Defining cardinal addition by $\preceq$-formulas

by

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Abstract. It is well-known that in Zermelo–Fraenkel set theory strengthened by the axiom of choice, cardinal addition of infinite sets can be expressed by the supremum which is defined by a first-order formula in the cardinal ordering $\preceq$. The same holds in presence of the weaker axiom stating that every infinite set is immeasurable, i.e. $\forall x(x < \omega \lor x = \omega \cdot 2)$. On the other hand, in Zermelo–Fraenkel set theory a definition of cardinal addition of infinite sets by a $\preceq$-formula is not possible. We show that this is also impossible for the following two kinds of extensions:

First, those extensions which are consistent with the existence of a Dedekind set $\varepsilon$, i.e. $\varepsilon$ infinite and $\varepsilon \neq \varepsilon + 1$,其次, extensions which are consistent with the existence of an infinite set $\mathcal{A}$, i.e. $\mathcal{A}$ infinite, $\mathcal{A} \not\subset \mathcal{A} + \mathcal{A}$, $\forall x, y (x + y = \mathcal{A} + x = \mathcal{A} \cup y = \mathcal{A})$.

§ 1. In Zermelo–Fraenkel set theory ZF with the axiom of choice cardinal addition $x + y = z$ of infinite sets can be expressed by the supremum $\sup(x, y, z)$ which is defined by the $\preceq$-formula $\forall x, y \leq z \land \forall u (x, y \leq u \rightarrow z \leq u)$. In [2], p. 55 A. Tarski showed that the weaker axiom $\forall x (x < \omega \lor x = \omega + x)$ — meaning that every infinite set is immeasurable — still implies within ZF the following equivalence for cardinal addition of infinite sets

$$\forall x, y, z \left[ \inf(x) \land \inf(y) \land \inf(z) \rightarrow (x + y = z \leftrightarrow \sup(x, y, z)) \right] ,$$

where $\inf(x)$ abbreviates $\forall z < \omega x$.

In ZF itself — provided it is consistent — a definition by a (first-order) $\preceq$-formula cannot be given. To show this, we assume that for some $\preceq$-formula $\varphi(x, y, z)$ the following holds in ZF

$$\forall x, y, z \left[ \inf(x) \land \inf(y) \land \inf(z) \rightarrow (x + y = z \leftrightarrow \varphi(x, y, z)) \right] .$$

This leads to a contradiction as follows:

Consider the consistent extension of ZF in which the existence of a Dedekind set $\varepsilon$ ($\inf(\varepsilon) \land \varepsilon \neq \varepsilon + 1$) is postulated. Then the cardinals $\varepsilon + n$, $n \in \omega$, are definite and ordered like the integers. Let $\theta$ be the following function symbol ($= m$ is the set-theoretical equality)

$$\theta(x) = \begin{cases} x + 1, & \text{if } \exists n \in \omega (x = \varepsilon + n \lor x + n = \varepsilon) \\ x, & \text{otherwise} \end{cases}$$

* It is my sad duty to inform the reader that Alexander Häussler died of cancer on August 8, 1982. — H. Läuchli.
and note that $\theta(s) = s + 1$, but $\theta(s + 1) = s + 2$, hence $\theta(s) + \theta(s) \neq \theta(s + 1)$. One can check by examining several cases that $\theta$ is an $\aleph$-automorphism of the universe, preserving the order $\leq$ and infinity, thus we have

$$\forall x, \exists y (x \leq y \Leftrightarrow \theta(x) = \theta(y)),$$
$$\exists y \exists \theta(x) = y.$$

$\forall x, \exists y \exists z (x \leq y \leq z \Leftrightarrow \theta(x) \leq \theta(y) \leq \theta(z))$.

$\forall x (\theta(x) \in \text{Inf}(\omega))$.

Induction on the complexity yields

$$\forall x (\theta(x) \in \text{Inf}(\omega))$$

for every $\leq$-formula $\mathcal{A}$.

By applying this to $\mathcal{A}(x, y, z)$ in the assumed equivalence for cardinal addition of infinite sets we thus obtain in $ZF + \text{Inf}(\omega) \land \omega \neq \epsilon + 1$ the following

$$\forall x, y, z (\text{Inf}(x) \land \text{Inf}(y) \land \text{Inf}(z) \rightarrow \theta(x) + \theta(y) = \theta(x + y)),$$

hence for $x = e$, $y = e$ and $z = e + 1$ we deduce $\theta(e) + \theta(e) = \theta(e + 1)$ — a contradiction to the consistency of $ZF + \text{Inf}(\omega) \land \omega \neq \epsilon + 1$.

The same proof is valid for a slightly more general situation:

**Theorem 1.** In a theory which is compatible with $ZF + \exists(x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ no $\leq$-formula defines cardinal addition of infinite sets.

**Corollary 2.** Let theory $T$ be an extension of $ZF$ in which a $\leq$-formula defines cardinal addition of infinite sets. Then $\forall x (x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ can be deduced from $T$.

