Defining cardinal addition by $\leqslant$-formulas

by

Alexander F. Häussler * (Zürich)

Abstract. It is well-known that in Zermelo–Fraenkel set theory strengthened by the axiom of choice, cardinal addition of infinite sets can be expressed by the supremum which is defined by a first-order formula in the cardinal ordering $\leqslant$. The same holds in presence of the weaker axiom stating that every infinite set is idemmultiple, i.e. $\forall x(x<\omega \vee x = x+x)$. On the other hand, in Zermelo–Fraenkel set theory a definition of cardinal addition of infinite sets by a $\leqslant$-formula is not possible. We show that this is also impossible for the following two kinds of extensions: First, those extensions which are consistent with the existence of a Dedekind set ε, i.e. ε infinite and $\varepsilon \neq \varepsilon+1$, secondly, extensions which are consistent with the existence of an infinite unit A, i.e. A infinite, $A<A+A$, $\forall x, y(x+y = A \to x = A \vee y = A)$.

§ 1. In Zermelo–Fraenkel set theory ZF with the axiom of choice cardinal addition $x+y = z$ of infinite sets can be expressed by the supremum $\sup(x, y, z)$ which is defined by the $\leqslant$-formula $x, y \leqslant z \wedge \forall u(x, y \leqslant u \to z \leqslant u)$. In [2], p. 55 A. Tarski showed that the weaker axiom $\forall x(x<\omega \vee x = x+x)$ — meaning that every infinite set is idemmultiple — still implies within ZF the following equivalence for cardinal addition of infinite sets

$$\forall x, y, z\big(\mathrm{Inf}(x) \wedge \mathrm{Inf}(y) \wedge \mathrm{Inf}(z) \to (x+y = z \leftrightarrow \sup(x, y, z))\big),$$

where $\mathrm{Inf}(x)$ abbreviates $\neg x<\omega$.

In ZF itself — provided it is consistent — a definition by a (first-order) $\leqslant$-formula cannot be given. To show this, we assume that for some $\leqslant$-formula $\mathscr{A}(x, y, z)$ the following holds in ZF

$$\forall x, y, z\big(\mathrm{Inf}(x) \wedge \mathrm{Inf}(y) \wedge \mathrm{Inf}(z) \to (x+y = z \leftrightarrow \mathscr{A}(x, y, z))\big).$$

This leads to a contradiction as follows:

Consider the consistent extension of ZF in which the existence of a Dedekind set ε $(\mathrm{Inf}(\varepsilon) \wedge \varepsilon \neq \varepsilon+1)$ is postulated. Then the cardinals $\varepsilon+n$, $n \in \omega$, are definite and ordered like the integers. Let θ be the following function symbol ($\equiv$ is the set-theoretical equality)

$$\theta(x) \equiv \begin{cases} x+1, & \text{if } \exists n \in \omega(x = \varepsilon+n \vee x+n = \varepsilon), \\ x, & \text{otherwise} \end{cases}$$

* It is my sad duty to inform the reader that Alexander Häussler died of cancer on August 8, 1982. — H. Läuchli.

and note that $\theta(\varepsilon) = \varepsilon+1$, but $\theta(\varepsilon+\varepsilon) = \varepsilon+\varepsilon$, hence $\theta(\varepsilon)+\theta(\varepsilon) \neq \theta(\varepsilon+\varepsilon)$. One can check by examining several cases that θ is an $=$-automorphism of the universe, preserving the order $\leqslant$ and infinity, thus we have

$$\forall x, x'(x = x' \leftrightarrow \theta(x) = \theta(x')),$$
$$\forall y \exists x \theta(x) = y,$$
$$\forall x, x'(x \leqslant x' \leftrightarrow \theta(x) \leqslant \theta(x')),$$
$$\forall x(\mathrm{Inf}(x) \leftrightarrow \mathrm{Inf}(\theta(x))).$$

Induction on the complexity yields

$$\forall \underline{x}(\mathscr{A}(\underline{x}) \leftrightarrow \mathscr{A}(\theta(\underline{x})))$$

for every $\leqslant$-formula $\mathscr{A}(\underline{x})$.

By applying this to $\mathscr{A}(x, y, z)$ in the assumed equivalence for cardinal addition of infinite sets we thus obtain in $\mathrm{ZF}+\mathrm{Inf}(\varepsilon) \wedge \varepsilon \neq \varepsilon+1$ the following

$$\forall x, y, z\big(\mathrm{Inf}(x) \wedge \mathrm{Inf}(y) \wedge \mathrm{Inf}(z) \rightarrow (x+y = z \leftrightarrow \theta(x)+\theta(y) = \theta(z))\big),$$

hence for $x = \varepsilon$, $y = \varepsilon$ and $z = \varepsilon+\varepsilon$ we deduce $\theta(\varepsilon)+\theta(\varepsilon) = \theta(\varepsilon+\varepsilon)$ — a contradiction to the consistency of $\mathrm{ZF}+\mathrm{Inf}(\varepsilon) \wedge \varepsilon \neq \varepsilon+1$.

The same proof is valid for a slightly more general situation:

THEOREM 1. *In a theory which is compatible with* $\mathrm{ZF}+\exists x(\mathrm{Inf}(x) \wedge x \neq x+1)$ *no* $\leqslant$*-formula defines cardinal addition of infinite sets.*

COROLLARY 2. *Let theory T be an extension of ZF in which a $\leqslant$-formula defines cardinal addition of infinite sets. Then $\forall x(x<\omega \vee x = x+1)$ can be deduced from T.*

Proof of Corollary 2. Assume that $\forall x(x<\omega \vee x = x+1)$ is not provable in T. Then $T+\exists x(\mathrm{Inf}(x) \wedge x \neq x+1)$ is consistent and hence compactible with $\mathrm{ZF}+\exists x(\mathrm{Inf}(x) \wedge x \neq x+1)$. Thus by Theorem 1 no $\leqslant$-formula defines cardinal addition of infinite sets.

The question arises whether the axiom $\forall x(x<\omega \vee x = x+1)$ — meaning that every infinite set is transfinite — is sufficient for defining cardinal addition of infinite sets by a (first-order) $\leqslant$-formula. The answer for even much stronger axioms like DC, $\mathrm{DC}_{\aleph_1}$, $\forall\lambda \mathrm{AC}_\lambda$ or even $\mathrm{DC}_{\aleph_1} \wedge \forall\lambda \mathrm{AC}_\lambda$ is negative (see Corollary 5).

