Homeotopy groups of punctured spheres with holes

by

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Abstract. Let $H(X_{m,n})$ denote the group of isotopy classes of homeomorphisms of the space $X_{m,n}$ which is obtained by removing m points and the interiors of n disjoint closed disks from the 2-sphere. Presentations of $H(X_{m,0})$ are known for all m. It has also been shown that $H(X_{0,m})$ is isomorphic to $H(X_{m,0})$. In this paper the groups $H(X_{m,n})$ are determined for all values of m and n.

1. Introduction. Let $p_1, \ldots, p_m$ be m points in S^2 and let $D_1, \ldots, D_n$ be n disjoint closed disks in S^2 with p_i not in D_j for any i or j. Let $X_{m,n}$ denote the manifold with boundary obtained by removing the points $p_1, \ldots, p_m$ and the interiors of the disks $D_1, \ldots, D_n$ from S^2. The homeotopy group of $X_{m,n}$, denoted $H(X_{m,n})$, is defined to be the group of all isotopy classes of homeomorphisms of $X_{m,n}$. The group $H(X_{m,0})$ has been studied extensively. Detailed descriptions of this group are given in [7] and [2]. In [9] it is shown that $H(X_{m,0})$ is isomorphic to $H(X_{0,m})$. The groups $H(X_{1,1})$ and $H(X_{1,2})$ are discussed in [6]. In this paper we use some of the techniques and results of [9] and [10] together with various facts about "dial" homeomorphisms to determine $H(X_{m,n})$ for any m and n.

The presentation of $H(X_{m,n})$ is obtained in three main steps. First a presentation is obtained for the group of isotopy classes of orientation preserving homeomorphisms of S^2 which send the set $P_m = \{p_1, \ldots, p_m\}$ to itself and also send the set $Q_n = \{q_1, \ldots, q_n\}$ to itself where q_i is in the interior of D_i, $1 \leqslant i \leqslant n$. The presentation of this group is then extended to a presentation of the isotopy group of the space of all homeomorphisms sending P_m to P_m and Q_n to Q_n. Finally, this last group is shown to be isomorphic to $H(X_{m,n})$. Since $H(X_{2,0})$ and $H(X_{1,1})$ are known, we will restrict attention to m and n with $m+n>2$.

2. Orientation preserving homeomorphisms. Let $F = P_m \cup Q_n$ and let $H^+(S^2, P_m, Q_n)$ denote the group of isotopy classes (rel F) of orientation preserving homeomorphisms of S^2 which send P_m to P_m and Q_n to Q_n. Define $H'(S^2, F)$ to be the group of isotopy classes (rel F) of orientation preserving homeomorphisms h of S^2 which are fixed on F, i.e. $h(x) = x$ for all x in F. Let S_p denote the symmetric group on p objects. Define $i\colon H'(S^2, F) \to H^+(S^2, P_m, Q_n)$ to be the inclusion map and define $r\colon H^+(S^2, P_m, Q_n) \to S_m \times S_n$ to be the map which sends the isotopy class of a homeomorphism h onto the restriction of h to F. Since the image of i is clearly the kernel of r we have the following lemma.

LEMMA 1. *The sequence*

$$1 \to H'(S^2, F) \xrightarrow{i} H^+(S^2, P_m, Q_n) \xrightarrow{r} S_m \times S_n \to 1 \text{ is exact}.$$

In order to use the above lemma to obtain a presentation of $H^+(S^2, P_m, Q_n)$ we need the following fact concerning group presentations and exact sequences.

LEMMA 2. *Given an exact sequence of groups*

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1.$$

Suppose A and C have presentations given by

$$A = \langle a_1, \ldots, a_s |\ r_i(a) = 1, 1 \leqslant i \leqslant p\rangle$$

and

$$C = \langle c_1, \ldots, c_t |\ r_i'(c) = 1, 1 \leqslant i \leqslant q\rangle.$$

Let $b_i = \alpha(a_i)$, $1 \leqslant i \leqslant s$ and choose $\bar{b}_i$ such that $\beta(\bar{b}_i) = c_i$, $1 \leqslant i \leqslant t$. Then B has a presentation with generators $\{b_1, \ldots, b_s, \bar{b}_1, \ldots, \bar{b}_t\}$ and a complete set of relations as follows:

(2.1) $$r_i(b) = 1, \quad 1 \leqslant i \leqslant p,$$

(2.2) $$r_i'(\bar{b}) = w(b), \quad 1 \leqslant i \leqslant q,$$

(2.3) $$\bar{b}_i b_j \bar{b}_i^{-1} = w(b), \quad 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q,$$

(2.4) $$\bar{b}_i^{-1} b_j \bar{b}_i = w(b), \quad 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q.$$

Proof. The relations in (2.3) and (2.4) exist since $\ker\beta$ is normal in B. Using (2.3) and (2.4) all the remaining relations in B can be put in the form $w(b) = w'(\bar{b})$, i.e. a word in the elements $b_1, \ldots, b_s$ equal to a word in $\bar{b}_1, \ldots, \bar{b}_t$. Relations of this form are in turn consequences of the relations in 2.1 and 2.2. In particular, relations of the form $w(b) = 1$ are consequences of the relations in 2.1.

In order to apply Lemma 2 to the short exact sequence in Lemma 1 we need presentations of $H'(S^2, F)$ and $S_m \times S_n$. Moreover, the generators of $S_m \times S_n$ should be given in a form which makes it convenient to find elements in $H^+(S^2, P_m, Q_n)$ which are sent to these generators by r. To this end, we relabel the points $q_1, \ldots, q_n$ as $p_{m+1}, \ldots, p_{m+n}$ and assume the points $p_1, \ldots, p_m, p_{m+1}, \ldots, p_{m+n}$ are arranged on the equator of S^2. Next define d_i to be the homeomorphism of S^2 which "dials" p_i to p_{i+1} in a counterclockwise direction and is fixed outside an annulus which contains p_i and p_{i+1} but no other points of F. In more detail, if A is an annulus in the Euclidean plane parametrized by (r, θ) where $1 \leqslant r \leqslant 2$ and θ is a real number mod 2π and e is an imbedding of A into S^2 with $e(\frac{3}{2}, 0) = p_{i+1}$ and $e(\frac{3}{2}, \pi) = p_i$, then $d_i | e(A) = ede^{-1}$ where d is the homeomorphism of A given by $d(r, \theta) = (r, \theta - 2\pi(r-1))$ for $1 \leqslant r \leqslant 1.5$ and $d(r, \theta) = (r, \theta - 2\pi(2-r))$ for $1.5 \leqslant r \leqslant 2$. A picture indicating the action of d is given in Figure 1d of [1].

LEMMA 3. *Let d_i' denote the restriction of d_i to F. $S_m \times S_n$ is generated by $\{d_i' \colon 1 \leqslant i \leqslant m-1, m+1 \leqslant i \leqslant n+m-1\}$. A complete set of relations is given as follows:*

(3.1) $$d_i' d_{i+1}' d_i' = d_{i+1}' d_i' d_{i+1}', \quad i < m-1 \text{ or } i \geqslant m+1,$$

(3.2) $$d_i' d_k' = d_k' d_i', \quad |i-k| \geqslant 2,$$

(3.3) $$(d_{n+m-1}')^2 = 1,$$

(3.4) $$(d_1')^2 = 1.$$

Proof. This presentation follows from the presentation given on page 64 of [3] by observing that d_i' denotes the transposition $(i, i+1)$.

Next we give a presentation of $H'(S^2, F)$. For $1 \leqslant j < k$, let α_{jk} denote a simple closed curve in S^2 which encloses the points p_j and $p_k, \ldots, p_{m+n}$. That is, one of the open disks bounded by α_{jk} contains these points and the other disk contains $F - \{p_j, p_k, \ldots, p_{m+n}\}$. Moreover, assume that α_{jk} is below the points $p_{j+1}, \ldots, p_{k-1}$ when $j < k-1$, i.e. assume the "neck" of α_{jk} is in the southern hemisphere. Let a_{jk} be the twist homeomorphism corresponding to α_{jk} as defined in [10]. Let $\bar{a}_{jk}$ denote the equivalence class of a_{jk} in $H'(S^2, F)$.

LEMMA 4. *If $m+n<2$, then*

$$H'(S^2, F) \text{ is generated by } \bigcup_{k=3}^{m+n} \{\bar{a}_{jk} \colon 1 \leqslant j < k\}$$

and in terms of these generators a complete set of relations is given by the relations in Theorem 4.1 of [10].

Proof. This lemma follows by Theorem 4.1 of [10].