Proof of Corollary 2. Assume that $\forall x (x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ is not provable in $T$. Then $T + \exists(x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ is consistent and hence compatible with $ZF + \exists(x \in \text{Inf}(\omega) \land \omega \neq x + 1)$. Thus by Theorem 1 no $\leq$-formula defines cardinal addition of infinite sets.

The question arises whether the axiom $\forall x (x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ — meaning that every infinite set is transfinite — is sufficient for defining cardinal addition of infinite sets by a (first-order) $\leq$-formula. The answer for even much stronger axioms like $\text{DC}_{\omega}$, $\text{DC}_{\omega}^{\text{AC}}$ or even $\text{DC}_{\omega}^{\text{AC}}$ is negative (see Corollary 4).

Thereby $\text{DC}_{\omega}$ and $\text{DC}_{\omega}^{\text{AC}}$ are axioms of dependent choices and $\text{DC}_{\omega}^{\text{AC}}$ means that any wellorderable set of nonempty elements has a choice function (see [1], p. 119).

In all these cases we cannot refer to a Dedekind set as we did in Theorem 1. However, it is compatible with these axioms that there exists a set $A$ such that the sets $A \times \omega$, $1 \leq k \in \omega$, have in a sense the same behaviour as the natural numbers (see Lemma 3 and Theorem 4).

**Definition.** A is an **Unit**, if $A \times A = A$ and $\forall x, y (x \leq y \rightarrow x = y \lor x = A \lor y = A)$.

**Lemma 3.** If $\mathcal{A}(x, z)$ is a $\leq$-formula, then there exists a numeral $s$ with $1 \leq s \in \omega$ such that in $ZF + \text{Unit}(A)$ the following holds:

$$\mathcal{A}(x, A, (s + 1) \times A) \\ \mathcal{A}(x, A, A \times A).$$

The proof of this lemma is given in §§ 2–5. We sketch the proof in § 2, in §§ 3 and 4 we work out the theory $ZF + \text{Unit}(A)$ and finally in § 5 we prove the lemma. Let us nevertheless give the conclusions:

**Theorem 4.** In a theory $T$ which is compatible with $ZF + \exists(x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ no $\leq$-formula defines cardinal addition of infinite sets.

**Proof of Theorem 4.** Let us assume that the $\leq$-formula $\mathcal{A}(x, y, z)$ defines in $T$ cardinal addition of infinite sets. Since $T$ has the consistency property, we can assume the existence of a set $A$ such that $T + \exists(x \in \text{Inf}(\omega) \land \omega \neq x + 1)$ is consistent; clearly in this extension of $T$ also the following holds:

$$\forall x, y, z (\text{Inf}(x) \land \text{Inf}(y) \land \text{Inf}(z) \rightarrow x + y = z \land \mathcal{A}(x, y, z)).$$

By applying Lemma 3 on the $\leq$-formula $\mathcal{A}(x, y, z)$ we obtain a numeral $s$ with $1 \leq s \in \omega$ such that

$$\mathcal{A}(x, A, s \times A, (s + 1) \times A) \land \mathcal{A}(x, A, s \times A, (s + s + 1) \times A)$$

holds in $ZF + \text{Unit}(A)$.

As $A$ is assumed to be infinite, $s + A = s \times A = (s + s + 1) \times A$ are infinite too; furthermore $s \times A \times A = (s + s + 1) \times A$ holds. Hence by the assumed equivalence for addition and the result from Lemma 3 we obtain $s \times A + s \times A = (s + s + 1) \times A$. Thus $A = A + A$ (see § 3) which contradicts the assumed consistency.

**Remark.** The same argument holds if we replace "infinite" by "finite". As 1 is a Unit, in any theory compatible with $ZF$ (for instance every consistent extension of $ZF$), no $\leq$-formula defines cardinal addition of finite sets.

**Corollary 5.** If $ZF$ is consistent, no $\leq$-formula defines cardinal addition of infinite sets in the theory $ZF + DC_{\omega} + \text{AC}_{\omega}$.

**Proof.** By Theorem 4 it suffices to show that this theory is compatible with $ZF + \exists(x \in \text{Inf}(\omega) \land \text{Unit}(\omega))$. In [1] Theorem 8.9 (p. 127) let $\alpha = 1$: Then $DC_{\omega}$ and $\text{AC}_{\omega}$ hold in the Fraenkel–Mostowski permutation model given there. Furthermore it is easy to see that the set of atoms is a Unit in the permutation model.

Using a refinement of the embedding theorem, this relative consistency result by means of a permutation model is transferred into $ZF$; hence

$$ZF + DC_{\omega} + \text{AC}_{\omega} + \exists(x \in \text{Inf}(\omega) \land \text{Unit}(\omega))$$

is relative consistent to $ZF$.

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**§ 2. Sketch of proof of Lemma 3.** We work in $ZF$ and furthermore assume $\text{Unit}(A)$ for a set $A$. $A \times A = A$ implies $k \times A = (k + 1) \times A$ for $k \in \omega$; thus the class $\mathcal{A}$ of all $k \times A$ with $1 \leq k \in \omega$ (including the cardinal equivalent sets) obviously has the same ordering as $\mathcal{A} = \{ \{ k \mid 1 \leq k \in \omega \}$.