Thereby DC and $\mathrm{DC}_{\aleph_1}$ are axioms of dependent choices and $\forall\lambda \mathrm{AC}_\lambda$ means that any wellorderable set of nonempty elements has a choice function (see [1], p. 119). In all these cases we cannot refer to a Dedekind set ε as we did in Theorem 1. However, it is compatible with these axioms to assume that there exists a set A such that the sets $k\times A$, $1\leqslant k \in \omega$, have in a sense the same behaviour as the natural numbers (see Lemma 3 and Theorem 4).

DEFINITION. A is an Unit, if $A<A+A$ and $\forall x, y(x+y = A \rightarrow x = A \vee y = A)$.

LEMMA 3. *If $\mathscr{D}(x, z)$ is a $\leqslant$-formula, then there exists a numeral s with $1\leqslant s \in \omega$ such that in $\mathrm{ZF}+\mathrm{Unit}(A)$ the following holds:*

$$\mathscr{D}(s\times A, (s+s)\times A) \leftrightarrow \mathscr{D}(s\times A, (s+s+1)\times A).$$

The proof of this lemma is given in §§ 2–5. We sketch the proof in § 2, in §§ 3 and 4 we work out the theory $\mathrm{ZF}+\mathrm{Unit}(A)$ and finally in § 5 we prove the lemma. Let us nevertheless give the conclusions now:

THEOREM 4. *In a theory T which is compatible with $\mathrm{ZF}+\exists x(\mathrm{Inf}(x) \wedge \mathrm{Unit}(x))$ no $\leqslant$-formula defines cardinal addition of infinite sets.*

Proof of Theorem 4. Let us assume that the $\leqslant$-formula $\mathscr{A}(x, y, z)$ defines in T cardinal addition of infinite sets. Since T has the compatibility property, we can assume the existence of a set A such that $T+\mathrm{ZF}+\mathrm{Inf}(A) \wedge \mathrm{Unit}(A)$ is consistent; clearly in this extension of T also the following holds:

$$\forall x, y, z\big(\mathrm{Inf}(x) \wedge \mathrm{Inf}(y) \wedge \mathrm{Inf}(z) \rightarrow (x+y = z \leftrightarrow \mathscr{A}(x, y, z))\big).$$

By applying Lemma 3 on the $\leqslant$-formula $\mathscr{A}(x, x, z)$ we obtain a numeral s with $1\leqslant s \in \omega$ such that

$$\mathscr{A}(s\times A, s\times A, (s+s)\times A) \leftrightarrow \mathscr{A}(s\times A, s\times A, (s+s+1)\times A)$$

holds in $\mathrm{ZF}+\mathrm{Unit}(A)$.

As A is assumed to be infinite, $s+A$, $(s+s)\times A$ and $(s+s+1)\times A$ are infinite too; furthermore $s\times A+s\times A = (s+s)\times A$ holds. Hence by the assumed equivalence for addition and the result from Lemma 3 we obtain $s\times A+s\times A = (s+s+1)\times A$. Thus $A = A+A$ (see § 3) which contradicts the assumed compatibility.

Remark. The same argument holds if we replace "infinite" by "finite". As 1 is a Unit, in any theory compatible with ZF (for instance every consistent extension of ZF), no $\leqslant$-formula defines cardinal addition of finite sets.

COROLLARY 5. *If ZF is consistent, no $\leqslant$-formula defines cardinal addition of infinite sets in the theory $\mathrm{ZF}+\mathrm{DC}_{\aleph_1}+\forall\lambda \mathrm{AC}_\lambda$.*

Proof. By Theorem 4 it suffices to show that this theory is compatible with $\mathrm{ZF}+\exists x(\mathrm{Inf}(x) \wedge \mathrm{Unit}(x))$. In [1] Theorem 8.9 (p. 127) let $\alpha \equiv 1$: Then $\mathrm{DC}_{\aleph_1}$ and $\forall\lambda \mathrm{AC}_\lambda$ hold in the Fraenkel–Mostowski permutation model given there. Furthermore it is easy to see that the set of atoms is a Unit in the permutation model.

Using a refinement of the embedding theorem, this relative consistency result by means of a permutation model is transferred into ZF; hence

$$\mathrm{ZF}+\mathrm{DC}_{\aleph_1}+\forall\lambda \mathrm{AC}_\lambda+\exists x(\mathrm{Inf}(x) \wedge \mathrm{Unit}(x))$$

is relative consistent to ZF.

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§ 2. Sketch of proof of Lemma 3. We work in ZF and furthermore assume $\mathrm{Unit}(A)$ for a set A. $A<A+A$ implies $k\times A<(k+1)\times A$ for $k \in \omega$; thus the class $\mathscr{K}$ of all $k\times A$ with $1\leqslant k \in \omega$ (including the cardinal equivalent sets) obviously has the same ordering as $\mathscr{N} \equiv \{k|\ 1\leqslant k \in \omega\}$. This isomorphism of the ordering $\leqslant$ can be transferred to $\leqslant$-formulas, provided that the quantifiers refer only to the classes $\mathscr{K}$

and $\mathcal{N}$ respectively. Hence, by the well-known theorem on natural numbers, Lemma 3 holds for $\mathcal{K}$-relativized $\leqslant$-formulas.

In order to treat quantifiers which refer to the whole universe, we investigate the ordering between elements of $\mathcal{K}$ and those of the universe. To do this, we introduce the notion of multiplicity $a(x)$ which intuitively counts the number of pairwise disjoint copies of A which can be embedded in a set x (§ 3). Then a set x has multiplicity k $(1 \leqslant k \in \omega)$ iff x is a sum $k \times A + y$ with $A \not\leqslant y$. In this decomposition the second summand y is not unique, however, $A+y$ is definite up to cardinal equivalence. Let $\mathcal{R}$ be the class of all sums $A+y$ with $A \not\leqslant y$. On the class $\mathcal{G}$ of all $k \times A + y$ $(1 \leqslant k \in \omega, A \not\leqslant y)$ we introduce the two projections $\varkappa$ onto $\mathcal{K}$ satisfying $\varkappa(k \times A + y) = k \times A$ and ϱ onto $\mathcal{R}$ satisfying $\varrho(k \times A + y) = A + y$.