We are now in position to determine $H^+(S^2, P_m, Q_n)$. Let $\bar{d}_i$ denote the equivalence class of d_i in $H^+(S^2, P_m, Q_n)$. For $m+n>2$, let $A = \bigcup_{k=3}^{m+n} \{\bar{a}_{jk} \colon 1 \leqslant j < k\}$ and let $B = \{\bar{d}_i \colon 1 \leqslant i < m+n, i \neq m\}$.

THEOREM 1. *If $m+n>2$, then $H^+(S^2, P_m, Q_n)$ is generated by $A \cup B$ and in terms of these generators a complete set of relations is given by:*

(1) *The relations involving the elements of A given by Lemma 4.*

(2) a) *The relations obtained from (1) and (2) of Lemma 3 by replacing d_j' by $\bar{d}_j$.*

b) $(\bar{d}_{m+n-1})^2 = \bar{a}_{m+n-1,m+n}$,

c) $(\bar{d}_1)^2 = \bar{a}_{34}^{-1}$,

(3) a) $\bar{d}_p \bar{a}_{jk} \bar{d}_p^{-1} = \bar{a}_{j+1,k}$, *if $j<k-1$ and $p=j$,*

b) $\bar{d}_p \bar{a}_{jk} \bar{d}_p^{-1} = \bar{a}_{jk}^{-1} \bar{a}_{j-1,k} \bar{a}_{jk}$, *if $p = j-1$,*

c) $\bar{d}_p \bar{a}_{jk} \bar{d}_p^{-1} = \bar{a}_{j,k-1} \bar{a}_{jk}^{-1} \bar{a}_{k-1,k}$, *if $k = m+n$ and $j<k-1$ and $p = m+n-1$,*

d) $\bar{d}_p \bar{a}_{jk} \bar{d}_p^{-1} = \bar{a}_{j,k-1} \bar{a}_{j,k+1} \bar{a}_{jk}^{-1} (\bar{a}_{k-1,k+1} \bar{a}_{k,k+1} \bar{a}_{k+1,k+2}^{-1} \bar{a}_{k-1,k}^{-1})$, *if $p = k-1$, $j<k-1$,*

e) $\bar{d}_p \bar{a}_{jk} \bar{d}_p^{-1} = \bar{a}_{jk}$ *for all remaining choices of j, k and p.*

Proof. The above theorem is obtained by applying Lemma 2 to the short exact sequence in Lemma 1 using the presentations in Lemma 3 and Lemma 4. The relations in (1), (2), and (3) of Theorem 1 correspond respectively to relations of Types 2.1, 2.2 and 2.3 in Lemma 2. Type 2.4 relations are consequences of the relations in (3). In particular, (3a) yields the relation $\bar{d}_j^{-1} a_{j+1,k} \bar{d}_j = \bar{a}_{jk}$ for $j+1<k$ and this relation can be combined with the others in (3) to yield all relations of Type 2.4. Thus in order to establish Theorem 1, it is enough to show that the relations given in (1), (2) and (3) are valid.

The relations in (1) can be obtained by using essentially the same isotopies used to obtain the corresponding relations in Theorem 4.1 of [10].

The relations in (2a) follow directly from the first two relations in Theorem 4.5 of [2] by noting that the homeomorphism d_j can be taken to be a representative of the element ω_j as defined in [2].

(2b) and (2c) hold since for any k, d_k^2 is isotopic (rel F) to a twist homeomorphism supported by a curve enclosing p_k and p_{k+1}. If $k = m+n-1$, this curve can be taken to be $\alpha_{m+n-1,m+n}$. If $k = 1$, this curve can be taken to be α_{34}.

The verification of the relations in (3) makes use of the fact that if a homeomorphism f is applied to a simple closed curve α, then the twist homeomorphism supported by $f(\alpha)$ is isotopic to the homeomorphism faf^{-1} where a is the twist homeomorphism supported by α. Moreover, if F is a finite subset of S^2 and $f(F) = F$ and α is in S^2-F, then all isotopies can be taken to be relative to F. We also will make use of the fact that ambient isotopic curves (rel F) support isotopic (rel F) twist homeomorphisms. A full discussion of twist homeomorphisms is given in [2]. In particular, the above comments follow from Lemmas 4.6.7 and 4.6.1 of [2].

The relations in (3a) are established by noting that for j less than $k-1$ the dial homeomorphism d_j applied to the curve α_{jk} yields a curve which is ambient isotopic (rel F) to $\alpha_{j+1,k}$.