This isomorphism of the ordering $\leq$ can be transferred to $\leq$-formulas, provided that the quantifiers refer only to the classes $\mathcal{A}$
and $\mathcal{J}$ respectively. Hence, by the well-known theorem on natural numbers, Lemma 3 holds for $\mathcal{X}$-relativized $\leq$-formulas.

In order to treat quantifiers which refer to the whole universe, we investigate the ordering between elements of $\mathcal{X}$ and those of the universe. To do this, we introduce the notion of multiplicity $a(x)$ which intuitively counts the number of pairwise disjoint copies of $A$ which can be embedded in a set $x$ (§3). Then a set $x$ has multiplicity $k$ ($1 \leq k \leq \omega$) if $x$ is a sum $k \times A + y$ with $A \not\subseteq y$. In this decomposition the second summand $y$ is not unique, however, $A + y$ is defined up to cardinal equivalence. Let $\mathcal{B}$ be the class of all sums $A + y$ with $A \subseteq y$. On the class $\mathcal{B}$ of all $k \times A + y$ ($1 \leq k \leq \omega$, $A \not\subseteq y$) we introduce the two projections $\pi$ onto $\mathcal{X}$ satisfying $\pi(k \times A + y) = k \times A$ and $\pi$ onto $\mathcal{B}$ satisfying $\pi(k \times A + y) = A + y$.

It can be shown that $x \leq y \leftrightarrow \exists x_1 \leq x_2 \leq x \pi(x_1) \leq \pi(x_2)$ holds for $x_1, x_2$ in $\mathcal{B}$. In order to compare $x'$ in $\mathcal{Y}$ to $x''$ in $\mathcal{Y}$, it suffices to compare $\pi(x')$ to $\pi(x'')$.

Introducing $\mathcal{Y}$ by $\mathcal{Y} \cup \mathcal{Y}$ we see (§4), that a predicate given by a $\leq$-formula $\phi(x', y')$ is equivalent to a propositional combination of $\mathcal{X}$-relativized $\leq$-formulas in $\pi(x')$ and $\pi(y')$ and $\mathcal{Y}$-relativized $\leq$-formulas in $\pi(x')$ and $\pi(y')$ provided that all $x_1', ..., x_2'$ are in $\mathcal{Y}$ and $x_1'', ..., x_2''$ are in $\mathcal{Y}$. Assuming all $x_1, x_2, ..., x_n$ to be in $\mathcal{B}$ (hence none in $\mathcal{Y}$), then $x_1, x_2, ..., x_n$ in $\mathcal{B}$ and thus $\phi(x')$ is equivalent to a propositional combination of $\mathcal{X}$-relativized $\leq$-formulas in $x$ and some $\leq$-sentences in $A$. This is what we need for the proof of Lemma 3 in §5, as mentioned at the beginning.

§ 3. Unit and multiplicity. Working in ZF $+$ Unit($A$) we may assume that there exists a set $A$ with the following two properties:

$$A \leq A + A,$$

$$\forall x, y \left[ x + y = A \rightarrow x = A \lor y = A \right].$$

Then the following holds for $A$:

$$\forall x \left[ x + x = A \rightarrow x < A \right],$$

$$\forall x \left[ x < A \rightarrow x + x = A \right].$$

Furthermore (3.3) is equivalent to (3.1) and assuming (3.1) or (3.3) respectively, then (3.2) and (3.4) are equivalent too.

Proofs. (3.1) $\Rightarrow$ (3.3). Let $\pi + x = A$ for some $x$. Then $x \leq A$, hence $x < A$ because $x$ is $\leq A$ is excluded by (3.1). (3.3) $\Rightarrow$ (3.1). With $x = A$ in (3.3) the assumption $A = A + A$ yields the contradiction $A < A$, hence $A < A + A$. (3.2) $\Rightarrow$ (3.4): Let $x < A$ for some $x$; then there exists $x + y = A$ and $y = A$ by (3.2), thus $x + A = A$. (3.4) $\land$ (3.1) $\Rightarrow$ (3.2): Let $x = y = A$, but $x, y < A$. By (3.4) $A + (x + y) = A$, hence $A + A = A$ in contradiction to (3.1).

Before continuing, let us note a theorem on cardinal algebra provable in ZF:

$$k \in \omega \land k \times x + y \leq k \times x + w \rightarrow u + v \leq u + w.$$  

For a proof refer to Corollary 4 in [3], p. 81.

From Unit($A$) we deduce

$$0 < A,$$

$$k \in \omega \rightarrow k \times A \leq (k + 1) \times A,$$

$$k \in \omega \cap (k + 1) \times A \leq k \times A + y \rightarrow A \leq y,$$

$$k \in \omega \times k \times A \rightarrow A \leq k \times A + x = A,$$

$$k \in \omega \times k \times A \leq A \leq x \times y \rightarrow A \leq x \times y + k \times A \leq x \times y.$$

Proofs. (3.6) and (3.7) follow by (3.1) and (3.5).