It can be shown that $x_1 \leqslant x_2 \leftrightarrow \varkappa(x_1) \leqslant \varkappa(x_2) \wedge \varrho(x_1) \leqslant \varrho(x_2)$ holds for x_1, x_2 in $\mathcal{G}$. In order to compare x' in $\mathcal{G}$ to x'' in $\neg\mathcal{G}$, it suffices to compare $\varrho(x')$ to x''. Introducing $\mathcal{M}$ by $\mathcal{R} \vee \neg\mathcal{G}$ we see (§ 4), that a predicate given by a $\leqslant$-formula $\mathcal{A}(\underline{x}', \underline{x}'')$ is equivalent to a propositional combination of $\mathcal{K}$-relativized $\leqslant$-formulas in $\varkappa(x_1'), \ldots, \varkappa(x_{n'}')$ and $\mathcal{M}$-relativized $\leqslant$-formulas in $\varrho(x_1'), \ldots, \varrho(x_{n'}')$, $x_1'', \ldots, x_{n''}''$ — provided that all $x_1', \ldots, x_{n'}'$ are in $\mathcal{G}$ and $x_1'', \ldots, x_{n''}''$ in $\neg\mathcal{G}$. Assuming all $x_1, x_2, \ldots, x_n$ to be in $\mathcal{K}$ (hence none in $\neg\mathcal{G}$), then $\varkappa(x_i) = x_i$, $\varrho(x_i) = A$ and thus $\mathcal{A}(\underline{x})$ is equivalent to a propositional combination of $\mathcal{K}$-relativized $\leqslant$-formulas in $\underline{x}$ and some $\leqslant$-sentences in A. This is what we need for the proof of Lemma 3 in § 5, as mentioned at the beginning.

§ 3. **Unit and multiplicity.** Working in ZF+Unit(A) we may assume that there exists a set A with the following two properties:

(3.1) $$A < A + A,$$

(3.2) $$\forall x, y(x+y = A \rightarrow x = A \vee y = A).$$

Then the following holds for A:

(3.3) $$\forall x(A+x = A \rightarrow x < A),$$

(3.4) $$\forall x(x < A \rightarrow A + x = A).$$

Furthermore (3.3) is equivalent to (3.1) and assuming (3.1) or (3.3) respectively, then (3.2) and (3.4) are equivalent too.

Proofs. (3.1) ⇒ (3.3). Let $A+x = A$ for some x. Then $x \leqslant A$, hence $x < A$ because $x = A$ is excluded by (3.1). (3.3) ⇒ (3.1): With $x = A$ in (3.3) the assumption $A = A+A$ yields the contradiction $A < A$, hence $A < A+A$. (3.2) ⇒ (3.4): Let $x < A$ for some x; then there exists y with $x+y = A$ and $y = A$ by (3.2), thus $x+A = A$. (3.4) ∧ (3.1) ⇒ (3.2): Let $x+y = A$, but $x, y < A$. By (3.4) $A+(x+y) = A$, hence $A+A = A$ in contradiction to (3.1).

Before continuing, let us note a theorem on cardinal algebra provable in ZF:

(3.5) $$k \in \omega \wedge k \times u + v \leqslant k \times u + w \rightarrow u + v \leqslant u + w.$$

For a proof refer to Corollary 4 in [3], p. 81.

From Unit(A) we deduce

(3.6) $$0 < A,$$

(3.7) $$k \in \omega \rightarrow k \times A < (k+1) \times A,$$

(3.8) $$k \in \omega \wedge (k+1) \times A \leqslant k \times A + y \rightarrow A \leqslant y,$$

(3.9) $$k \in \omega \wedge x \leqslant k \times A \rightarrow A \leqslant x \vee A + x = A,$$

(3.10) $$k \in \omega \wedge x \leqslant k \times A + y \rightarrow A \leqslant x \vee A + x \leqslant A + y.$$

Proofs. (3.6) and (3.7) follow by (3.1) and (3.5).

(3.8): Let $(k+1) \times A \leqslant k \times A + y$, then by applying (3.5) we obtain $A + A \leqslant A + y$. Hence there exist a', a'', y', y'' with $A = a' + y' = a'' + y''$, $a' + a'' \leqslant A$ and $y' + y'' \leqslant y$. By (3.2) $a' = A \vee y' = A$ and $a'' = A \vee y'' = A$. In the case $A = a' = a''$, we obtain $A + A = a' + a'' \leqslant A$ — a contradiction to (3.1). In all other cases $A \leqslant y' + y'' \leqslant y$.

(3.9): By induction on $k \in \omega$: For $k \equiv 0$ it holds trivially. Let $x \leqslant (k+1) \times A$, then there exist x', x'' with $x = x' + x''$, $x' \leqslant A$, $x'' \leqslant k \times A$, thus $x' < A$ or $x' = A$ and by induction hypothesis $A \leqslant x''$ or $A + x'' = A$. In the case of $x' < A$ and $A + x'' = A$, 3.4 yields $A + x = A + x' + x'' = A$, in all other cases $A \leqslant x' + x'' = x$.

(3.10): Let $x \leqslant k \times A + y$, then there exist x', x'' with $x = x' + x''$, $x' \leqslant k \times A$, $x'' \leqslant y$. By 3.9 it is $A \leqslant x'$ or $A + x' = A$. In the first case $A \leqslant x' \leqslant x$, in the latter $A + x = A + x' + x'' = A + x'' \leqslant A + y$.

We introduce a function symbol assigning to a set x the maximal number $a(x)$ of pairwise disjoint copies of A which can be embedded in x. This notion of multiplicity is helpful for investigating the ordering $\leqslant$.

Let a be defined formally by the following:

(3.11) $$a(x) \equiv \{n \mid n \in \omega \wedge (n+1) \times A \leqslant x\}$$

$a(x)$ is an initial segment of ω, hence $a(x) \in \omega$ or $a(x) \equiv \omega$. Furthermore multiplicity is monotone, thus the following three propositions hold and are trivial:

(3.12) $$a(x) \in \omega^+, \quad \text{where} \quad \omega^+ \equiv \omega \cup \{\omega\},$$

(3.13) $$x_1 \leqslant x_2 \rightarrow a(x_1) \leqslant a(x_2),$$

(3.14) $$x_1 = x_2 \rightarrow a(x_1) \equiv a(x_2).$$

If $a(x)$ is finite then, as intended, the following holds:

(3.15) $$k \in \omega \rightarrow (a(x) \equiv k \leftrightarrow k \times A \leqslant x \wedge (k+1) \times A \not\leqslant x).$$

Proof. Let $k \in \omega$. The case $k \equiv 0$ is obvious, because $A \not\leqslant x$ iff $a(x)$ is empty $(a(x) \equiv 0)$. Let $1 \leqslant k \in \omega$: If $a(x) \equiv k$, then $k-1 \in a(x)$, hence by (3.11) $k \times A \leqslant x$; the assumption $(k+1) \times A \leqslant x$ however leads to the contradiction $k \in k$. For the other implication assume that $k \times A \leqslant x$, but $(k+1) \times A \not\leqslant x$; (3.11) yields $k-1 \in a(k)$, but $k \notin a(x)$; hence $a(x) \equiv k$ follows.

