On the shape of pointed compact connected subsets of E^3

by

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Abstract. The shape of pointed subcontinua of E^3 is studied. It is proved that pointed FANR-spaces in E^3 have polyhedral shapes. Some characterizations of shapes in E^3 are given. Decompositions of a given shape into a one-point union and a Cartesian product are studied.

Introduction. The structure of the shape of plane continua is very simple. It is known (see [2] p. 221) that if $(X, x_0) \subset E^2$ is a pointed continuum (or an unpointed one), then (X, x_0) has the shape of a bouquet of 1-spheres. Therefore, the shape of a plane continuum is uniquely determined by its Betti numbers.

In the space E^3 the situation is more complicated, but the shape structure of a pointed continuum $(X, x_0) \subset E^3$ is in a sense still simple.

The following theorem will play a fundamental role in the theory of the shape of compact connected pointed 1-movable subsets of E^3:

THEOREM. *If $(X, x_0) \subset E^3$ is a pointed 1-movable continuum, then there are pointed continua (Y, y_0) and $(Z, z_0) \subset E^3$ such that:*

(i) $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0) + \mathrm{Sh}(Z, z_0)$,

(ii) *(Z, z_0) is a bouquet of 2-spheres,*

(iii) $(Y, y_0) = \bigcap_{i=1}^{\infty} (Y_i, y_0)$ *where $(Y_{i+1}, y_0) \subset (Y_i, y_0) \subset E^3$ is an aspherical polyhedron for every $i = 1, 2, \ldots$*

(We say that a topological space (X, x_0) is *aspherical* if $\pi_n(X, x_0) = 0$ for every $n \geqslant 2$.)

Almost all the results of this paper are fairly simple consequences of the above theorem.

In many proofs the techniques applied in the topology of 3-manifolds will be used. In Section I we will recall and prove some facts concerning 3-manifolds which are going to be used later.

In Section II the above basic theorem will be proved.

The main theorem of Section III states that if $(X, x_0) \subset E^3$ is a pointed FANR-space, then there is a polyhedron (P, p_0) such that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(P, p_0)$. D. A. Edwards and R. Geoghegan in [11] have constructed a 2-dimensional FANR which is not of polyhedral shape. Our theorem implies the impossibility of realizing such a construction in E^3.

In Section IV some necessary and sufficient conditions are given for two continua $(X, x_0), (Y, y_0) \subset E^3$ to have the same shape. We also give an algebraic condition for a given continuum $(X, x_0) \subset E^3$ to be FANR. In particular, we prove that $(X, x_0) \subset E^3$ is FAR if and only if (X, x_0) is pointed 1-movable and the groups $\check{H}_1(X, Z)$ and $\check{H}_2(X, Z)$ are trivial.

Section V contains some results concerning the shape of a suspension of the continuum $(X, x_0) \subset E^3$.

In Sections VI and VII the decomposition of the shape of $(X, x_0) \subset E^3$ into simple and prime shapes is studied.

All spaces in this paper are metric spaces and all the maps are continuous. All the inverse sequences of topological spaces considered in this paper are ANR-sequences. The Čech homology and cohomology groups with coefficients in a group G are denoted by $\check{H}_n(X, G)$ and $\check{H}^n(X, G)$ respectively. Borsuk's fundamental group is denoted by $\underline{\pi}_1(X, x_0)$. By a manifold we mean in this paper a PL-3-manifold.

We assume that the reader is familiar with the shape theory for metric compacta (see [2]), and with the basic concepts of PL-topology (see [25]).

I. Auxiliary results. This section contains all the definitions and theorems concerning PL-3-manifolds which will be needed later.

(1.1) Convention. If X is a subset of E^3, then the closure of the union of all bounded components of $E^3 \setminus X$ is denoted by $\mathrm{bc}X$. If $y \notin X$, then the closure of that component of $E^3 \setminus X$ which contains point y is denoted by $X(y)$. The manifold obtained from M by capping off each 2-sphere component of ∂M with a 3-cell is denoted by $\hat{M}$.

(1.2) Definition. Let M, M_1, M_2 be connected manifolds (not necessarily compact). We say that M is a *connected sum of* M_1 and M_2 and we denoted it by $M = M_1 \# M_2$ if there are 3-cells $B_i \subset \mathrm{int} M_i$ $(i = 1, 2)$ and PL-embeddings $h_i\colon M_i \setminus \mathrm{int} B_i \to M$ such that

$$h_1(M_1 \setminus \mathrm{int} B_1) \cup h_2(M_2 \setminus \mathrm{int} B_2) = M$$

and

$$h_1(\partial B_1) = h_2(\partial B_2) = h_1(M_1 \setminus \mathrm{int} B_1) \cap h_2(M_2 \setminus \mathrm{int} B_2).$$

If M, M_1, M_2 are oriented, we require that h_i be orientation preserving.

(1.3) Definition. We say that a manifold M is *prime* if $M \neq S^3$ and $M = M_1 \# M_2$ implies that either M_1 or M_2 is a 3-sphere.

The following theorem holds:

(1.4) Theorem ([13] Theorem 3.15 p. 31 and Theorem 3.21 p. 35). *Each compact oriented manifold can be expressed as a connected sum of a finite number of prime factors and such a decomposition is unique.*

The uniqueness of decomposition into a connected sum ought to be understood as follows: if $M = M_1 \# M_2 \# \ldots \# M_n = N_1 \# N_2 \# \ldots \# N_m$ and all M_i and N_j are prime, then $n = m$ and there is a bijection $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that M_i is PL-homeomorphic with $N_{\sigma(i)}$, and such homeomorphisms preserve orientation.

(1.5) Remark. Let us observe that the operation of a connected sum does not depend on the particular choice of cells B_i, and hence we can always assume that 2-spheres $S_i = \mathrm{cl}(M_1 \# \ldots \# M_i \setminus B_i) \cap \mathrm{cl}(M_{i+1} \setminus B_{i+1})$ which realize the decomposition of M into a connected sum are pairwise disjoint.

(1.6) Definition. A finite collection of 2-spheres $\{S_1, S_2, \ldots, S_k\}$ in the interior of a compact manifold $M \subset E^3$ is a *full system of* 2-*spheres in* M provided the following conditions are satisfied:

(i) $\mathrm{bc}S_i \cap \mathrm{bc}S_j = \emptyset$ for $i \neq j$,

(ii) spheres $S_1, S_2, \ldots, S_k$ realize the decomposition of M into a connected sum of prime factors.

(1.7) Lemma. *For every compact manifold $M \subset E^3$ there is a full system of 2-spheres in M.*

Proof. Let $S_1, S_2, \ldots, S_k$ be 2-spheres in $\mathrm{int}M$ which realize the decomposition of M into a connected sum of prime factors. By Remark (1.5) we may assume that these spheres are pairwise disjoint. Let

$$A = \{(i, j)\colon \mathrm{bc}S_i \subset \mathrm{bc}S_j\}$$

and let us choose a pair $(i, j) \in A$ such that no sphere S_n lies in $\mathrm{int}\,\mathrm{bc}S_j \setminus \mathrm{bc}S_i$. We denote the regular neighbourhoods of S_i and S_j in M by R_i and R_j respectively. We may assume $R_i \cap R_j = \emptyset$. Let J be a PL-arc in $\mathrm{bc}S_j$ such that:

$$\mathrm{int}J \subset \mathrm{bc}S_j \setminus (R_i \cup R_j),$$
$$J \cap R_i \neq \emptyset \neq J \cap R_j.$$

Let R be a regular neighbourhood of J in $\mathrm{cl}[\mathrm{bc}S_j \setminus (R_i \cup R_j)]$. It is clear that the set

$$\{[(\mathrm{bc}S_j \setminus \mathrm{bc}S_i) \cap \partial(R_i \cup R_j)] \setminus R\} \cup \mathrm{cl}[\partial R \setminus (R_i \cup R_j)]$$

is a 2-sphere. We denote it by S_j'. Consider the collection $S_1, \ldots, S_{j-1}, S_j', S_{j+1}, \ldots, S_k$ of 2-spheres in M. These spheres realize the decomposition of M into a connected sum of prime factors and the set of pairs (i, j) for which $\mathrm{bc}S_i \subset \mathrm{bc}S_j$ for $S_i, S_j \in \{S_1, \ldots, S_j', \ldots, S_k\}$ contains fewer elements than A. Hence, after finitely many modifications of the above type, we obtain a full system of 2-spheres in M.