(3b) is obtained by noting that $d_{j-1}(\alpha_{jk})$ is ambient isotopic (rel F) to $a_{jk}^{-1}(\alpha_{j-1,k})$.

(3c) is established by first noting that for the given values of k, j and p the twist homeomorphism supported by $d_p(\alpha_{jk})$ followed by the inverse of $a_{j,k-1} a_{jk}^{-1} a_{k-1,k}$ is isotopic (rel F) to a homeomorphism which is the identity on the equator. We then apply a technique of J. W. Alexander (see Theorem 5.2 of [4]) to show that this homeomorphism is isotopic (rel F) to the identity on all of S^2.

(3d) is verified in a manner similar to that used for (3c).

Finally, (3e) holds since for those values of p, j and k not covered in (3a) through (3d) the annuli used to define d_p and a_{jk} can be chosen to be disjoint and hence the homeomorphisms commute.

3. Orientation reversing homeomorphisms. Let $H(S^2, P_m, Q_n)$ denote the space of isotopy classes (rel F) of all homeomorphisms h of S^2 with $h(P_m) = P_m$ and $h(Q_n) = Q_n$. Let f be the orientation reversing homeomorphism of S^2 defined by reflection through the equator, i.e. $f(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Let $\bar{f}$ denote the equivalence class of f in $H(S^2, P_m, Q_n)$.

THEOREM 2. *Let $m+n>2$. A complete set of generators of $H(S^2, P_m, Q_n)$ is formed by adjoining $\bar{f}$ to the generators of $H^+(S^2, P_m, Q_n)$ given in Theorem 1. A complete set of relations is given as follows:*

(1) *The relations given in Theorem 1.*
(2) $\bar{f}^2 = 1$,
(3) (a) $\bar{f}\bar{d}_p\bar{f}^{-1} = \bar{d}_p^{-1}$,
(b) $\bar{f}\bar{a}_{jk}\bar{f}^{-1} = \bar{a}_{jk}^{-1}$, *if* $j = k-1$,
(c) $\bar{f}\bar{a}_{jk}\bar{f}^{-1} = \bar{a}_{j+1,j+2}^{-1}\bar{a}_{jk}^{-1}\bar{a}_{j+1,j+2}$, *if* $j<k-1$.

Proof. We apply Lemma 2 to the short exact sequence

$$1 \to H^+(S^2, P_m, Q_n) \xrightarrow{i} H(S^2, P_m, Q_n) \xrightarrow{\gamma} Z_2 \to 1,$$

where γ is the projection of $H(S^2, P_m, Q_n)$ onto $H(S^2, P_m, Q_n)/H^+(S^2, P_m, Q_n) \cong Z_2$. Since $\gamma(\bar{f})$ generates Z_2 and i is an imbedding, by Lemma 2 $\{\bar{f}\} \cup A \cup B$ generates $H(S^2, P_m, Q_n)$. Also the relations given in (1), (2) and (3) of Theorem 2 correspond respectively to the relations of Types 2.1, 2.2 and 2.3 of Lemma 2. Since $\bar{f} = \bar{f}^{-1}$, the relations of Type 2.4 are also covered by the relations in (3). Thus in order to prove Theorem 2 it suffices to show that the relations in (1), (2) and (3) of Theorem 2 are valid.

The relations in (1) hold since i is an imbedding.

(2) and (3a) follow directly from the definitions of the homeomorphisms involved. That is, the homeomorphism f^2 is the identity and $fd_pf^{-1} = d_p^{-1}$.

(3b) and (3c) are verified by using the properties of twist homeomorphisms mentioned in the proof of Theorem 1. (3b) is established by showing that for $j = k-1$ the curve $f(\alpha_{jk})$ is ambient isotopic (rel F) to the curve α_{jk}^{-1} and hence the corresponding twist homeomorphisms are isotopic. (3c) is established by showing that for $j<k-1$ the curve $f(\alpha_{jk})$ is ambient isotopic (rel F) to the curve $a_{j+1,j+2}^{-1}(a_{jk}^{-1})$.

4. The homeotopy group of $X_{m,n}$. We will obtain the desired presentation of $H(X_{m,n})$ by showing $H(X_{m,n})$ is isomorphic to $H(S^2, P_m, Q_n)$. Let Y_n be the space obtained by removing the interiors of the disks $D_1, \ldots, D_n$ from S^2. Let $H(Y_n, P_m)$ denote the space of isotopy classes (rel P_m) of homeomorphisms of Y_n which send P_m to itself. Note that $Y_n - P_m = X_{m,n}$.

