca}(X), d_{\mathrm{ca}})$ separable? A negative answer is given in:

(4.5) Example. *Let E^3 denote euclidean 3-space. Then $(\mathrm{ca}(E^3), d_{\mathrm{ca}})$ is not separable.*

Proof. By [G], there is an uncountable family $\{S_\alpha \mid \alpha\in A\}$ of solenoids in E^3 such that $\alpha\neq\beta$ implies $\mathrm{Sh}(S_\alpha)\neq\mathrm{Sh}(S_\beta)$. Since solenoids are calm [Cl, 4.11], it follows from (4.4) that $\{E^3[S_\alpha] \mid \alpha\in A\}$ is an uncountable family of nonempty, pairwise disjoint open sets in $\mathrm{ca}(E^3)$. Hence $(\mathrm{ca}(E^3), d_{\mathrm{ca}})$ is not separable.

We remark that in (4.3), c) implies a) even if A_0 is not calm: For c) clearly implies $\lim_{n\to\infty} d_F(A_n, A_0) = 0$, and d_F and d_{CF} coincide on pairs of non-calm compacta. However, if A_0 is not calm, then a) does not imply c), as the following shows:

Suppose A_0 is the usual "middle-third" Cantor set of real numbers. For $n = 1, 2, \ldots$, let A_n be the set of endpoints of the 2^n intervals remaining after the nth step in the construction of A_0, i.e., $A_n = \{x\in A_0 \mid x = m/(3^n)$ for some integer $m\}$.

(4.6) Example. *If $\{A_n\}_{n=0}^\infty$ is as described above, then $\lim_{n\to\infty} d_{\mathrm{CF}}(A_n, A_0) = 0$ in $2^{A_0}_{\mathrm{CF}}$.*

Proof. In [Bx–Sh] it was shown that $\lim_{n\to\infty} d_F(A_n, A_0) = 0$. Since A_0 has infinitely many components, it is not calm [Ce1, (4.6)]. Thus $i(A_0) = 0$ (we are assuming A_0 is Z-embedded in Q). Therefore we must show $\lim_{n\to\infty} i(A_n) = 0$. But since $n\geqslant 1$ implies A_n is discrete, the fact that

$$\lim_{n\to\infty}\min\{d(x,y) \mid x \text{ and } y \text{ are distinct points of } A_n\} = 0$$

and the easily-shown fact that h-Comp(V, A_n) implies no component of V contains distinct points of A_n imply $\lim_{n\to\infty} i(A_n) = 0$.

§ 5. On restricting d_{CF} to certain subsets of 2^X.

We have seen that for non-calm members of 2^X, $d_{\mathrm{CF}} = d_F$. In this section, we examine d_{CF} for the following subsets of 2^X:

$\mathrm{FAR}^X = \{Y\in 2^X \mid Y\in\mathrm{FAR}\}$;

$\mathrm{ANR}^X = \{Y\in 2^X \mid Y\in\mathrm{ANR}\}$ (the latter only in the case where $\dim X<\infty$).

(5.1) Lemma. *Let $A\in Z^Q$, $A\in\mathrm{FAR}$. Then h-Comp(Q, A).*

Proof. This follows easily from the contractibility of Q and the fact that A is a fundamental retract of Q.

(5.2) Corollary. *The metrics d_F and d_{CF} induce the same topology on FAR^X.*

Proof. This follows from (5.1) and the equivalence of a) and b) in (4.3).

In the remainder of this section we assume X is a finite-dimensional compactum Z-embedded in Q. Borsuk [B1] defined the *homotopy metric* d_h on ANR^X (the resulting space is denoted 2^X_h in the literature) and showed that it has the property that if $d_h(A, B)$ is sufficiently small, then A and B have the same homotopy type. Since the latter is an analogue of (4.3), and since there is a certain similarity in the forms of the definitions of d_{CF} and d_h, it seems reasonable to investigate the relation between these metrics.

The following theorem characterizes the topology of 2^X_h:

(5.3) Theorem [B1]. *Let $\{A_n\}_{n=0}^\infty\subset 2^X_h$. Then $\lim_{n\to\infty} d_h(A_n, A_0) = 0$ if and only if*

a) $\lim_{n\to\infty} d_H(A_n, A_0) = 0$, and

b) given $\varepsilon>0$, there is a $\delta>0$ such that for all n, every subset of A_n with diameter less than δ contracts to a point within a subset of A_n of diameter less than ε.

The following is a weak version of [B1, Lemma on p. 188 and Theorem on p. 196].

(5.4) Theorem. *Suppose $\lim_{n\to\infty} d_h(A_n, A_0) = 0$ in $2^{I^m}_h$ (I^m is the m-cube). Then there is a neighborhood U of A_0 in I^m and a positive integer p such that $n\geqslant p$ implies A_n is a retract of U.*

We have:

(5.5) Theorem. *Let $\{A_n\}_{n=0}^\infty\subset\mathrm{ANR}^X$. If $\lim_{n\to\infty} d_h(A_n, A_0) = 0$, then*

$$\lim_{n\to\infty} d_{\mathrm{CF}}(A_n, A_0) = 0 .$$

Proof. Since X is a finite-dimensional compactum, it embeds in I^m for some positive integer m. We may regard 2^X_h as a subspace of $2^{I^m}_h$ [B1, Corollary 5, p. 198]. We may regard I^m as a Z-set in Q by identifying it with $I^m\times\{(0, 0, \ldots)\}\subset I^m\times Q_{m+1} = Q$. By (5.4) there is a neighborhood U of X in I^m and a positive integer n_1 such that $n\geqslant n_1$ implies there is a retraction $r_n\colon U\to A_n$. There is a compact ANR neighborhood P of $A_0\cup\bigcup_{n=n_1}^\infty A_n$ in Q such that $P\subset U\times Q_{m+1}$.