(1.8) Lemma. *Let $\{S_1, S_2, \ldots, S_k\}$ be a full system of 2-spheres in a manifold $M \subset E^3$ and let $x \notin M$. If there is a 2-sphere S in $\mathrm{int}M$ which divides E^3 between the unbounded component of $E^3 \setminus X$ and $M(x)$, then there is an index i such that $M(x) \subset \mathrm{bc}S_i$.*

Proof. Let S be a 2-sphere in $\mathrm{int}M$ which divides E^3 between $M(x)$ and the unbounded component of $E^3 \setminus M$, and suppose $M(x) \cap \mathrm{bc}S_i = \emptyset$ for $i = 1, 2, \ldots, k$.

Then S divides E^3 between sets $N(x)$ and the unbounded component of $E^3\setminus N$ where $N = \bigcup_{i=1}^{k} \mathrm{bc} S_i \cup M$. This means that N is not a prime manifold. On the other hand, N is homeomorphic with one of the factors of the decomposition of M into a connected sum of prime factors by spheres $S_1, S_2, \ldots, S_k$. This contradiction finishes the proof of Lemma (1.8).

(1.9) Lemma. *Let M be a manifold in E^3 and $x \notin M$. Suppose that B_1, B_2 are PL-3-cells in $\mathrm{int} M$ such that ∂B_1 and ∂B_2 are in general position and $M(x) \subset \mathrm{bc}(B_1 \cup B_2)$. Then there is a 2-sphere S in $\mathrm{int} M$ which divides E^3 between $M(x)$ and the unbounded component of $E^3\setminus M$.*

Proof. Since ∂B_1 and ∂B_2 are in general position, $\partial B_1 \cap \partial B_2$ is a finite family of simple closed curves (see [13] Chapter I). Let y be a point in the unbounded component of $E^3\setminus M$ and let J be a PL-arc with ends x and y such that $J \cap B_1 = \emptyset$. Moreover, let us suppose that J meets ∂B_2 transversely and that the number of points of the set $J \cap \partial B_2$ is minimal (i.e. that, for any other arc J' in M with ends x, y and with $J' \cap B_1 = \emptyset$, the set $J \cap \partial B_2$ does not contain fewer points than $J' \cap \partial B_2$ does).

Let $p \in \partial B_2 \cap J$ be a point such that $p \in \mathrm{cl}$ (unbounded component of $E^3\setminus(B_1 \cup B_2)$) and no point which lies between p and x in J belongs to cl (unbounded component of $E^3\setminus(B_1 \cup B_2)$).

By C we denote the closure of that component of $\partial B_2\setminus\partial B_1$ which contains point p. Then C is a disc with wholes. Let us observe that $M(x) \subset \mathrm{bc}(C \cup B_1)$. This follows from the fact that $C \cup B_1$ decomposes E^3 and the arc J has only one point common with $C \cup B_1$.

Let R be a regular neighbourhood of ∂B_1 in M such that $p \in (C\setminus R)$, $\mathrm{cl}(C\setminus R)$ is a strong deformation retract of C and $\partial R \cap C$ is a finite family of closed simple curves, and $J \cap R = \emptyset$.

Let $D \subset \partial R$ be a disc such that $D \cap C = \partial D$. Let K be a component of $C \cap R$ such that $\partial D \subset K$. Consider the set

$$C_1 = D \cup (C\setminus K).$$

C_1 is a disc with wholes such that $\partial B_1 \cup C_1$ divides E^3 between $M(x)$ and the unbounded component of $E^3\setminus M$, $J \cap C_1$ contains only one point and $C_1 \cap \partial B_1$ contains fewer components than $C \cap \partial B_1$ does.

Hence after finitely many steps we obtain a disc C_k without wholes such that $C_k \cap \partial B_1$ is a simple closed curve, and $C_k \cup B_1$ divides E^3 between $E^3\setminus M$ and $M(x)$, and $J \cap C_k$ contains exactly one point.

Let D_1, D_2 be 2-discs in ∂B_1 such that $D_1 \cup D_2 = \partial B_1$ and $D_1 \cap D_2 = C_k \cap B_1 = \partial C_k$. Let us consider a 2-sphere $S = D_1 \cup C_k$ or $D_2 \cup C_k$. It is easy to see that in both cases $M(x) \subset \mathrm{bc} S$. The proof is completed.

(1.10) Definition. Let $N \subset M$ be a submanifold in a manifold $M \subset E^3$. We say that N is *well embedded in M* if the following conditions are satisfied:

(i) if $x \notin M$ and $M(x)$ is bounded set, then $N(x)$ is bounded set;

(ii) if $x, y \notin M$ and there is a 2-sphere S in $\mathrm{int} M$ which divides E^3 between $M(x)$ and $M(y)$, then there is a 2-sphere S' in $\mathrm{int} N$ which divides E^3 between $N(x)$ and $N(y)$.

(1.11) Lemma. *Let N be a well embedded submanifold in a manifold $M \subset E^3$. If $\{S_1, S_2, \ldots, S_k\}$ is a full system of 2-spheres in M, then there is a full system $\{S_1', S_2', \ldots, S_n'\}$ of 2-spheres in N and an isotopy $h_t\colon M \to M$ such that*

$$h_t|\partial M = \mathrm{id}_{\partial M} \quad \textit{for every } t \in I$$

and

$$S_i \in \{h_1(S_1'), h_1(S_2'), \ldots, h_1(S_n')\} \quad \textit{for every } i = 1, 2, \ldots, k.$$

Proof. Let $\{S_1'', S_2'', \ldots, S_n''\}$ be any full system of 2-spheres in N. Let us observe that the definition of a full system of 2-spheres and the definition of well embedding imply that the following condition is satisfied (after a possible change of numbering of S_i''):

(1.12) $$\mathrm{bc} S_i \cap \partial M \subset \mathrm{bc} S_i'' \quad \text{for} \quad i = 1, 2, \ldots, k.$$

First of all let us prove that there is a full system $\{S_1', S_2', \ldots, S_n'\}$ of 2-spheres in N such that the following condition is satisfied:

(1.13) $$\mathrm{bc}(\mathrm{bc} S_1 \cup \mathrm{bc} S_1') \cap \partial M = \emptyset \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Suppose there is $x \notin M$ such that

$$M(x) \subset \mathrm{bc}(\mathrm{bc} S_1 \cup \mathrm{bc} S_1'').$$

Lemmas (1.9) and (1.8) imply that there is a $j \neq 1$ such that $M(x) \subset \mathrm{bc} S_j''$. Let J be a PL-arc in N which joins S_j'' and the unbounded component of $\mathrm{cl}(E^3\setminus N)$ such that $\mathrm{int} J \cap (\mathrm{bc} S_j'' \cup \mathrm{bc} S_1) = \emptyset$. We denote the regular neighbourhood of J in $N\setminus\mathrm{int}(\mathrm{bc} S_j'' \cup \mathrm{bc} S_1)$ by R. Let us observe that $R \cup \mathrm{bc} S_j$ is a 3-cell.