(3.8): Let $(k + 1) \times A \leq A + x + y$, then by applying (3.5) we obtain $A + A \leq x + y$. Hence there exist $a', a''$, $y', y''$ with $A = a' + y' = a'' + y''$, $a' \leq a''$ and $y' \leq y''$. By (3.2) $a = a' \lor y' = A$ and $a'' = a' \lor y'' = A$. In the case $A = a''$, we obtain $A + A = a'' + a'' \leq A$ — a contradiction to (3.1). In all other cases $y' \leq y''$.

(3.9): By induction on $k \in \omega$: For $k = 0$ it holds trivially. Let $x \leq (k + 1) \times A$, then there exist $x', x''$ with $x = x' + x''$, $x' \leq A$, $x'' \leq k \times A$ thus $x' < A$ or $x' = A$ and by induction hypothesis $A \leq x''$ or $A + x'' = A$. In the case of $x' < A$ and $A + x'' = A$, 3.4 yields $A + x = A + x' + x'' = A$, in all other cases $A \leq x' < A$.

(3.10): Let $x \leq k \times A + y$, then there exist $x', x''$ with $x = x' + x''$, $x' \leq k \times A$, $x'' \leq y$. By 3.9 it is $A \leq x'$ or $A + x' = A$. In the first case $A \leq x'$, in the latter $A + x' = A + x' + x'' = A + x'' \leq A + y$.

We introduce a function symbol assigning to a set $x$ the maximal number $\omega(x)$ of pairwise disjoint copies of $A$ which can be embedded in $x$. This notion of multiplicity is helpful for investigating the ordering $\leq$.

Let $a$ be defined formally by the following:

(3.11) $\omega(x) = \{ \varphi \mid \varphi \in \omega \land (\omega + 1) \times A \leq \varphi \}$

$\omega(x)$ is an initial segment of $\omega$, hence $\omega(x) \in \omega$ or $\omega(x) = \omega$. Furthermore multiplicity is monotone, thus the following three propositions hold and are trivial:

(3.12) $a(x) \in \omega^{+}$, where $\omega^{+} = \omega \cup \{ \omega \}$

(3.13) $x_1 \leq x_2 \rightarrow a(x_1) \leq a(x_2)$

(3.14) $x_1 = x_2 \rightarrow a(x_1) = a(x_2)$

If $a(x)$ is finite then, as indicated, the following holds:

(3.15) $k \in \omega \rightarrow (a(x) = k \leftrightarrow A \leq k \times x \times (k + 1) \times A \leq x)$

Proof. Let $k \in \omega$. The case $k = 0$ is obvious, because $A \not\subseteq x$ if $a(x)$ is empty ($a(x) = 0$). Let $1 \leq k \in \omega$: If $a(x) = k$, then $k = 1 + a(x)$, hence by (3.11) $x \times A \leq x$. The assumption $k < k \times x \times A$ however leads to the contradiction $k < k$. For the other implication assume that $A \leq k \times x \times A$; (3.11) yields $k - 1 \in a(x)$, but $k \not\in a(x)$, hence $a(x) = k$ follows.

Remark. In the case $k = \omega$ we have only the equivalence between $a(x) = \omega$ and $\forall x (A \times x \leq A)$ in general this does not imply $A \times \omega \subseteq A$. 


The function $a(x)$ decomposes the universe in a* many classes. In the following we give, in some sense, a more explicit characterization of sets with finite multiplicity and the ordering $\leq$ between such sets. For this purpose we introduce the monadic predicate $\mathcal{F}$ in (3.16). By (3.15) the proposition (3.17) is then obvious.

(3.16) \[ \mathcal{F}(x) \iff a(x) = 0, \]
(3.17) \[ \mathcal{F}(x) \iff A \not\subseteq x. \]

Now the following statements hold:

(3.18) \[ k \in o \iff a(k) \equiv k \iff 3 \{\mathcal{F}(y) \land x = k \times A + y\}, \]
(3.19) \[ k \in o \iff a(k \times A) = k, \]
(3.20) \[ 1 \leq k_1, k_2 \in \omega \wedge \mathcal{F}(y_1) \wedge k_2 \times A + y_2 \iff k_1 \leq k_2 \times A + y_1 \iff k_2 \times A + y_2 \]

Remark. It cannot be expected that (3.20) holds with $k_1 \leq k_2 \times A$. For (3.18) there exists $y_1$ such that $x = k \times A + y$ holds. (3.6) follows. If, on the other hand, $x = k \times A + y$ for some $y_1$ with $A \not\subseteq y_1$, we have $k \times A \subseteq x$. The assumption $(k + 1) \times A \subseteq y$, however, yields the contradiction $A \not\subseteq x$ by (3.8). Hence $a(k) = k$ by (3.15).

(3.19): Let $y = 0$ in (3.18), then $A \not\subseteq y$ by (3.6), hence $A \not\subseteq x$ holds.

(3.20): Assume $1 \leq k_1, k_2 \in \omega \wedge \mathcal{F}(y_1) \wedge \mathcal{F}(y_2)$. If $k_1 \leq k_2$ and $A \not\subseteq k_2 \times A + y_2$, then obviously $k_1 \times A + y_1 \leq A \not\subseteq k_2 \times A + y_2$. If, on the other hand, $k_1 \times A + y_1 \leq k_2 \times A + y_2$, then $k_1 \leq k_2$ by (3.15) and (3.18). Furthermore we have $y_1 \leq k_2 \times A + y_2$, hence $A \not\subseteq y_1$ or $A \not\subseteq y_2$ by (3.10). Because of $\mathcal{F}(y_2)$ the second case holds.