Remark. In the case $k \equiv \omega$ we have only the equivalence between $a(x) \equiv \omega$ and $\forall k \in \omega\ k \times A \leqslant x$. In general this does not imply $\omega \times A \leqslant x$.

The function $a(x)$ decomposes the universe in ω^+ many classes. In the following we give, in some sense, a more explicit characterization of sets with finite multiplicity and the ordering $\leqslant$ between such sets. For this purpose we introduce the monadic predicate $\mathscr{W}$ in (3.16). By (3.15) the proposition (3.17) is then obvious.

(3.16) $\qquad \mathscr{W}(x) \leftrightarrow a(x) \equiv 0\,,$

(3.17) $\qquad \mathscr{W}(x) \leftrightarrow A \nleqslant x\,.$

Now the following statements hold:

(3.18) $\quad k \in \omega \to \big(a(x) \equiv k \leftrightarrow \exists y\big(\mathscr{W}(y) \wedge x = k\times A+y\big)\big)\,,$

(3.19) $\quad k \in \omega \to a(k\times A) \equiv k\,,$

(3.20) $\quad 1 \leqslant k_1, k_2 \in \omega \wedge \mathscr{W}(y_1) \wedge \mathscr{W}(y_2)$
$\qquad \to (k_1\times A+y_1 \leqslant k_2\times A+y_2 \leftrightarrow k_1 \subseteq k_2 \wedge A+y_1 \leqslant A+y_2)\,.$

Remark. It cannot be expected that (3.20) holds with $k_1 \subseteq k_2 \wedge y_1 \leqslant y_2$.

Proofs. (3.18): Let $k \in \omega$. If $a(x) \equiv k$, then by (3.15) there exists y such that $x = k\times A+y$ and $(k+1)\times A \nleqslant x$ hold, hence $A \nleqslant y$ follows. If, on the other hand, $x = k\times A+y$ for some y with $A \nleqslant y$, we have $k\times A \leqslant x$. The assumption $(k+1)\times A \leqslant x = k\times A+y$, however, yields the contradiction $A \leqslant y$ by (3.8). Hence $a(x) \equiv k$ by (3.15).

(3.19): Let $y = 0$ in (3.18), then $A \nleqslant y$ by (3.6), hence $a(k\times A) \equiv k$.

(3.20): Assume $1 \leqslant k_1, k_2 \in \omega$, $\mathscr{W}(y_1)$, $\mathscr{W}(y_2)$. If $k_1 \subseteq k_2$ and $A+y_1 \leqslant A+y_2$, then obviously $k_1\times A+y_1 \leqslant k_2\times A+y_2$. If, on the other hand, $k_1\times A+y_1 \leqslant k_2\times A+y_2$, then $k_1 \subseteq k_2$ by (3.13) and (3.18). Furthermore we have $y_1 \leqslant k_2\times A+y_2$, hence $A \leqslant y_1$ or $A+y_1 \leqslant A+y_2$ by (3.10). Because of $\mathscr{W}(y_1)$ the second case holds.

Let us introduce the monadic predicates $\mathscr{K}$, $\mathscr{G}$ and $\mathscr{R}$ in (3.21) to (3.23). In (3.24) and (3.25) equivalent forms, obvious by (3.18), are added:

(3.21) $\quad \mathscr{K}(x) \leftrightarrow \exists k(1 \leqslant k \in \omega \wedge x = k\times A)\,,$

(3.22) $\quad \mathscr{G}(x) \leftrightarrow 1 \leqslant a(x) \in \omega\,,$

(3.23) $\quad \mathscr{R}(x) \leftrightarrow a(x) \equiv 1\,,$

(3.24) $\quad \mathscr{G}(x) \leftrightarrow \exists k \exists y(1 \leqslant k \in \omega \wedge \mathscr{W}(y) \wedge x = k\times A+y)\,,$

(3.25) $\quad \mathscr{R}(x) \leftrightarrow \exists y\big(\mathscr{W}(y) \wedge x = A+y\big)\,.$

The elements of $\mathscr{G}$ are sums. By (3.20) not the summands $k\times A$ and y but $k\times A$ and $A+y$ are unique up to cardinal equivalence. This allows us to introduce the following two cardinal function symbols $\varkappa$ and ϱ on the class $\mathscr{G}$, i.e. function symbols with respect to cardinal equality only:

(3.26) If $\mathscr{G}(x)$ — thus $x = k\times A+y$ for some k, y with $1 \leqslant k \in \omega$ and $\mathscr{W}(y)$ — then let $\varkappa(x) = k\times A$ and $\varrho(x) = A+y$.

The following list of propositions formally expresses that the class $\mathscr{G}$ with the ordering $\leqslant$ is the "product" of the subclasses $\mathscr{K}$ and $\mathscr{R}$ with projections $\varkappa$ and ϱ.

The proofs are easy applications of previously shown propositions which we leave to the reader:

(3.27) $\quad \mathscr{K}(x) \to \mathscr{G}(x)\,,$

(3.28) $\quad \mathscr{R}(x) \to \mathscr{G}(x)\,,$

(3.29) $\quad \mathscr{G}(x) \to \mathscr{K}\big(\varkappa(x)\big) \wedge \mathscr{R}\big(\varrho(x)\big)\,,$

(3.30) $\quad \mathscr{G}(z) \to \exists u \exists v\big(\mathscr{K}(u) \wedge \mathscr{R}(v) \wedge \varkappa(z) = u \wedge \varrho(z) = v\big)\,,$

(3.31) $\quad \mathscr{K}(u) \wedge \mathscr{R}(v) \to \exists z\big(\mathscr{G}(z) \wedge \varkappa(z) = u \wedge \varrho(z) = v\big)\,,$