Let p be any point in $R \cap \partial(N \cup \mathrm{bc} S_j'')$ and let R' be a regular neighbourhood of p in $R\setminus\mathrm{bc} S_1''$. The uniqueness of regular neighbourhoods implies that there is an isotopy

$$f_t\colon N \cup \mathrm{bc} S_j'' \to N \cup \mathrm{bc} S_j''$$

which is constant outside any given open neighbourhood of R' and is such that $R \subset \mathrm{int} f_1(R')$. It is clear that the family $\{f_1(S_1''), \ldots, f_1(S_{j-1}''), S_j'', f_1(S_{j+1}''), \ldots, f_1(S_m'')\}$ forms a full system of 2-spheres in N and the set

$$\mathrm{bc}(\mathrm{bc} S_1 \cup \mathrm{bc} f_1(S_1'')) \cap \partial M$$

contains fewer components than

$$\mathrm{bc}(\mathrm{bc} S_1 \cup \mathrm{bc} S_1'') \cap \partial M.$$

After finitely many steps we can obtain a new family $\{S_1', \ldots, S_m'\}$ which is a full system of 2-spheres in N such that

$$\mathrm{bc}(\mathrm{bc} S_1 \cup \mathrm{bc} S_1') \cap \partial M = \emptyset.$$

Suppose that N is well embedded in M and we are given full systems of 2-spheres $\{S_1, \ldots, S_k\}$, $\{S'_1, \ldots, S'_m\}$ in M and N respectively. We assume that these families satisfy conditions (1.12) and (1.13). Without loss of generality we may assume that S_1 and S'_1 are in general position. Then $S_1 \cap S'_1$ is a sum of finitely many simple closed curves. Suppose that this set has m components. Let us choose a point $q \in S'_1 \setminus \operatorname{bc} S_1$ in such a way that there is a curve $F \subset S_1 \cap S'_1$ which divides S'_1 between q and all the other components of $S_1 \cap S'_1$. Let D be a 2-disc in S'_1 which contains q and $\partial D = F$. The curve F bounds in S_1 two discs D_1 and D_2 such that $D_1 \cup D_2 = S_1$ and $D_1 \cap D_2 = F$. Consider two 2-spheres, $\hat{S}_1 = D \cup D_1$ and $\hat{S}_2 = D \cup D_2$. Conditions (1.12) and (1.13) imply that $\operatorname{bc}\hat{S}_1$ or $\operatorname{bc}\hat{S}_2$ lies in M. Suppose that $M \cap \operatorname{bc}\hat{S}_1$ is a 3-cell. Then there is an isotopy

$$h_t\colon M \to M$$

such that h_t is constant outside any given open neighbourhood of $\operatorname{bc}\hat{S}_1$ and $h_1(\operatorname{bc}\hat{S}_1) \subset \operatorname{bc} S_1$. The spheres S_1 and $h_1(S'_1)$ have fewer components of intersection than $S_1 \cap S'_1$. After finitely many steps of the above type we find an isotopy $\bar{h}_t\colon M \to M$ which is constant on ∂M and such that $\bar{h}_1(S'_1) \subset \operatorname{bc} S_1$. Since $\operatorname{bc} S_1 \setminus \operatorname{bc}\bar{h}_1(S'_1) \subset M$, there is an isotopy $g_t\colon M \to M$ which is constant on ∂M and such that $f_1 \bar{h}_1(S'_1) = S_1$. In order to finish the construction of the required isotopy we repet the above construction with respect to manifolds $M \setminus \operatorname{int}\operatorname{bc} S_1$ and $f_1 \bar{h}_1(N) \setminus \operatorname{int}\operatorname{bc} S_1$ and so on. After finitely many steps we find an isotopy of M onto itself which is constant on ∂M and which satisfies all the conditions of Lemma (1.11).

(1.14) Definition. Let $\omega\colon [0,1] \to M$ be a PL-embedding such that $\omega(1) \in \partial M$. Let R_1 be a regular neighbourhood of $\omega(I)$ in M and let R_2 be a regular neighbourhood of $\omega(I)$ such that $\partial R_1 \cap \partial R_2 \subset \partial M$. Let D be a PL-disc in $\operatorname{int}(\partial R_1 \cap \partial R_2)$ such that $\omega(1) \in \partial D$. We denote the component of $\partial R_1 \cap \partial R_2$ which contains D by D_1. Let $h_t\colon R_1 \to R_1$ be a small isotopy such that

$$h_t|\operatorname{cl}(\partial R_1 \setminus D_1) \cup D \cup \omega(I) = \mathrm{id},$$

$$h_1(\operatorname{int} D_1 \setminus D) \subset \operatorname{int} R_1 .$$

By an *almost regular neighbourhood of* $\omega(I)$ *in* M we mean the set $R = h_1(R_2)$.

(1.15) Lemma. *Let $M \subset E^3$ be a manifold. Suppose we have:*
$S_1, S_2, \ldots, S_k$ — different 2-spheres in ∂M,
$x_1, x_2, \ldots, x_k$ — points in $\partial M \setminus \bigcup_{i=1}^{k} S_i$ (not necessarily different),
a point $y_i \in S_i$ for every $i = 1, 2, \ldots, k$,
PL-embeddings $\eta_i, \omega_i\colon I \to M$ such that

$$\eta_i(1) = \omega_i(1) = x_i, \quad \eta_i(0) = \omega_i(0) = y_i,$$

$$\eta_i(\operatorname{int} I) \cup \omega_i(\operatorname{int} I) \subset \operatorname{int} M,$$

$$\eta_i(\operatorname{int} I) \cap \Big(\bigcup_{j=1}^{k} \omega_j(\operatorname{int} I) \cup \bigcup_{\substack{j=1\\ j\neq 1}}^{k} \eta_j(\operatorname{int} I)\Big) = \emptyset$$

for every $i = 1, 2, \ldots, k$.

If the embeddings ω_i and η_i are homotopic rel. ∂I in M, then there is an isotopy $h_t\colon M \to M$ such that $h_t(x) = x$ for any $x \in \partial M$ and $t \in I$ and $h_1\omega_i = \eta_i$.

Proof. The case of $k = 1$ is proved in [20] p. 1290–1292. Let us observe that, if h_t is an isotopy of M onto itself such that $h_1\omega_1 = \eta_1$, then the embeddings $h_1\omega_2$ and η_2 are homotopic in $\operatorname{cl}(M \setminus R)$ where R is a sufficiently small almost regular neighbourhood of $\eta_1(I)$ disjoint with $\bigcup_{i=2}^{k} [\omega_i([0,1)) \cup \eta_i([0,1))]$. This follows from the fact that $\operatorname{cl}(M \setminus R)$ is a strong deformation retract of $M \cup F$ where F is the bounded component of $E^3 \setminus S_1$. Now the proof of (1.15) may be obtained from the case of $k = 1$ by an easy induction.

Now let us define the operation of taking out 2-spheres from a manifold along the paths. This operation and Lemmas (1.11), (1.18) and (1.17) will play the crucial role in the proof of the main theorem, which is formulated in the introduction.