Let us introduce the monadic predicates $\mathcal{X}$, $\mathcal{G}$ and $\mathcal{R}$ in (3.21) to (3.23). In (3.24) and (3.25) equivalent forms, obviously by (3.18), are added:

(3.21) \[ \mathcal{X}(x) \iff \exists k (1 \leq k \in o \wedge x = k \times A), \]
(3.22) \[ \mathcal{G}(x) \iff 1 \leq a(x) \in o, \]
(3.23) \[ \mathcal{R}(x) \iff a(x) = 1, \]
(3.24) \[ \mathcal{G}(x) \iff 3 \exists \{1 \leq k \in o \wedge \mathcal{F}(y) \land x = k \times A + y\}, \]
(3.25) \[ \mathcal{R}(x) \iff 3 \exists \{\mathcal{F}(y) \land x = A + y\}. \]

The elements of $\mathcal{G}$ are sums. By (3.20) not the summands $k \times A$ and $y$ but $k \times A + y$ are unique up to cardinal equivalence. This allows us to introduce the following two cardinal function symbols $x$ and $q$ on the class $\mathcal{G}$, i.e. function symbols with respect to cardinal equality only:

(3.26) If $\mathcal{G}(x)$—thus $x = k \times A + y$ for some $k, y$ with $1 \leq k \in o$ and $\mathcal{F}(y)$—then let $x(a) = k \times A$ and $q(a) = A + y$.

The following list of propositions formally expresses that the class $\mathcal{G}$ with the ordering $\leq$ is the "product" of the subclasses $\mathcal{X}$ and $\mathcal{R}$ with projections $x$ and $q$.

The proofs are easy applications of previously shown propositions which we leave to the reader:

(3.27) \[ \mathcal{X}(x) \Rightarrow \mathcal{G}(x), \]
(3.28) \[ \mathcal{R}(x) \Rightarrow \mathcal{G}(x), \]
(3.29) \[ \mathcal{G}(x) \Rightarrow \mathcal{X}(a(x)) \wedge \mathcal{R}(a(x)), \]
(3.30) \[ \mathcal{G}(x) \Rightarrow \exists \exists z (\mathcal{R}(z) \land \mathcal{G}(z) \land x = z \times A + y), \]
(3.31) \[ \mathcal{X}(a) \land \mathcal{R}(a) \Rightarrow \mathcal{G}(a) \land \mathcal{X}(a) \land \mathcal{R}(a) = 0, \]
(3.32) \[ \mathcal{G}(x) \wedge \mathcal{X}(z) \rightarrow (x_1 \leq x_2 \Rightarrow x_1 \times A + y_1 \leq x_2 \times A + y_2), \]
(3.33) \[ \mathcal{G}(x) \rightarrow x(a) \iff x \times A + y = A. \]

Having established this "product" property of $\mathcal{G}$, we consider now the ordering between elements of $\mathcal{G}$ and its complementary class $\mathcal{S}$. Of an element in $\mathcal{G}$ only its o-projection is involved in this. By (3.12) and (3.22) the following holds for $\mathcal{G}$:

(3.34) \[ \neg \mathcal{G}(x) \iff a(x) = 0 \wedge a(x) = 0, \]
(3.35) \[ \mathcal{G}(x_1) \wedge \neg \mathcal{G}(x_2) \rightarrow (x_1 \leq x_2 \Rightarrow q(x_1) \leq q(x_2)), \]
(3.36) \[ \mathcal{G}(x_2) \wedge \neg \mathcal{G}(x_1) \rightarrow (x_1 \leq x_2 \Rightarrow q(x_1) \leq q(x_2)). \]

Proofs. (3.35): Let $\mathcal{G}(x_1)$, $\neg \mathcal{G}(x_2)$, hence $x_1 \leq k_1 \times A + y_1$ for some $k_1, y_1$ with $1 \leq k_1 \in o$ and $\mathcal{F}(y_1)$. Furthermore $q(x_1) = A + y_1$ by (3.28). Assuming $x_1 \leq x_2$, we get $q(x_1) = A + y_1 \leq k_1 \times A + y_1 = x_2 \times A + y_2$ because $1 \leq k_1$. On the other hand, let $q(x_1) \leq y_2$, hence $A \not\subseteq y_2$, thus $a(x_1) = 0$ by $\neg \mathcal{G}(x_1)$ (and 3.34). By (3.11) $k_1 \times A + y_2$, thus there exists a set $u$ with $k_1 \times A + u = x_2$. Furthermore we have $y_1 \leq A + y_2$ and $y_2 = k_1 \times A + u$ and by applying (3.10) we obtain $A \not\subseteq y_1$ or $A + y_2 \leq A + u$. The first case contradicts $\mathcal{G}(y_1)$, hence by the latter $x_1 \leq k_1 \times A + y_1 \leq k_1 \times A + u = u_2$, because $1 \leq k_1$ holds.