(3.32) $\quad \mathscr{G}(x_1) \wedge \mathscr{G}(x_2) \to \big(x_1 \leqslant x_2 \leftrightarrow \varkappa(x_1) \leqslant \varkappa(x_2) \wedge \varrho(x_1) \leqslant \varrho(x_2)\big)\,,$

(3.33) $\quad \mathscr{K}(x) \to \varkappa(x) = x \wedge \varrho(x) = A\,.$

Having established this "product" property of $\mathscr{G}$, we consider now the ordering between elements of $\mathscr{G}$ and its complementary class $\neg\mathscr{G}$. Of an element in $\mathscr{G}$ only its ϱ-projection is involved in this. By (3.12) and (3.22) the following holds for $\mathscr{G}$:

(3.34) $\quad \neg\mathscr{G}(x) \leftrightarrow a(x) \equiv o \vee a(x) \equiv \omega\,,$

(3.35) $\quad \mathscr{G}(x_1) \wedge \neg\mathscr{G}(x_2) \to \big(x_1 \leqslant x_2 \leftrightarrow \varrho(x_1) \leqslant x_2\big)\,,$

(3.36) $\quad \mathscr{G}(x_2) \wedge \neg\mathscr{G}(x_1) \to \big(x_1 \leqslant x_2 \leftrightarrow x_1 \leqslant \varrho(x_2)\big)\,.$

Proofs. (3.35): Let $\mathscr{G}(x_1)$, $\neg\mathscr{G}(x_2)$, hence $x_1 = k_1\times A+y_1$ for some k_1, y_1 with $1 \leqslant k_1 \in \omega$ and $\mathscr{W}(y_1)$. Furthermore $\varrho(x_1) = A+y_1$ by (3.26). Assuming $x_1 \leqslant x_2$, we get $\varrho(x_1) = A+y_1 \leqslant k_1\times A+y_1 = x_1 \leqslant x_2$ because $1 \leqslant k_1$. On the other hand, let $\varrho(x_1) \leqslant x_2$, hence $A \leqslant x_2$, thus $a(x_2) \equiv \omega$ by $\neg\mathscr{G}(x_2)$ and (3.34). By (3.11) $k_1\times A \leqslant x_2$, thus there exists a set u with $k_1\times A+u = x_2$. Furthermore, we have $y_1 \leqslant A+y_1 = \varrho(x_1) \leqslant x_2 = k_1\times A+u$ and by applying (3.10) we obtain $A \leqslant y_1$ or $A+y_1 \leqslant A+u$. The first case contradicts $\mathscr{W}(y_1)$, hence by the latter $x_1 = k_1\times A+{}$ $+y_1 \leqslant k_1\times A+u = x_2$, because $1 \leqslant k_1$ holds.

(3.36): Let $\mathscr{G}(x_2)$, $\neg\mathscr{G}(x_1)$, hence $x_2 = k_2\times A+y_2$ for some k_2, y_2 with $1 \leqslant k_2 \in \omega$ and $\mathscr{W}(y_2)$. Furthermore $\varrho(x_2) = A+y_2$. Assuming $x_1 \leqslant x_2$, we obtain $a(x_1) \subseteq k_2 \in \omega$ by (3.13) and (3.18). Hence $a(x_1) \equiv 0$ by $\neg\mathscr{G}(x_1)$ and (3.34). By (3.10) $x_1 \leqslant x_2 = k_2\times A+y_2$ yields $A \leqslant x_1$ or $A+x_1 \leqslant A+y_2$; but $A \leqslant x_1$ is excluded by $a(x_1) \equiv 0$, thus the latter holds and we obtain $x_1 \leqslant \varrho(x_2)$. On the other hand, if $x_1 \leqslant \varrho(x_2)$, we have $x_1 \leqslant \varrho(x_2) = A+y_2 \leqslant k_2\times A+y_2 = x_2$ by $1 \leqslant k_2$.

Finally we introduce in (3.37) the class $\mathscr{M}$, priviously mentioned in § 2, and note two properties obvious by (3.12) and the definitions (3.22), (3.23) of $\mathscr{G}$ and $\mathscr{R}$.

(3.37) $\quad \mathscr{M}(x) \leftrightarrow a(x) \equiv 0 \vee a(x) \equiv 1 \vee a(x) \equiv \omega\,,$

(3.38) $\quad \mathscr{G}(x) \vee \mathscr{M}(x)\,,$

(3.39) $\quad \mathscr{G}(x) \wedge \mathscr{M}(x) \leftrightarrow \mathscr{R}(x)\,.$

§ 4. Analysing $\leqslant$-formulas. In (3.32), (3.35) and (3.36) we gave, for the atomic formula $x_1 \leqslant x_2$, equivalent formulas depending on whether x_1, x_2 are in $\mathscr{G}$ or $\neg\mathscr{G}$. In the following we generalize this to $\leqslant$-formulas.

$L(\leqslant, \mathscr{P}_1, \ldots, \mathscr{P}_s)$ denotes the first order language built by means of the logical connectives $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$ starting with the primitive symbols $\leqslant$ and the predicates $\mathscr{P}_1, \ldots, \mathscr{P}_s$. If $\mathscr{A}(\underline{x})$ is a formula of $L(\leqslant, \mathscr{P}_1, \ldots, \mathscr{P}_s)$ and $\mathscr{P}$ any monadic predicate, we write $[\mathscr{A}(\underline{x})]^{\mathscr{P}}$ for the relativization of the formula $\mathscr{A}(\underline{x})$ to the class $\mathscr{P}$. $[\mathscr{A}(\underline{x})]^{\mathscr{P}}$ is recursively defined on the complexity of $\mathscr{A}(\underline{x})$ as follows:

$$[x \leqslant y]^{\mathscr{P}} \text{ is } x \leqslant y,\ [\mathscr{P}_r(\underline{x})]^{\mathscr{P}} \text{ is } \mathscr{P}_r(\underline{x}) \text{ for } r = 1, 2, \ldots, s,$$
$$[\mathscr{B}(\underline{x}) \wedge \mathscr{C}(\underline{x})]^{\mathscr{P}} \text{ is } [\mathscr{B}(\underline{x})]^{\mathscr{P}} \wedge [\mathscr{C}(\underline{x})]^{\mathscr{P}},$$
$$[\mathscr{B}(\underline{x}) \vee \mathscr{C}(\underline{x})]^{\mathscr{P}} \text{ is } [\mathscr{B}(\underline{x})]^{\mathscr{P}} \vee [\mathscr{C}(\underline{x})]^{\mathscr{P}},$$
$$[\neg \mathscr{B}(\underline{x})]^{\mathscr{P}} \text{ is } \neg [\mathscr{B}(\underline{x})]^{\mathscr{P}},$$
$$[\exists z \mathscr{B}(\underline{x}, z)]^{\mathscr{P}} \text{ is } \exists z \big(\mathscr{P}(z) \wedge [\mathscr{B}(\underline{x}, z)]^{\mathscr{P}}\big),$$
$$[\forall z \mathscr{B}(\underline{x}, z)]^{\mathscr{P}} \text{ is } \forall z \big(\neg \mathscr{P}(z) \vee [\mathscr{B}(\underline{x}, z)]^{\mathscr{P}}\big).$$