(1.16) Definition. Suppose that M is a manifold in E^3, $S_M = \{S_1, S_2, \ldots, S_k\}$ is a finite collection of 2-spheres in $\operatorname{int} M$ such that $\operatorname{bc} S_i \cap \operatorname{bc} S_j = \emptyset$ for $i \neq j$ and $\operatorname{bc} S_i \cap (E^3 \setminus M) \neq \emptyset$ for $i = 1, 2, \ldots, k$. Suppose that for every $j = 1, 2, \ldots, k$ two paths $\omega_i^1, \omega_i^2\colon I \to M$ are given such that:

(i) ω_j^i is a PL-embedding for $i = 1, 2$, $j = 1, 2, \ldots, k$,

(ii) $\omega_j^i(\operatorname{int} I) \subset \operatorname{int} M \setminus \bigcup_{j=1}^{k} S_i$,

(iii) $\omega_j^1(0) = \omega_j^2(0) \in S_j$ for $j = 1, 2, \ldots, k$,

(iv) $\omega_j^1(1) \in \operatorname{bc} S_j \cap \partial M$,

(v) $\omega_j^2(1) \in \partial M \setminus \operatorname{cl}(\text{unbounded component of } E^3 \setminus M)$.

Let $\Omega_M = \{\omega_j^i\colon i = 1, 2,\ j = 1, 2, \ldots, k\}$. Let R_j be the union of almost regular neighbourhoods of $\omega_j^1(I)$, $\omega_j^2(I)$ and a regular neighbourhood of S_j in M such that R_j is homeomorphic to $S^2 \times I$ for every $j = 1, 2, \ldots, k$ and $R_i \cap R_j \in \{\omega_i^1(1), \omega_j^1(1)\}$ for $i \neq j$. The set

$$M(S_M, \Omega_M) = \operatorname{cl}\Big(M \setminus \bigcup_{i=1}^{k} R_i\Big) \cup \bigcup_{j=1}^{k} [\omega_j^1(I) \cup S_j \cup \omega_j^2(I)]$$

will be called a *manifold M with 2-spheres $S_1, \ldots, S_k$ taken out along the paths $\omega_1^1, \ldots, \omega_k^1, \omega_k^2, \ldots, \omega_k^2$.*

The set of 2-spheres $S_1, S_2, \ldots, S_k$ and PL-arcs ω_j^i $(i = 1, 2, j = 1, 2, \ldots, k)$ in M which satisfy all the assumption of Definition (1.16) we denote shortly by (S_M, Ω_M).

(1.17) Lemma. *Inclusion $M(S_M, \Omega_M) \subset M$ is a homotopy equivalence.*

Proof. One can see that $M(S_M, \Omega_M)$ is a strong deformation retract of M.

(1.18) Lemma. *Suppose $N \subset M$ are manifolds in E^3. Let (S_M, Ω_M) (S_N, Ω_N) be sets of spheres and paths in N and M, respectively, which satisfy all the conditions* *of definition* (1.16). *Suppose* $S_M = \{S_1, S_2, \ldots, S_k\}$, $S_N = \{S_1, S_2, \ldots, S_k, \ldots, S_n\}$, $\Omega_M = \{\omega_j^i, i = 1, 2, j = 1, 2, \ldots, k\}$, $\Omega_N = \{\eta_j^i, i = 1, 2, j = 1, 2, \ldots, n\}$. *Moreover, let us assume that*

$$\eta_j^i(0) = \omega_j^i(0), \quad \eta_j^i(1) = \omega_j^i(1) \quad \text{for} \quad i = 1, 2, \quad j = 1, 2, \ldots, k$$

and

$$\eta_j^i(\operatorname{int} I) \cap \omega_j^i(\operatorname{int} I) = \emptyset \quad \text{for} \quad i = 1, 2, \quad j = 1, 2, \ldots, k.$$

If $\eta_j^i \simeq \omega_j^i$ rel. ∂I *in* M *for every* $i = 1, 2$ *and* $j = 1, 2, \ldots, k$, *then there is an embedding* $g\colon N(S_N, \Omega_N) \to M(S_M, \Omega_M)$ *such that*

$$g(S_i) = S_i \quad \text{for} \quad i = 1, 2, \ldots, k,$$
$$g|S_i \text{ is homotopically trivial for } i > k,$$

and the following diagram commutes up to homotopy:

$$\begin{array}{ccc} M(S_M, \Omega_M) & \overset{g}{\leftarrow} & N(S_N, \Omega_N) \\ \cap & & \cap \\ M & \supset & N \end{array}$$

Proof. Let $M_0, M_1, \ldots, M_k$ be the closure of the components of

$$M(S_M, \Omega_M) \setminus \bigcup_{i=1}^{k} [\omega_i^1(I) \cup S_i \cup \omega_i^2(I)].$$

We assume that $M_0 \cap$ (unbounded component of $E^3 \setminus M) \neq \emptyset$. Similarly we define the sets $N_0, N_1, \ldots, N_n$. The embedding g we define in the following way:

$$g|N_i = \mathrm{id}_{N_i} \quad \text{for} \quad i = 1, 2, \ldots, k,$$
$$g|\eta_j^i(I) = \omega_j^i \cdot (\eta_j^i)^{-1} \quad \text{for} \quad i = 1, 2, \quad j = 1, 2, \ldots, k.$$

Let h_t be an isotopy of $\operatorname{cl}(M \setminus \bigcup_{i=1}^{k} \operatorname{bc} S_i)$ onto itself such that $h_1 \eta_j^1 = \omega_j^1$ for $j = 1, 2, \ldots, k$. Such an isotopy exists by Lemma (1.15). Without loss of generality we may assume that h_1 maps the almost regular neighbourhoods of $\eta_j^1(I)$ for $j = 1, 2, \ldots, k$ which we use in the operation of taking out 2-spheres from N onto the almost regular neighbourhoods of $\omega_j^1(I)$ which we use in the operation of taking out 2-spheres from M. So $h_1|N_0$ maps N_0 into M_0.

For each point $\eta_j^1(1)$, $j = k+1, k+2, \ldots, n$, let us choose a 3-cell B_j in M_0 such that

$$B_j \cap \partial M_0 \quad \text{is a 2-disc in } \partial M \cap \partial M_0,$$
$$\eta_j^1(1) \in \partial B_j,$$
$$B_i \cap B_j = \emptyset \quad \text{if} \quad \eta_i^1(1) \neq \omega_j^1(1).$$

Let $f_t\colon M_0 \to M_0$ be an isotopy such that f_t is constant outside a small neighbourhood of $\bigcup_{i=k+1}^{n} B_i$ and $f_1(\operatorname{int} B_i) \cap B_i = \emptyset$. We define

$$g|N_0 = f_1 h_1 | N_0$$

and

$$g|\eta_j^2(I) \cup S_j \cup \eta_j^1(I) = \text{an arbitrary embedding into } B_j \text{ such that}$$
$$g\eta_j^1(1) = \eta_j^1(1) \text{ for } j = k+1, k+2, \ldots, n.$$

It is obvious that g has all the required properties.

Let us formulate the following version of the well known "sphere theorem":

(1.19) THEOREM ([13] Theorem 4.11, p. 50). *Let* M *be an oriented manifold. If the group* $\pi_2(M)$ *is non-trivial, then there is a homotopically non-trivial embedding* $h\colon S^2 \to \operatorname{int} M$.

This theorem implies the following

(1.20) COROLLARY. *Let* $M \subset E^3$ *be a manifold and let* $S_M = \{S_1, S_2, \ldots, S_k\}$ *be a full system of 2-spheres in* M. *For any collection of paths*

$$\Omega_M = \{\omega_j^i, i = 1, 2, j = 1, 2, \ldots, k\},$$

such that $M(S_M, \Omega_M)$ *is well defined, the closure of every component of*

$$M(S_M, \Omega_M) \setminus \bigcup_{i=1}^{k} [\omega_i^1(I) \cup S_i \cup \omega_i^2(I)]$$

is an aspherical manifold.