(3.36): Let $\mathcal{G}(x_2)$, $\neg \mathcal{G}(x_1)$, hence $x_1 = k_1 \times A + y_1$ for some $k_1, y_1$ with $1 \leq k_1 \in o$ and $\mathcal{F}(y_1)$. Furthermore $q(x_2) = A + y_2$. Assuming $x_1 \leq x_2$, we obtain $a(x_1) \leq k_1 \in o$ by (3.13) and (3.18). Hence $a(x_1) = 0$ by $\neg \mathcal{G}(x_1)$ and (3.34). By (3.10) $x_1 \leq x_2 \iff x_1 \leq k_1 \times A + y_2 \Rightarrow A \not\subseteq y_1$ or $A + y_2 \leq A + u$, but $A \not\subseteq y_1$ is excluded by $a(x_1) = 0$, thus the latter holds and we obtain $x_1 \leq q(x_2)$. On the other hand, if $x_1 \leq q(x_2)$, we have $x_1 \leq q(x_1) = A + y_1 \leq k_1 \times A + y_1 = x_2 \times A + y_2$ by $1 \leq k_1$.

Finally we introduce in (3.37) the class $\mathcal{R}$, previously mentioned in § 2, and note two properties obvious by (3.12) and the definitions (3.22) of $\mathcal{G}$ and $\mathcal{R}$.

(3.37) \[ 0 \in \mathcal{R}(x) \iff 0 \leq a(x) \iff 1 \leq a(x) = 0, \]
(3.38) \[ \mathcal{G}(x) \vee \mathcal{R}(x), \]
(3.39) \[ \mathcal{G}(x) \wedge \mathcal{R}(x) \iff \mathcal{R}(x). \]

§ 4. Analyzing $\lambda$-formulas. In (3.32), (3.35) and (3.36) we gave, for the atomic formula $x_1 \leq x_2$, equivalent formulas depending on whether $x_1, x_2$ are in $\mathcal{G}$ or $\mathcal{S}$. In the following we generalize this to $\lambda$-formulas.
L(≤, ϕ₁, ..., ϕₖ) denotes the first order language built by means of the logical connectives ∧, ∨, ¬, ∈, ⊂ starting with the symbols ≤ and the predicates ϕ₁, ..., ϕₖ. If \( \mathfrak{a}(g) \) is a formula of \( L(≤, ϕ₁, ..., ϕₖ) \) and \( ϕ \) any monadic predicate, we write \( \mathfrak{a}(g)^{m} \) for the relativization of the formula \( \mathfrak{a}(g) \) to the class \( ϕ \). \( \mathfrak{a}(g)^{m} \) is recursively defined on the complexity of \( \mathfrak{a}(g) \) as follows:

\[
\begin{align*}
[x ≤ y]^m & \equiv x ≤ y, \\
[ϕ(0), y]^m & \equiv ϕ(0)^m, \\
[ϕ(1), y]^m & \equiv [ϕ(0), y]^m \land [ϕ(1), y]^m, \\
[ϕ(2), y]^m & \equiv [ϕ(0), y]^m \lor [ϕ(2), y]^m, \\
[¬ϕ(1), y]^m & \equiv [¬ϕ(1), y]^m, \\
[ϕ(1), ϕ(2), y]^m & \equiv [ϕ(1), y]^m \land [ϕ(2), y]^m, \\
[ϕ(2), ϕ(1), y]^m & \equiv [ϕ(2), y]^m \lor [ϕ(1), y]^m.
\end{align*}
\]

Thereby we write \( ϕ(x₁, x₂, ..., xₖ) \) — shorter \( ϕ(g) \), to indicate that the free variables in the formula \( ϕ \) are among \( x₁, x₂, ..., xₖ \) — shorter \( g \). Furthermore we abbreviate \( ϕ(1), ϕ(2), ..., ϕ(ₖ) \) by \( ϕ(g) \) and \( ¬ϕ(1), ¬ϕ(2), ..., ¬ϕ(ₖ) \) by \( ¬ϕ(g) \).

**Proposition 4.1.** Let \( \mathfrak{a}(g) \) be a formula of \( L(≤) \) with all its free variables among \( x₁, x₂, ..., xₖ \). For every decomposition \( (g', g'') \) of the set \( g \) of variables, there exist formulas \( \mathfrak{a}(g') \) of \( L(≤, ϕ, ψ) \) such that

\[
\mathfrak{a}(g') \land ¬\mathfrak{a}(g'') \Rightarrow \mathfrak{a}(g) \iff \bigwedge_{i=1}^{m} \left( ϕ(0)^{i}(x₁, x₂, ..., xₖ)^{i} \land \left[ ϕ(1)^{i}(x₁, x₂, ..., xₖ)^{i} \lor ϕ(2)^{i}(x₁, x₂, ..., xₖ)^{i} \right] \right).
\]

In the following discussion this type of disjunction will be referred to as the "normal form" of \( \mathfrak{a}(g) \).