Thereby we write $\mathscr{A}(x_1, x_2, \ldots, x_n)$ — shorter $\mathscr{A}(\underline{x})$, to indicate that the free variables in the formula $\mathscr{A}$ are among $\{x_1, x_2, \ldots, x_n\}$ — shorter $\{\underline{x}\}$. Furthermore we abbreviate $\mathscr{P}(x_1) \wedge \mathscr{P}(x_2) \wedge \ldots \wedge \mathscr{P}(x_n)$ by $\underline{\mathscr{P}}(\underline{x})$ and $\neg \mathscr{P}(x_1) \wedge \neg \mathscr{P}(x_2) \wedge \ldots \wedge \neg \mathscr{P}(x_n)$ by $\underline{\neg \mathscr{P}}(\underline{x})$.

PROPOSITION 4.1. *Let $\mathscr{A}(x)$ be a formula of $L(\leqslant)$ with all its free variables among $\{x_1, \ldots, x_n\}$. For every decomposition $(\{\underline{x}'\}, \{\underline{x}''\})$ of the set $\{\underline{x}\}$ of variables, there exist formulas $\mathscr{A}_i'(\underline{x}')$ of $L(\leqslant)$ and $\mathscr{A}_i''(\underline{x}', \underline{x}'')$ of $L(\leqslant, \mathscr{R})$ $(i = 1, 2, \ldots, m)$ such that*

$$\underline{\mathscr{G}}(\underline{x}') \wedge \underline{\neg \mathscr{G}}(\underline{x}'') \to \Big(\mathscr{A}(\underline{x}) \leftrightarrow \bigvee_{i=1}^{m} \big([\mathscr{A}_i'(\varkappa(\underline{x}'))]^{\mathscr{K}} \wedge [\mathscr{A}_i''(\varrho(\underline{x}'), \underline{x}'')]^{\mathscr{M}}\big)\Big).$$

In the following discussion this type of disjunction will be referred to as the "normal form" of $\mathscr{A}(\underline{x})$.

The proof is by induction on the complexity of $\mathscr{A}(\underline{x})$. It is sufficient to consider the logical connectives $\vee$, $\neg$, $\exists$. Let $\mathscr{A}(\underline{x})$ be an atomic formula, hence $x_1 \leqslant x_2$. There are four possible decompositions of the variables, namely $(\{x_1, x_2\}, \Lambda)$, $(\{x_1\}, \{x_2\})$, $(\{x_2\}, \{x_1\})$ and $(\Lambda, \{x_1, x_2\})$. By (3.32), (3.35) and (3.36) and the definition of relativation, the following holds:

$$\mathscr{G}(x_1) \wedge \mathscr{G}(x_2) \to \big(x_1 \leqslant x_2 \leftrightarrow [\varkappa(x_1) \leqslant \varkappa(x_2)]^{\mathscr{K}} \wedge [\varrho(x_1) \leqslant \varrho(x_2)]^{\mathscr{M}}\big),$$
$$\mathscr{G}(x_1) \wedge \neg \mathscr{G}(x_2) \to \big(x_1 \leqslant x_2 \leftrightarrow [\varrho(x_1) \leqslant x_2]^{\mathscr{M}}\big),$$
$$\mathscr{G}(x_2) \wedge \neg \mathscr{G}(x_1) \to \big(x_1 \leqslant x_2 \leftrightarrow [x_1 \leqslant \varrho(x_2)]^{\mathscr{M}}\big),$$
$$\neg \mathscr{G}(x_1) \wedge \neg \mathscr{G}(x_2) \to \big(x_1 \leqslant x_2 \leftrightarrow [x_1 \leqslant x_2]^{\mathscr{M}}\big).$$

Let $\mathscr{A}(\underline{x})$ be the disjunction $\mathscr{B}(\underline{x}) \vee \mathscr{C}(\underline{x})$ and $(\{\underline{x}'\}, \{\underline{x}''\})$ a decomposition of $\{\underline{x}\}$. Then by induction hypothesis there exists a normal form for $\mathscr{B}(\underline{x})$ ($\mathscr{C}(\underline{x})$ respectively). Assuming $\underline{\mathscr{G}}(\underline{x}') \wedge \underline{\neg \mathscr{G}}(\underline{x}'')$, it is equivalent to $\mathscr{B}(\underline{x})$ ($\mathscr{C}(\underline{x})$ respectively). The disjunction of these two normal forms is a normal form for $\mathscr{A}(\underline{x})$.

Let $\mathscr{A}(\underline{x})$ be $\neg \mathscr{B}(\underline{x})$ and $(\{\underline{x}'\}, \{\underline{x}''\})$ a decomposition of $\{\underline{x}\}$. By induction hypothesis there is a normal form for $\mathscr{B}(\underline{x})$, therefore — provided $\underline{\mathscr{G}}(\underline{x}') \wedge \underline{\neg \mathscr{G}}(\underline{x}'')$ — the negation of this normal form is equivalent to $\mathscr{A}(\underline{x})$, thus

$$\mathscr{A}(\underline{x}) \leftrightarrow \neg \bigvee_{i=1}^{m} \big(\{\mathscr{B}_i'(\varkappa(\underline{x}'))]^{\mathscr{K}} \wedge [\mathscr{B}_i''(\varrho(\underline{x}'), \underline{x}'')]^{\mathscr{M}}\big).$$

By referring to propositional calculus and the definition of relativation we obtain the following normal form for $\mathscr{A}(\underline{x})$:

$$\bigvee_{I \subseteq \{1, \ldots, m\}} \big([\bigwedge_{i \in I} \neg \mathscr{B}_i'(\varkappa(\underline{x}'))]^{\mathscr{K}} \wedge [\bigwedge_{i \notin I} \neg \mathscr{B}_i''(\varrho(\underline{x}'), \underline{x}'')]^{\mathscr{M}}\big).$$