In order to prove the next lemma we need the following two facts:

(1.21) THEOREM ([13] Theorem 8.6, p. 73). *If* M *is a manifold with* $\pi_1(M)$ *finitely generated then there is a compact manifold* $Q \subset M$ *with* $i_\#\colon \pi_1(Q) \to \pi_1(M)$ *an isomorphism.*

(1.22) THEOREM ([24] Lemma 2). *If a manifold* M *is a connected sum* $M_1 \# M_2$ *where* M_i *is not a homotopy sphere for* $i = 1, 2$, *then the embedding* $i\colon S^2 \to M_1 \# M_2$ *onto* $\operatorname{cl} M_1 \cap \operatorname{cl} M_2$ *generates a non-trivial element of* $\pi_2(M)$. *Moreover, if* S^2 *embedded in* $\operatorname{int} M$ *does not separate* M, *then it generates a non-trivial element of* $\pi_2(M)$.

(1.23) LEMMA. *Suppose that* M *is a compact oriented aspherical manifold. If* H *is a finitely generated subgroup of the group* $\pi_1(M)$, *then there is a compact aspherical connected manifold* N *with* $\pi_1(N) \approx H$.

Proof. Let L be a covering space over M such that $\pi_1(L) \approx H$. If the index of H in $\pi_1(M)$ is finite, then we set $N = L$. Suppose that the index of H in $\pi_1(M)$ is infinite. Then L is a non compact manifold. From Theorem (1.21) we infer that there is a compact submanifold $N \subset L$ such that $i_\#\colon \pi_1(N) \to \pi_1(L)$ is an isomorphism. Since L is non-compact, we have $\partial N \neq \emptyset$. In order to finish the proof it is sufficient to prove that N may be chosen in such a way that $\pi_2(N) = 0$. Since L is aspherical, Theorem (1.22) implies that each 2-sphere S^2 in L bounds a homotopy 3-cell in L and we may assume that $N = \hat{N}$, which means that N does not have any 2-sphere in ∂N. Suppose $\pi_2(N) \neq 0$. Then by Theorem (1.19) there is a 2-sphere $S \subset \operatorname{int} N$ which is not contractible in N. This means that in the 3-cell B which is bounded by S in L there is a point which does not belong to N. This implies that $B \cap \partial N$ is non-empty. Since $N = \hat{N}$, $\pi_1(B \cap N)$ is non-trivial. Moreover, $\pi_1(N) \approx \pi_1(\operatorname{cl}(N \setminus B)) *$

$*\pi_1(B \cap N)$. But the inclusion $B \cap N \to L$ induces a trivial homomorphism, which contradicts the assumption that $i\colon N \to L$ induces the isomorphism of fundamental groups.

Let us also formulate the following lemma:

(1.24) LEMMA ([13] Lemma 6.7, p. 63). *Suppose that M is a compact oriented manifold. If $\partial\hat{M} \neq \emptyset$, then $\check{H}_1(M, Z)$ is infinite.*

II. Basic theorem. In this section we will use the notation introduced in the preceding chapter.

Let $(X_i, x_0) = \underleftarrow{\lim}\{(X_n^i, x_0), f_{i,n}^{n+1}\}$ be a pointed continuum for every $i = 1, 2, \ldots$ By a bouquet $\bigvee_{i=1}^{\infty}(X_i, x_0)$ we mean a pointed continuum (X, x_0) defined as follows:

$$(X, x_0) = \underleftarrow{\lim}\{(X_n, x_0), g_n^{n+1}\}$$

where

$$(X_n, x_0) = (X_n^1, x_0) \vee (X_n^2, x_0) \vee \ldots \vee (X_n^n, x_0),$$
$$g_n^{n+1}|X_{n+1}^i = f_{i,n}^{n+1} \quad \text{for} \quad i = 1, 2, \ldots, n,$$
$$g_n^{n+1}(X_{n+1}^{n+1}) = \{x_0\}.$$

The continua (X_i, x_0) are called the *leaves of the bouquet* (X, x_0).

(2.1) Remark. If we additionally assume that the set $X_i \setminus \{x_0\}$ is connected for every $i = 1, 2, \ldots$, then our definition will be equivalent to the definition of a disperse bouquet introduced by A. Gmurczyk in [12].

The following theorem will play an important role in the shape theory of compact connected pointed 1-movable subsets of E^3.

(2.2) THEOREM. *Suppose that $(X, x_0) \subset E^3$ is a pointed 1-movable continuum. There are pointed continua (Y, y_0), $(Z, z_0) \subset E^3$ such that*

(i) $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0) + \mathrm{Sh}(Z, z_0)$,

(ii) $(Y, y_0) = \bigcap_{i=1}^{\infty}(P_n, p_0)$ *where* $(P_{n+1}, p_0) \subset (P_n, p_0) \subset E^3$ *is an aspherical polyhedron for every* $n = 1, 2, \ldots$,

(iii) (Z, z_0) *is a bouquet of 2-spheres.*

We start from the following lemma:

(2.3) LEMMA. *Let $X \subset E^3$ be a continuum. Then there is a continuum $Y \subset E^3$ of the same unpointed shape and a sequence of manifolds M_n in E^3 such that the following conditions are satisfied:*

(i) $Y \subset M_{n+1} \subset M_n$ *for every* $n = 1, 2, \ldots$,

(ii) $Y = \bigcap_{n=1}^{\infty} M_n$,

(iii) M_{n+1} *is well embedded in* M_n *for every* $n = 1, 2, \ldots$,

(iv) *there is a sequence (finite or not) $S_1, S_2, \ldots$ of 2-spheres in E^3 and a sequence of integers $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots$ such that $\{S_1, S_2, \ldots, S_{i_n}\}$ is a full system of 2-spheres in M_n for every $n = 1, 2, \ldots$*

Proof. Let N_n be any sequence of manifolds in E^3 which satisfies the following conditions:

$$X \subset N_{n+1} \subset N_n \quad \text{for} \quad n = 1, 2, \ldots,$$
$$X = \bigcap_{n=1}^{\infty} N_n.$$

Let N_{n_i} be a subsequence of a sequence N_n which has the following property:

if $x \notin N_{n_i}$ and $N_{n_i}(x)$ is bounded, then $N_{n_j}(x)$ is a bounded set for every $j \geqslant i$ or $N_{n_i}(x)$ lies in the unbounded component of $E^3 \setminus N_{n_{i+1}}$.

The existence of such a sequence follows from the fact that the set $E^3 \setminus N_n$ has finitely many components for every $n = 1, 2, \ldots$

Let F_i be the sum of all bounded components of $E^3 \setminus N_{n_i}$ which lay in the unbounded component of $E^3 \setminus N_{n_{i+1}}$.

We define a new sequence N'_n by setting

$$N'_i = N_{n_i} \cup F_i \quad \text{for} \quad i = 1, 2, \ldots$$

It is clear that $X \subset N'_{i+1} \subset N'_i$ and that $X = \bigcap_{i=1}^{\infty} N'_i$. Let N'_{n_i} be a subsequence of a sequence N'_n which satisfies the following condition:

if $x, y \notin N'_{n_i}$ and there is a 2-sphere S in $\mathrm{int} N'_{n_i}$ such that $N'_{n_i}(x) \subset \mathrm{bc} S$ and $N'_{n_i}(y) \cap \mathrm{bc} S = \emptyset$, then there is no 2-sphere in $\mathrm{int} N'_{n_{i+1}}$ which divides E^3 between $N'_{n_{i+1}}(x)$ and $N'_{n_{i+1}}(y)$, or for every $k > n_i$ there is a 2-sphere S_k in $\mathrm{int} N'_k$ which divides E^3 between $N'_k(x)$ and $N'_k(y)$.