The proof is by induction on the complexity of \( \mathfrak{a}(g) \). It is sufficient to consider the logical connectives ∨, ¬, ∈, ⊂. Let \( \mathfrak{a}(g) \) be an atomic formula, hence \( x₁ ≤ x₂ \). There are four possible decompositions of the variables, namely \( (x₁, x₂) \), \( (x₂, x₁) \), \( (x₁, x₂, x₃) \) and \( (x₁, x₂, x₃, x₄) \). By (3.32), (3.35) and (3.36) and the definition of relativation, the following holds:

\[
\begin{align*}
ϕ(x₁, x₂) & \iff [ϕ(x₁, x₂)]^{m} \land [ϕ(x₂, x₁)]^{m}, \\
¬ϕ(x₁, x₂) & \iff ¬ϕ(x₁, x₂)^{m} \land [ϕ(x₂, x₁)]^{m}, \\
[ϕ(x₁, x₂)]^{m} & \iff [ϕ(x₁, x₂)]^{m} \land [ϕ(x₂, x₁)]^{m}, \\
[¬ϕ(x₁, x₂)]^{m} & \iff ¬ϕ(x₁, x₂)^{m} \land [ϕ(x₂, x₁)]^{m}.
\end{align*}
\]

Let \( \mathfrak{a}(g) \) be the disjunction \( ϕ(g) \lor ψ(g) \) and \( (g', g'') \) a decomposition of \( g \). Then by induction hypothesis there exists a normal form for \( ϕ(g) \) (resp. \( ψ(g) \)). It follows that \( ϕ(g)^{m} \) is equivalent to \( ϕ(g)^{m} \lor ψ(g)^{m} \) (resp. \( ϕ(g)^{m} \lor ψ(g)^{m} \)). The disjunction of these two normal forms is a normal form for \( ϕ(g)^{m} \).

Let \( \mathfrak{a}(g) \) be \( ¬ϕ(g) \) and \( (g', g'') \) a decomposition of \( g \). By induction hypothesis there is a normal form for \( ϕ(g) \), therefore — provided \( ϕ(g') \land ¬ϕ(g'') \) — the negation of this normal form is equivalent to \( ϕ(g)^{m} \), thus

\[
\mathfrak{a}(g) \iff \bigwedge_{i=1}^{m} \left( [ϕ(g')]^{i} \land [¬ϕ(g'')]^{i} \right).
\]

By referring to propositional calculus and the definition of relativation we obtain the following normal form for \( \mathfrak{a}(g) \):

\[
\bigwedge_{i=1}^{m} \left( [¬ϕ(g')]^{i} \land [ϕ(g'')]^{i} \right).
\]

Let \( \mathfrak{a}(g) \) be \( ϕ(g) \lor ψ(g) \) and \( (g', g'') \) a decomposition of \( g \). \( \mathfrak{a}(g) \) is then equivalent to \( ϕ(g) \lor ψ(g) \lor (¬ϕ(g') \land ψ(g'')) \). It is sufficient to find a normal form for each of these two disjuncts:

By induction hypothesis there exists for the decomposition \( (g', g'') \lor (g'', g'') \) a normal form for \( ϕ(g') \) (resp. \( ϕ(g'') \)). Let \( \mathfrak{a}(g) \) be \( ϕ(g) \lor ψ(g) \) and \( (g', g'') \) a decomposition of \( g \). \( \mathfrak{a}(g) \) is then equivalent to \( ϕ(g') \lor ψ(g') \lor (¬ϕ(g'') \land ψ(g'')) \). It is sufficient to find a normal form for each of these two disjuncts:

By induction hypothesis there exists for the decomposition \( (g', g'') \lor (g'', g') \) a normal form for \( ϕ(g') \) (resp. \( ϕ(g'') \)).
Provided $\mathcal{G}(x') \land \neg \mathcal{G}(y')$, it can be shown by predicate calculus that
\[ \exists z (\neg \mathcal{G}(x') \land \mathcal{G}(y, z)) \] is equivalent to
\[ \neg \mathcal{G}(x') \land \exists z (\neg \mathcal{G}(y', z') \land \mathcal{G}(y, z')) \].

With $\neg \mathcal{G}(x') \equiv \neg \mathcal{G}(x') \land \mathcal{G}(z)$, the definition of relativation, the normal form
\[ \neg \exists z (\neg \mathcal{G}(x') \land \mathcal{G}(z')) \] can finally be obtained.

Corollary 4.2 is obtained by specifying all free variables to be in $\mathcal{G}$:

**Corollary 4.2.** Let $\alpha(x)$ be a formula of $L(\mathcal{G})$ with all its free variables among $\{x\}$. Then there exist formulas $\beta(x)$ of $L(\mathcal{G})$ and $\gamma(x)$ of $L(\mathcal{G}, \mathcal{A})$ ($i = 1, 2, \ldots, m$) such that
\[ \mathcal{G}(x) \equiv \{\beta(x) \equiv (\mathcal{G}(x) | \mathcal{G}(x')) \} \land \mathcal{G}(x') \].