Let $\mathscr{A}(\underline{x})$ be $\exists z \mathscr{B}(\underline{x}, z)$ and $(\{\underline{x}'\}, \{\underline{x}''\})$ a decomposition of $\{\underline{x}\}$. $\mathscr{A}(\underline{x})$ is then equivalent to $\exists z(\mathscr{G}(z) \wedge \mathscr{B}(\underline{x}, z)) \vee \exists z(\neg \mathscr{G}(z) \wedge \mathscr{B}(\underline{x}, z))$. It is sufficient to find a normal form for each of these two disjuncts:

By induction hypothesis there exists for the decomposition $(\{\underline{x}'\} \cup \{z\}, \{\underline{x}''\})$ a normal form for $\mathscr{B}(\underline{x}, z)$, hence

$$\mathscr{B}(\underline{x}, z) \leftrightarrow \bigvee_{j} \big([\mathscr{B}_j'(\varkappa(\underline{x}'), \varkappa(z))]^{\mathscr{K}} \wedge [\mathscr{B}_j''(\varrho(\underline{x}'), \varrho(z), \underline{x}'')]^{\mathscr{M}}\big)$$

provided $\underline{\mathscr{G}}(\underline{x}') \wedge \mathscr{G}(z) \wedge \underline{\neg \mathscr{G}}(\underline{x}'')$. Predicate calculus shows that $\exists z(\mathscr{G}(z) \wedge \mathscr{B}(\underline{x}, z))$ is equivalent to

$$\bigvee_{j} \exists z \big(\mathscr{G}(z) \wedge [\mathscr{B}_j'(\varkappa(\underline{x}'), \varkappa(z))]^{\mathscr{K}} \wedge [\mathscr{B}_j''(\varrho(\underline{x}'), \varrho(z), \underline{x}'')]^{\mathscr{M}}\big)$$

provided $\underline{\mathscr{G}}(\underline{x}') \wedge \underline{\neg \mathscr{G}}(\underline{x}'')$. By (3.30) each disjunct of $\bigvee_j$ is equivalent to

$$\exists z \exists u \exists v \big(\mathscr{G}(z) \wedge \mathscr{K}(u) \wedge \mathscr{R}(v) \wedge \varkappa(z) = u \wedge \varrho(z) = v \wedge$$
$$\wedge [\mathscr{B}_j'(\varkappa(\underline{x}'), \varkappa(z))]^{\mathscr{K}} \wedge [\mathscr{B}_j''(\varrho(\underline{x}'), \varrho(z), \underline{x}'')]^{\mathscr{M}}\big).$$

With $\varkappa(z) = u$, $\varrho(z) = v$ and using the fact that the cardinal equivalence is an equality for all occurring predicates, we obtain:

$$\exists z \exists u \exists v \big(\mathscr{G}(z) \wedge \mathscr{K}(u) \wedge \mathscr{R}(v) \wedge \varkappa(z) = u \wedge \varrho(z) = v \wedge$$
$$\wedge [\mathscr{B}_j'(\varkappa(\underline{x}'), u)]^{\mathscr{K}} \wedge [\mathscr{B}_j''(\varrho(\underline{x}'), v, \underline{x}'')]^{\mathscr{M}}\big).$$

By (3.31) each disjunct of $\bigvee_j$ is consequently equivalent to

$$\exists u \exists v \big(\mathscr{K}(u) \wedge \mathscr{R}(v) \wedge [\mathscr{B}_j'(\varkappa(\underline{x}'), u)]^{\mathscr{K}} \wedge [\mathscr{B}_j''(\varrho(\underline{x}'), v, \underline{x}'')]^{\mathscr{M}}\big);$$

together with $\mathscr{R}(v) \leftrightarrow \mathscr{R}(v) \wedge \mathscr{M}(v)$ (3.39) and the definition of relativation, this finally yields

$$[\exists u \mathscr{B}_j'(\varkappa(\underline{x}'), u)]^{\mathscr{K}} \wedge [\exists v \big(\mathscr{R}(v) \wedge \mathscr{B}_j''(\varrho(\underline{x}'), v, \underline{x}'')\big)]^{\mathscr{M}}.$$

Hence we have a normal form for $\exists z(\mathscr{G}(z) \wedge \mathscr{B}(\underline{x}, z))$. For the disjunct $\exists z(\neg \mathscr{G}(z) \wedge \mathscr{B}(\underline{x}, z))$ we use induction with the decomposition $(\{\underline{x}'\}, \{\underline{x}''\} \cup \{z\})$, hence — provided $\underline{\mathscr{G}}(\underline{x}') \wedge \underline{\neg \mathscr{G}}(\underline{x}'') \wedge \neg \mathscr{G}(z)$ — we have

$$\mathscr{B}(\underline{x}, z) \leftrightarrow \bigvee_{h} \big([\overline{\mathscr{B}}_h'(\varkappa(\underline{x}'))]^{\mathscr{K}} \wedge [\overline{\mathscr{B}}_h''(\varrho(\underline{x}'), \underline{x}'', z)]^{\mathscr{M}}\big).$$

Provided $\underline{\mathscr{G}}(\underline{x}') \wedge \underline{\neg \mathscr{G}}(\underline{x}'')$, it can be shown by predicate calculus that

$$\exists z(\neg \mathscr{G}(z) \wedge \mathscr{B}(\underline{x}, z))$$

is equivalent to

$$\bigvee_h \left([\mathscr{B}_h'(\varkappa(\underline{x}'))]^{\mathscr{K}} \wedge \exists z(\neg \mathscr{G}(z) \wedge [\mathscr{B}_h''(\varrho(\underline{x}'), \underline{x}'', z)]^{\mathscr{M}})\right).$$

With $\neg \mathscr{G}(z) \leftrightarrow \neg \mathscr{R}(z) \wedge \mathscr{M}(z)$ ((3.38), (3.39)) and the definition of relativation, the normal form

$$\bigvee_h \left([\mathscr{B}_h'(\varkappa(\underline{x}'))]^{\mathscr{K}} \wedge [\exists z(\neg \mathscr{R}(z) \wedge \overline{\mathscr{B}}_h''(\varrho(\underline{x}'), \underline{x}'', z))]^{\mathscr{M}}\right)$$

can finally be obtained.