We use the sequence N'_{n_i} to obtain a new sequence M'_n defined by induction in the following way:

Let us divide the set $\{F_0, F_1, \ldots, F_k\}$ of components of $E^3 \setminus N'_{n_1}$, where F_0 is the unbounded component, into disjoint subsets $C_1, C_2, \ldots, C_s$ such that $F_i, F_j \in C_l$ if and only if no 2-sphere in $\mathrm{int} N'_{n_2}$ divides E^3 between F_i and F_j. If no 2-sphere in $\mathrm{int} N'_{n_1}$ divides E^3 between F_i and F_j for $F_i, F_j \in C_l$ and $l = 1, 2, \ldots, p$, then we set

$$M'_1 = N'_{n_1}.$$

If in N'_{n_1} there is a 2-sphere S which divides E^3 between F_i and F_j for $F_i, F_j \in C_l$, then we remove from C_l the set F_i or the set F_j where $i \neq 0 \neq j$ (but only one). After finitely many steps we can find for every $l = 1, 2, \ldots, s$ a non-empty subset $C'_l \subset C_l$ such that no 2-sphere in $\mathrm{int} N'_{n_1}$ divides E^3 between F_i and F_j for $F_i, F_j \in C'_l$ and $l = 1, 2, \ldots, s$.

Now we set

$$M'_1 = N'_{n_1} \cup \{F_i\colon F_i \notin \bigcup_{i=1}^{s} C'_i\}.$$

Suppose we have defined manifolds $M'_1, \ldots, M'_k$ such that

$$M'_i \supset N'_{n_i} \supset M'_{i+1}$$

and

for any $x, y \notin M'_i$ such that there is a 2-sphere S in $\operatorname{int} M'_i$ which divides E^3 between $M'_i(x)$ and $M'_i(y)$ there is a 2-sphere in $\operatorname{int} N'_{n_i+1}$ which divides E^3 between $N'_{n_i+1}(x)$ and $N'_{n_i+1}(y)$.

If for every $x, y \notin N'_{n_k+1}$ and a 2-sphere in $\operatorname{int} N'_{n_k+1}$ which divides E^3 between $N'_{n_k+1}(x)$ and $N'_{n_k+1}(y)$, there is a 2-sphere in $\operatorname{int} N'_{n_k+2}$ which divides E^3 between $N'_{n_k+2}(x)$ and $N'_{n_k+2}(y)$, then we set

$$M'_{k+1} = N'_{n_k+1}.$$

Otherwise let us divide, as above, the set of all components of $E^3 \setminus N'_{n_k+1}$ into disjoint sets $C_1, C_2, \ldots, C_p$ such that if $F_i, F_j \in C_l$ then no 2-sphere in $\operatorname{int} N'_{n_k+2}$ divides E^3 between F_i and F_j for $l = 1, 2, \ldots, p$. Let us observe that

(2.4) if $F_i, F_j \in C_l$ and $F_i \cap \operatorname{bc} M'_k \neq \emptyset \neq F_j \cap \operatorname{bc} M'_k$, then there is no 2-sphere in $\operatorname{int} N'_{n_k+1}$ which divides E^3 between F_i and F_j.

For every $l = 1, 2, \ldots, p$ let us choose a subset $C'_l \subset C_l$ such that

if there is an $F_i \in C_l$ with $F_i \cap \operatorname{bc} M'_k \neq \emptyset$, then $F_i \in C'_l$; if there is a 2-sphere S in $\operatorname{int} N'_{n_k+1}$ which divides E^3 between F_i and F_j and $F_i \cap \operatorname{bc} M'_k = F_j \cap \operatorname{bc} M'_k = \emptyset$, then C'_l contains exactly one of these sets.

We set

$$M'_{k+1} = N'_{n_k+1} \cup \{F_i\colon F_i \notin \bigcup_{i=1}^{p} C'_i\}.$$

Condition (2.4) implies that $M'_{k+1} \supset N'_{n_k} \supset M'_k$ for every $k = 1, 2, \ldots$ and hence $\bigcap_{k=1}^{\infty} M'_k = X$. Therefore M'_k is a sequence of manifolds which satisfies conditions (i)–(iii) of Lemma (2.3) for $Y = X$.

The sequence M_k and the family $\{S_1, S_2, \ldots\}$ of spheres will be defined inductively as follows:

$M_1 = M'_1$ and $\{S_1, S_2, \ldots, S_{i_1}\}$ is any full system of 2-spheres in M_1. Such a system exists by Lemma (1.7).

Suppose we have defined $M_1, M_2, \ldots, M_k$, $S_1, S_2, \ldots, S_{i_k}$ such that $M_1 \supset M_2 \supset \ldots \supset M_k$, $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k$, $\{S_1, S_2, \ldots, S_{i_j}\}$ is a full system of 2-spheres in M_j, M_{i+1} is well embedded in M_i, and there is a homeomorphism $h_i\colon M'_i \to M_i$ such that the diagram

$$\begin{array}{ccc} M_i & \supset & M_{i+1} \\ \uparrow{\scriptstyle h_i} & & \uparrow{\scriptstyle h_{i+1}} \\ M'_i & \supset & M'_{i+1} \end{array}$$

commutes up to homotopy for every $i = 1, 2, \ldots, k-1$.

Since M'_{k+1} is well embedded in M'_k, $h_k(M'_{k+1})$ is well embedded in $h_k(M'_k) = M_k$. Therefore by Lemma (1.11) there is a full system $\{S'_1, S'_2, \ldots, S'_{i_{k+1}}\}$ of 2-spheres in $h_k(M'_{k+1})$ and an isotopy $f_t\colon M_k \to M_k$ which is constant onto ∂M_k and such that

$$f_1(S'_i) = S_i \quad \text{for} \quad i = 1, 2, \ldots, i_k.$$

We set

$$M_{k+1} = f_1 h_k(M'_{k+1})$$

and

$$h_{k+1} = f_1 h_k,$$

and

$$S_{i_k+1} = h_{k+1}(S'_{i_k+1}), \ldots, S_{i_{k+1}} = h_{k+1}(S'_{i_{k+1}}).$$

Since f_t is an isotopy, the above diagram is commutative up to homotopy, which implies that $Y = \bigcap_{k=1}^{\infty} M_k$ has the same unpointed shape as X. The proof is finished.

(2.5.) Corollary. *A pointed continuum $(X, x_0) \subset E^3$ has a non-trivial pro-$\pi_2(X, x_0)$ if and only if $\operatorname{Sh}(X) \geqslant \operatorname{Sh}(S^2)$.*

Proof. Let Y be a pointed continuum such that $\operatorname{Sh}(Y) = \operatorname{Sh}(X)$ and there is a sequence M_n which satisfies all the conditions given in Lemma (2.3). If (X, x_0) is not approximatively 2-connected, then by the "sphere theorem" (Theorem 1.18) the full system of 2-spheres in M_n is non-empty for almost all n and hence Y contains a 2-sphere S which is a retract of Y. Therefore $\operatorname{Sh}(X) \geqslant \operatorname{Sh}(S^2)$.

In the proof of Theorem (2.2) we will need a characterization of pointed 1-movability in terms of approximative paths (see [18]).