Again by specifying all free variables to be in $\mathcal{G}$ and $\mathcal{A}$ (3.27), (3.33) the following holds:

**Corollary 4.3.** Let $\alpha(x)$ be a formula of $L(\mathcal{G})$ with all its free variables among $\{x\}$. Then there exist formulas $\beta(x)$ of $L(\mathcal{G})$ and sentences $\mathcal{S}_i$ of $L(\mathcal{G}, \mathcal{A}, \mathcal{M}, \mathcal{A})$ ($i = 1, 2, \ldots, m$) such that
\[ \mathcal{G}(x) \equiv \{\beta(x) \equiv (\mathcal{G}(x) | \mathcal{G}(x')) \} \land \mathcal{S}_i \].

The restriction on the class $\mathcal{A}$ of a predicate given in the universe by a $\mathcal{G}$-formula is thus equivalent to a predicate essentially defined within $\mathcal{G}$.

§ 5. Proof of Lemma 3. Let $\mathcal{A}$ be the monadic predicate defined by $\mathcal{A}(x', z) \equiv 1 \leq x \leq a$. By (3.21), (3.19) and (3.13) the multiplicity $a$ introduced in § 3 gives an order preserving isomorphism from $(\mathcal{A}, \sqcup, \sqsubset, \mathcal{A})$ onto $(\mathcal{A}, \sqcup, \subseteq, \mathcal{A})$ respectively. By induction this isomorphism of the ordering $\sqsubset$ can be extended to $\mathcal{G}$-formulas, provided all quantifiers are restricted, hence

(5.1) If $\alpha(x)$ is a formula of $L(\mathcal{G})$ with all its free variables among $\{x\}$, then
\[ \mathcal{G}(x) \equiv \{ (\alpha(x) \equiv (\mathcal{G}(x) | \mathcal{G}(x')) \} \land \mathcal{G}(x') \].

We are now in the position to prove Lemma 3 of § 1 without much difficulty. The only assumption on $A$ we have used so far is $\mathcal{G}(A)$. Let $\mathcal{G}(x, z)$ be a $\mathcal{G}$-formula of $L(\mathcal{G})$ and sentences $\mathcal{S}_i$ of $L(\mathcal{G}, \mathcal{A}, \mathcal{M}, \mathcal{A})$ ($i = 1, 2, \ldots, m$) such that
\[ \mathcal{G}(x, z) \equiv \{ \mathcal{G}(x, z) \equiv (\mathcal{G}(x, z) | \mathcal{G}(x, z')) \} \land \mathcal{S}_i \],
\[ \mathcal{G}(x, z) \equiv \{ (\mathcal{G}(x, z) \equiv (\mathcal{G}(x, z) | \mathcal{G}(x, z')) \} \land \mathcal{S}_i \] provided $\mathcal{G}(x) \land \mathcal{A}(z)$.

By the well-known analysis of $\mathcal{G}$-formulas on natural numbers using elimination of quantifiers we obtain for each $\mathcal{G}$-formula $\mathcal{G}(x, z)$ two numerals $g_i$ and $p_i$ such that for any numeral $x$, $z$ with $g_i < x$ and $p_i < z$ the following holds:
\[ \mathcal{G}(x, z) \equiv \{ (\mathcal{G}(x, z) \equiv (\mathcal{G}(x, z) | \mathcal{G}(x, z')) \} \land \mathcal{S}_i \].

**Remark.** If $x$ and $z$ are far away from the first element and the distance between them is large enough, then the $\mathcal{G}$-formula $\mathcal{G}(x, z)$ does not distinguish between the two pairs $(x, z)$ and $(x, z + 1)$.

Let $q = \max \{g_i\}$, $p = \max \{p_i\}$, and $s = \max \{p_i, p_i + 1\}$.

Then $q < x$ and $p < x$ hold for all $i = 1, 2, \ldots, m$, hence for all $i = 1, 2, \ldots, m$ simultaneously hold
\[ \mathcal{G}(x, z) \equiv \{ (\mathcal{G}(x, z) \equiv (\mathcal{G}(x, z) | \mathcal{G}(x, z')) \} \land \mathcal{S}_i \].

In (5.1) let $x = x \times A$, $z = x \times A$. Then $\mathcal{G}(x \times A)$, $\mathcal{G}(x \times A)$, $x \times A = z$, $a \times A = t$ yields
\[ \mathcal{G}(x \times A, x \times A) \equiv \{ (\mathcal{G}(x \times A) \equiv (\mathcal{G}(x \times A) | \mathcal{G}(x \times A)) \} \land \mathcal{S}_i \].

Similarly by $x = x \times A$ and $z = (x \times A) \times A$ we obtain
\[ \mathcal{G}(x \times A, (x \times A) \times A) \equiv \{ (\mathcal{G}(x \times A, (x \times A) \times A) \equiv (\mathcal{G}(x \times A, (x \times A) \times A)) \} \land \mathcal{S}_i \].

Hence by (5.1) and $t = z + x$:
\[ \mathcal{G}(x \times A, (x \times A) \times A) \equiv \{ (\mathcal{G}(x \times A, (x \times A) \times A) \equiv (\mathcal{G}(x \times A, (x \times A) \times A)) \} \land \mathcal{S}_i \]