Corollary 4.2 is obtained by specifying all free variables to be in $\mathscr{G}$:

COROLLARY 4.2. *Let $\mathscr{A}(\underline{x})$ be a formula of $L(\leqslant)$ with all its free variables among $\{\underline{x}\}$. Then there exist formulas $\mathscr{A}_i'(\underline{x})$ of $L(\leqslant)$ and $\mathscr{A}_i''(\underline{x})$ of $L(\leqslant, \mathscr{R})$ $(i = 1, 2, \ldots, m)$ such that*

$$\underline{\mathscr{G}}(\underline{x}) \rightarrow \left(\mathscr{A}(\underline{x}) \leftrightarrow \bigvee_i \left([\mathscr{A}_i'(\varkappa(\underline{x}))]^{\mathscr{K}} \wedge [\mathscr{A}_i''(\varrho(\underline{x}))]^{\mathscr{M}}\right)\right).$$

Again by specifying all free variables to be in $\mathscr{K}$ and (3.27), (3.33) the following holds:

COROLLARY 4.3. *Let $\mathscr{A}(\underline{x})$ be a formula of $L(\leqslant)$ with all its free variables among $\{\underline{x}\}$. Then there exist formulas $\mathscr{A}_i(\underline{x})$ of $L(\leqslant)$ and sentences $\mathscr{S}_i$ of $L(\leqslant, \mathscr{R}, \mathscr{M}, A)$ $(i = 1, 2, \ldots, m)$ such that*

$$\mathscr{K}(\underline{x}) \rightarrow \left(\mathscr{A}(\underline{x}) \leftrightarrow \bigvee_i \left([\mathscr{A}_i(\underline{x})]^{\mathscr{K}} \wedge \mathscr{S}_i\right)\right).$$

The restriction on the class $\mathscr{K}$ of a predicate given in the universe by a $\leqslant$-formula is thus equivalent to a predicate essentially defined within $\mathscr{K}$.

§ 5. Proof of Lemma 3. Let $\mathscr{N}$ be the monadic predicate defined by $\mathscr{N}(k) \leftrightarrow 1 \leqslant k \in \omega$. By (3.21), (3.19) and (3.13) the multiplicity a introduced in § 3 gives an order preserving isomorphism from $(\mathscr{K}, =, \leqslant)$ onto $(\mathscr{N}, \equiv, \subseteq)$, $(\mathscr{N}, =, \leqslant)$ respectively. By induction this isomorphism of the ordering $\leqslant$ can be extended to $\leqslant$-formulas, provided all quantifiers are restricted, hence

(5.1) If $\mathscr{A}(\underline{x})$ is a formula of $L(\leqslant)$ with all its free variables among $\{\underline{x}\}$, then

$$\underline{\mathscr{K}}(\underline{x}) \rightarrow \left([\mathscr{A}(\underline{x})]^{\mathscr{K}} \leftrightarrow [\mathscr{A}(a(\underline{x}))]^{\mathscr{N}}\right).$$

We are now in the position to prove Lemma 3 of § 1 without much difficulty. The only assumption on A we have used so far is Unit(A). Let $\mathscr{D}(x, z)$ be a $\leqslant$-formula. By Corollary 4.3 there exist formulas $\mathscr{D}_i(x, z)$ of $L(\leqslant)$ and sentences $\mathscr{S}_i$ of $L(\leqslant, \mathscr{R}, \mathscr{M}, A)$ $(i = 1, 2, \ldots, m)$ such that

$$\mathscr{K}(x) \wedge \mathscr{K}(z) \rightarrow \left(\mathscr{D}(x, z) \leftrightarrow \bigvee_i \left([\mathscr{D}_i(x, z)]^{\mathscr{K}} \wedge \mathscr{S}_i\right)\right),$$

hence by (5.1)

$$(*) \qquad \mathscr{D}(x, z) \leftrightarrow \bigvee_i \left([\mathscr{D}_i(a(x), a(z))]^{\mathscr{N}} \wedge \mathscr{S}_i\right)$$

provided $\mathscr{K}(x) \wedge \mathscr{K}(z)$.

By the well-known analysis of $\leqslant$-formulas on natural numbers using elimination of quantifiers we obtain for each $\leqslant$-formula $\mathscr{D}_i(x, z)$ two numerals q_i and p_i such that for any numeral x, z with $q_i < x$ and $p_i < z - x$ the following holds:

$$[\mathscr{D}_i(x, z)]^{\mathscr{N}} \leftrightarrow [\mathscr{D}_i(x, z+1)]^{\mathscr{N}}.$$

Remark. If x and z are far away from the first element and the distance between them is large enough, then the $\leqslant$-formula $[\mathscr{D}_i(\cdot, \cdot)]^{\mathscr{N}}$ does not distinguish between the two pairs (x, z) and $(x, z+i)$.

Let $q = \max\{q_i |\ i = 1, 2, \ldots, m\}$, $p = \max\{p_i |\ i = 1, 2, \ldots, m\}$,

$$s = \max\{q, p\} + 1 \quad \text{and} \quad t = s + s.$$

Then $q_i < s$ and $p_i < t - s$ hold for all $i = 1, 2, \ldots, m$, hence for all $i = 1, 2, \ldots, m$ simultaneously hold

$$(**) \qquad [\mathscr{D}_i(s, t)]^{\mathscr{N}} \leftrightarrow [\mathscr{D}_i(s, t+1)]^{\mathscr{N}}.$$

In (*) let $x = s \times A$, $z = t \times A$. Then $\mathscr{K}(s \times A)$, $\mathscr{K}(t \times A)$, $a(s \times A) = s$, $a(t \times A) = t$ yields

$$\mathscr{D}(s \times A, t \times A) \leftrightarrow \bigvee_i \left([\mathscr{D}_i(s, t)]^{\mathscr{N}} \wedge \mathscr{S}_i\right).$$

Similarly by $x = s \times A$ and $z = (t+1) \times A$ we obtain

$$\mathscr{D}(s \times A, (t+1) \times A) \leftrightarrow \bigvee_i \left([\mathscr{D}_i(s, t+1)]^{\mathscr{N}} \wedge \mathscr{S}_i\right).$$

Hence by (**) and $t = s + s$:

$$\mathscr{D}(s \times A, (s+s) \times A) \leftrightarrow \mathscr{D}(s \times A, (s+s+1) \times A).$$

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MAX-PLANCK-INST. FÜR BIOPHYS. CHEMIE
Postfach 968
D-3400 Göttingen

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