Suppose that X_n is a sequence of ANR's such that $X_{n+1} \subset X_n$ for every $n = 1, 2, \ldots$ and let $X = \bigcap_{n=1}^{\infty} X_n$. Let $x, y \in X$. By an approximative path from x to y we mean a family $\underline{\omega} = \{\omega_n\}$ of maps such that

$$\omega_n\colon (I, 0, 1) \to (X_n, x, y),$$
$$\omega_n \simeq \omega_m \operatorname{rel} \partial I \quad \text{in } X_{\min(n,m)}$$

for every $n, m = 1, 2, \ldots$

The following theorem is a special case of Theorem 3.1, p. 151 in [18].

(2.6) Theorem. *Let $X = \bigcap_{n=1}^{\infty} X_n$ where $X_{n+1} \subset X_n$ and X_n is a connected compact ANR for every $n = 1, 2, \ldots$ Then X is pointed 1-movable if and only if for every points $x, y \in X$ there is an approximative path from x to y.*

We will also need the following result:

(2.7) Theorem ([8] Proposition, p. 60). *If $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$ and (X, x_0) is pointed 1-movable, then for any $y \in Y$, the pointed continuum (Y, y_0) is pointed 1-movable and $\operatorname{Sh}(X, x_0) = \operatorname{Sh}(Y, y_0)$.*

Now we are ready to prove Theorem (2.2).

Proof of Theorem (2.2). Let $(X, x_0) \subset E^3$ be a pointed 1-movable continuum. By Theorem (2.7) and Lemma (2.3) we may assume that there is a sequence M_n of manifolds in E^3 which satisfies conditions (i)–(v) given in Lemma (2.3).

In each bounded component of $E^3 \setminus X$ let us choose one point. Let $\{x_1, x_2, \ldots\}$ be the set of all these points. Let x_∞ be any point in the unbounded component of $E^3 \setminus X$. Let us choose a point $y_n \in X$ such that the segment $I_n = \{z \in E^3\colon$ there is $t \in [0, 1]$ such that $tx_n + (1-t)y_n = z\}$ has only one point common with X for every $n = 1, 2, \ldots$

Let D be a PL-disc in $E^3 \setminus X$ such that $x_\infty \in \operatorname{int} D$ and $I_\infty \cap D = \{x_\infty\}$. Let us consider the continuum

$$X' = X \cup \bigcup_{n=1}^{\infty} I_n \cup D \cup I_\infty .$$

It is clear that X is a strong deformation retract of X' and hence has the same pointed shape. Moreover, one can see that by adding some PL-3-cells to each M_n we can obtain a sequence M'_n of manifolds in E^3 such that

$$\bigcap_{n=1}^{\infty} M'_n = X' ,$$

a sequence M'_n satisfies conditions (i)–(iv) of (2.3),

$$D \subset \partial M'_n \quad \text{for every } n = 1, 2, \ldots ,$$

$$x_n \in \partial M'_n \cup \operatorname{int} M_n .$$

Let $S_1, S_2, \ldots$ be a family of 2-spheres in X' and let $i_1 \leqslant i_2 \leqslant \ldots$ be a sequence of integers such that $\{S_1, \ldots, S_{i_k}\}$ is a full system of 2-spheres in M'_k.

We choose one point $z_i \in S_i$ for each $i = 1, 2, \ldots$ Since $\operatorname{Sh}(X) = \operatorname{Sh}(X')$ and (X, x_0) is pointed 1-movable, then by (2.7) (X', x) is pointed 1-movable for any $x \in X'$ and hence we can find approximative paths $\underline{\omega}_n = \{\omega_n^i\}$, $\underline{\eta}_n = \{\eta_n^i\}$ such that:

$\underline{\omega}_n$ is an approximative path from z_n to x_∞,
$\underline{\eta}_n$ is an approximative path from z_n to x_n,

for every $n = 1, 2, \ldots$ Without loss of generality we may assume that

ω_n^i is a PL-embedding,
$\omega_n^i(I) \cap \partial M'_i = \omega_n^i(1) = x_\infty$,
$\omega_n^k(I) \cap \omega_m^k(I) = \{x_\infty\}$,
$\omega_m^k(\operatorname{int} I) \subset M'_k \setminus \bigcup_{j=1}^{k} \operatorname{bc} S_j$,
$\omega_n^k \simeq \omega_n^l \operatorname{rel} \partial I$ in $M'_{\min(k,l)} \setminus \bigcup_{s=1}^{\min(k,l)} \operatorname{int} \operatorname{bc} S_s$,
η_n^i is a PL-embedding,
$\eta_n^i(I) \subset \operatorname{bc} S_n \cap M'_i$,
$\eta_n^i(I) \cap (S_n \cup \partial M'_i) = \{z_n, x_n\}$,
$\eta_n^i \simeq \eta_n^k \operatorname{rel} \partial I$ in $M'_{\min(i,k)} \cap \operatorname{bc} S_n$,

for every $i, k, l, m, n = 1, 2, \ldots$

Let $S_{M_n} = \{S_1, S_2, \ldots, S_{i_n}\}$, $\Omega_{M_n} = \{\omega_n^1, \ldots, \omega_n^{i_n}, \eta_n^1, \ldots, \eta_n^{i_n}\}$. Then by Lemma (1.18) there is an embedding

$$g_n\colon M'_{n+1}(S_{M'_{n+1}}, \Omega_{M'_{n+1}}) \to M'_n(S_{M'_n}, \Omega_{M'_n})$$

such that the following diagram commutes up to homotopy for every $n = 1, 2, \ldots$

$$\begin{array}{ccc} M'_n(S_{M'_n}, \Omega_{M'_n}) & \xleftarrow{g_n} & M'_{n+1}(S_{M'_{n+1}}, \Omega_{M'_{n+1}}) \\ \cap & & \cap \\ M'_n & \supset & M'_{n+1} \end{array}$$

Hence $\operatorname{Sh}(X') = \operatorname{Sh}(\underleftarrow{\lim} \{M'_n(S_{M'_n}, \Omega_{M'_n}), g_n\})$. If we now contract all $\omega_n^i(I)$ to a point x_∞, then the resulting continuum is, by Lemma (1.18) and Corollary (1.20), a bouquet each leaf of which is a 2-sphere or an intersection of aspherical manifolds. The proof is finished.

Remark. The assumption that (X, x_0) is pointed 1-movable is essential. In order to see that, it is sufficient to change S^1 into S^2 in T. Watanabe's example given in [29] p. 239.

The following corollary is an immediate consequence of the above theorem:

(2.8) Corollary. *A pointed continuum $(X, x_0) \subset E^3$ is pointed movable if and only if a pro-group pro-$\pi_1(X, x_0)$ is movable.*

Examples of locally connected (and therefore pointed 1-movable) non-movable E^3 subcontinua (see [1] and [21]) show that the assumption of movability of pro-$\pi_1(X, x_0)$ cannot be replaced by a weaker one where pro-$\pi_1(X, x_0)$ satisfies the Mittage–Leffler condition.

The following corollary does not require any proof either:

(2.9) Corollary. *Suppose (X, x_0) is a pointed 1-movable subcontinuum of E^3. Then (X, x_0) is approximatively n-connected for $n \geqslant 2$ if and only if $\operatorname{Sh}(X, x_0)$ does not dominate $\operatorname{Sh}(S^2, s_0)$.*

The next corollary describes all the approximatively 1-connected continua up to shape.

(2.10) Corollary. *If the pointed continuum $(X, x_0) \subset E^3$ is approximatively 1-connected and $\operatorname{Sh}(X, x_0)$ is non-trivial, then (X, x_0) is pointed movable and has the shape of a bouquet of 2-spheres.*

III. Shapes of pointed FANR-spaces in E^3. The question whether it is true that every FANR-space is of a polyhedral shape — is generally answered in the negative. D. A. Edwards and R. Geoghegan in [11] have constructed a pointed FANR-space of fundamental dimension two which is not of a polyhedral shape. The following problem, raised by K. Borsuk in [2] (Problem 8.1, p. 350), remains open:

Is it true that every FANR-set lying in E^3 is of a polyhedral shape?