Since $\lim_{n\to\infty} d_h(A_n, A_0) = 0$ implies $\lim_{n\to\infty} d_C(A_n, A_0) = 0$ [B1, (79), p. 190], there is a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$ whose limit is 0, and for $n\geqslant 1$ there are ε_n-maps $f_n\colon A_n\to A_0$ and $g_n\colon A_0\to A_n$. Since ε_n-maps induce ε_n-fundamental sequences, it follows from (4.2) that it suffices to show f_n and g_n are homotopy inverses, for almost all n. By [Bx–Sh, (3.14), p. 852], there is a positive integer n_2 such that $n\geqslant n_2$ implies $f_n\circ g_n\simeq 1_{A_0}$.

There is a positive integer n_3 such that $n \geqslant n_3$ implies $(2\varepsilon_n)$-close maps into P are homotopic. Let $n \geqslant \max\{n_1, n_2, n_3\}$ be fixed. Let $r\colon P \to A_n$ be the retraction defined by

$$r(x, q_{m+1}, q_{m+2}, \ldots) = r_n(x), \quad \text{for} \quad x \in U.$$

Since $g_n \circ f_n\colon A_n \to A_n \subset P$ is a $(2\varepsilon_n)$-map, our choice of n_3 implies there is a homotopy $F\colon A_n \times I \to P$ with

$$F(x, 0) = g_n \circ f_n(x) \quad \text{and} \quad F(x, 1) = x \quad \text{for all } x \in A_n.$$

Thus $r \circ F\colon A_n \times I \to A_n$ is a homotopy with

$$r \circ F(x, 0) = g_n \circ f_n(x) \quad \text{and} \quad r \circ F(x, 1) = x.$$

This completes the proof.

The converse of (5.5) is not true: Let $\{A_n\}_{n=0}^{\infty}$ be the sequence of [Bx, (4.9)], in which it was shown that $A_0 \neq \lim\limits_{n\to\infty} A_n$ in the topology of d_h. However, $A_0 = \lim\limits_{n\to\infty} A_n$ in the topology of d_C, hence in the topology of d_F, hence (by (5.2)) in the topology of d_{CF}.

Thus d_h induces a stronger topology on ANR^X than does d_{CF}.

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On non compact FANR's and MANR's

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Abstract. It is proved that a finite dimensional metrizable space X is a FANR if and only if X is a MANR and the set of points at which X is not locally contractible has the compact closure. As an application, for finite dimensional metrizable spaces X and Y, a necessary and sufficient condition under which $X \times Y$ be a FANR is obtained in terms of X and Y.

1. Introduction. The notion of FANR is introduced by K. Borsuk [2]. According to [2, p. 94] a metrizable space X is a FANR if for every metrizable space X' containing X as a closed subset, X is a fundamental neighborhood retract of X'. S. Godlewski [4] has introduced the concept of MANR. From the definition it is obvious that every FANR is a MANR. By [4] and [6] the properties "to be a FANR" are not generally shape invariants in the sense of Fox [3]. In this paper we shall show that a finite dimensional metrizable space X is a FANR if and only if X is a MANR and the set of points at which X are not locally contractible has the compact closure. Obviously the second condition is not a shape invariant.

All spaces under considerations are metrizable and maps are continuous. AR and ANR mean those for metrizable spaces.

2. Theorems. Let X be a space and let $x \in X$. If for every neighborhood U of x in X there exists a neighborhood V of x such that V is contractible in U, then X is said to be *locally contractible at* x. Put $L'(X) = \{x\colon x \in X$ and X is locally contractible at $x\}$ and $L(X) = \mathrm{Cl}(X - L'(X))$, where Cl means the closure in X.

THEOREM 1. *A finite dimensional space X is a* FANR *if and only if X is a* MANR *and $L(X)$ is compact.*

Proof. "If part". Let M be an AR containing X as a closed set. It is assumed by [7] that M is finite dimensional and X is unstable in M in the sense of Sher [9, p. 346]. Since X is a MANR, there is a closed neighborhood W of X in M and a mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}(W, M) \to \boldsymbol{U}(X, M)$. Here $\boldsymbol{U}(A, M)$ means the family of all open neighborhoods of A in M. (See [3] and [5] for notations and definitions.) Let d be a metric in M. Choose an open cover $\mathcal{U}$ of the set $M - L(X)$ such that if $d(x_i, L(X)) \to 0$ $(i \to \infty)$ for $x_i \in M - L(X)$ then diameter $\mathrm{St}(x_i, \mathcal{U}) \to 0$ $(i \to \infty)$, where $\mathrm{St}(x, \mathcal{U}) = \bigcup \{U\colon x \in U$ and $U \in \mathcal{U}\}$. Since X is locally contractible at each point of the set $X - L(X)$ and M is finite dimensional, by [1, Theorem (9.1), p. 80]