LEMMA 5. $H(Y_n, P_m) \cong H(X_{m,n})$.

Proof. Let ψ: $H(Y_n, P_m) \to H(X_{m,n})$ be the map which sends the isotopy class (rel P_m) of a homeomorphism h of Y_n with $h(P_m) = P_m$ to the isotopy class of the restriction of h to $Y_n - P_m$. Theorem 4.21 of [8] shows that ψ is an isomorphism from $H(Y_n, P_m)$ to $H(Y_n - P_m) = H(X_{m,n})$.

THEOREM 3. $H(S^2, P_m, Q_n) \cong H(X_{m,n})$.

Proof. By Lemma 5 it suffices to show that $H(S^2, P_m, Q_n)$ is isomorphic to $H(Y_n, P_m)$. Let π: $H(Y_n, P_m) \to H(S^2, P_m, Q_n)$ be the map which sends the isotopy class (rel P_m) of a homeomorphism h of Y_n with $h(P_m) = P_m$ to the isotopy class (rel $P_m \cup Q_n$) of the homeomorphism of S^2 formed by taking the cone of h over each of the boundary components of Y_n. In Theorem 6 of [9] it is shown that π is an isomorphism from the homeotopy group of Y_n to the group of isotopy classes (rel Q_n) of homeomorphisms h of S^2 with $h(Q_n) = Q_n$. If we restrict attention to those homeomorphisms which also send P_m to P_m, then it is possible to construct all isotopies used in the proof of Theorem 6 of [9] so that P_m is sent to P_m by these isotopies. Thus by a slight modification of the proof of Theorem 6 of [9] we have that π is an isomorphism from $H(Y_n, P_m)$ to $H(X_{m,n})$.

Remark. If m and n are such that $m+n = 3$, then the presentation obtained for $H(X_{m,n})$ yields the group $S_m \times S_n \times Z_2$. This follows since when F consists of three elements, the twist homeomorphisms a_{13} and a_{23} are isotopic (rel F) to the identity and each dial homeomorphism is its own inverse.

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Hyperspaces where convergence to a calm limit implies eventual shape equivalence

by

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Abstract. We introduce the *calm fundamental metric* as a means of topologizing the collection 2^X of nonempty subcompacta of a compactum X. The calm fundamental metric d_{CF} induces a topology stronger than that of Borsuk's fundamental metric and has the following property: if A_0 is calm and $\lim_{n\to\infty} d_{CF}(A_n, A_0) = 0$, then $\mathrm{Sh}(A_n) = \mathrm{Sh}(A_0)$ for almost all n. The relation between d_{CF} and other hyperspace metrics is explored for certain subsets of 2^X.

§ 1. Introduction. For a metric space X, let 2^X denote the collection of nonempty compact subsets of X. There have been several methods developed for imposing a metric topology on 2^X. The best-known is by use of the *Hausdorff metric* d_H. The Hausdorff metric has interesting properties, but is displeasing from the following standpoint: for fixed $A \in 2^X$, we may have $\lim_{n\to\infty} d_H(A_n, A) = 0$ and yet for all n, A_n and A may be very different topologically. For example, every member of 2^X is a limit of finite sets in the topology of d_H.

Metrics for 2^X that induce stronger topologies than that induced by d_H were introduced by Borsuk in [B1] and [B2]. The *fundamental metric* d_F defined in the latter paper was shown in [Ce-So] to have the following property: if $\lim_{n\to\infty} d_F(A_n, A) = 0$ and A is a calm compactum (see § 3 for the definition of calm) then $\mathrm{Sh}(A_n) \geqslant \mathrm{Sh}(A)$ for almost all n.

In this paper, we assume that X is a nonempty compactum. Our main results include the introduction of the *calm fundamental metric* d_{CF}, which induces on 2^X a topology stronger than that of d_F and has the following property: if $\lim_{n\to\infty} d_{CF}(A_n, A) = 0$ and A is calm, then $\mathrm{Sh}(A_n) = \mathrm{Sh}(A)$ for almost all n.

After submitting the first draft of this paper, the author received a preprint of [Ce2]. We show the notion of calmly regular convergence introduced there is essentially equivalent to convergence in the topology of d_{CF} and we answer a question raised in [Ce2].

We assume the reader is familiar with shape theory [B3] and the topology of the Hilbert cube [Ch].