The main theorem of this section states that in the case of pointed FANR-sets the answer to the above question is positive.

The proof of this theorem is based on the following two lemmas:

(3.1) Lemma. *If $(X, x_0)\subset E^3$ is a pointed 1-movable continuum with the countable $\underline{\pi}_1(X, x_0)$, then (X, x_0) is pointed movable.*

Proof. By Theorem (2.2) we may assume that $(X, x_0) = \bigcap_{n=1}^{\infty}(X_n, x_0)\vee(Z, z_0)$ where $(X_n, x_0)\supset(X_{n+1}, x_0)$ is an aspherical manifold in E^3 for $n = 1, 2, \ldots$ and (Z, z_0) is a bouquet of 2-spheres. Since the operation of one-point union preserves pointed movability, it is sufficient to show that (X, x_0) is pointed movable. Since pro-$\pi_1(X, x_0)$ satisfies the Mittage–Leffler condition and $\underline{\pi}_1(X, x_0)$ is countable, pro-$\pi_1(X, x_0)$ is stable (see [7], Corollary 2.19, p. 13). Hence this pro-group is movable. Now Lemma (3.1) follows from Corollary (2.8).

(3.2) Lemma. *Suppose (X, x_0) is a pointed 1-movable continuum in E^3 with the countable $\underline{\pi}_1(X, x_0)$. If pro-$\pi_2(X, x_0)$ is trivial then (X, x_0) is of a polyhedral shape.*

Proof. Theorem (2.2) implies $(X, x_0) = \bigcap_{n=1}^{\infty}(X_n, x_0)$ where $(X_{n+1}, x_0)\subset(X_n, x_0)$ is an aspherical manifold for $n = 1, 2, \ldots$ Lemma (3.1) implies that (X, x_0) is pointed movable. Hence (X, x_0) is a pointed FANR-space (see [7], Theorem 7.12, p. 37). This means that there is an index n_0 such that $\mathrm{Sh}(X_{n_0}, x_0)\geqslant\mathrm{Sh}(X, x_0)$. Let Q be an aspherical manifold such that $\pi_1(Q, q_0)\approx\underline{\pi}_1(X, x_0)$ (see Lemma 1.23). Let

$$\underline{f}\colon (X_{n_0}, x_0)\to(X, x_0) \quad \text{and} \quad \underline{g}\colon (X, x_0)\to(X_{n_0}, x_0)$$

be shape morphisms such that $\underline{fg}\simeq\underline{\mathrm{id}}_{(X,x_0)}$. Let $h\colon (Q, q_0)\to(X_{n_0}, x_0)$ be a map which induces an isomorphism $h_{\#}\colon \pi_1(Q, q_0)\to\mathrm{im}(\underline{g})_{\#}\subset\underline{\pi}_1(X, x_0)$ (see [26], Theorem 9, p. 427). The shape morphism $\underline{f}h$ induces an isomorphism

$$(\underline{f}h)_{\#,n}\colon \pi_n(Q, q_0)\to\underline{\pi}_n(X, x_0)$$

for all $n = 1, 2, \ldots$ and hence it is a shape equivalence (see [22] p. 250).

The main theorem is the following:

(3.3) Theorem. *Suppose $(X, x_0)\subset E^3$ is a pointed connected FANR-space. There is a polyhedron (P, p_0) such that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(P, p_0)$.*

Proof. Suppose $(X, x_0)\subset E^3$ is a pointed connected FANR-set. By Theorem (2.2) we may assume that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)+\mathrm{Sh}(Z, z_0)$ where (Z, z_0) is a bouquet of 2-spheres and $\mathrm{Sh}(Y, y_0)$ is approximatively 2-connected. Since $\mathrm{Sh}(Y, y_0)\leqslant\mathrm{Sh}(X, x_0)$, (Y, y_0) is a pointed FANR-set and hence by Lemma (3.2) is of a polyhedral shape. Since (X, x_0) is FANR-set, $\check{H}_2(X, Z)$ is finitely generated and hence (Z, z_0) is a finite bouquet of 2-spheres. The proof is finished.

Let us prove the following corollary:

(3.4) Corollary. *Suppose (X, x_0), $(Y, y_0)\subset E^3$ are pointed connected FANR-sets. Then $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$ if and only if $\underline{\pi}_1(X, x)\approx\underline{\pi}_1(Y, y)$ and $\check{H}_2(X, Z)\approx\check{H}_2(Y, Z)$.*

Proof. By Theorems (2.2) and (3.3) we may assume that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(X_1, x_0)+\mathrm{Sh}(X_2, x_0)$ and $\mathrm{Sh}(Y, y_0) = \mathrm{Sh}(Y_1, y_0)+\mathrm{Sh}(Y_2, y_0)$ where (X_1, x_0), (Y_1, y_0) are aspherical polyhedra and (X_2, x_0), (Y_2, y_0) are finite bouquets of 2-spheres. The condition $\underline{\pi}_1(X, x_0)\approx\underline{\pi}_1(Y, y_0)$ implies $\underline{\pi}_1(X_1, x_0)\approx\underline{\pi}_1(Y_1, y_0)$ and hence $(X_1, x_0)\simeq(Y_1, y_0)$. Since the groups $\check{H}_2(X, Z)$ and $\check{H}_2(Y, Z)$ are finitely generated and $(X_1, x_0)\simeq(Y_1, y_0)$, the condition $\check{H}_2(X, Z)\approx\check{H}_2(Y, Z)$ implies $\check{H}_2(X_2, Z)\approx\check{H}_2(Y_2, Z)$ and hence $(X_2, x_0)\simeq(Y_2, y_0)$. This implies $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$. The inverse implication is obvious. The proof is finished.

Theorem (3.3) states that a pointed FANR-set in E^3 has a polyhedral shape. The following problem remains open:

(3.5) Problem. Let (X, x_0) be a connected pointed FANR-set in E^3. Is it true that there is a pointed polyhedron $(P, p_0)\subset E^3$ of the same shape?

Suppose $(P, p_0)\subset E^3$ is a connected aspherical polyhedron. Let (X, x_0) be a pointed continuum (not necessarily in E^3) such that $\mathrm{Sh}(P, p_0)\geqslant\mathrm{Sh}(X, x_0)$. Then (X, x_0) is a pointed FANR-set and $\underline{\pi}_1(X, x_0)$ is a finitely generated subgroup of a group $\pi_1(P, p_0)$. By Lemma (1.23) we infer that there is an aspherical polyhedron (Q, q_0) with $\pi_1(Q, q_0)\approx\underline{\pi}_1(X, x_0)$. Hence $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Q, q_0)$. We have obtained the following theorem:

(3.6) Theorem. *Suppose $(P, p_0)\subset E^3$ is a connected aspherical polyhedron. If $\mathrm{Sh}(P, p_0)\geqslant\mathrm{Sh}(X, x_0)$, then there is a polyhedron (Q, q_0) such that $\mathrm{Sh}(Q, q_0) = \mathrm{Sh}(X, x_0)$.*

The following problem remains open:

(3.7) Problem. *Suppose $(P, p_0)\subset E^3$ is a connected polyhedron and $\mathrm{Sh}(P, p_0)\geqslant\mathrm{Sh}(X, x_0)$. Is it true that (X, x_0) is of a polyhedral shape?*

The above problem seems to be rather difficult. One can prove that it is equivalent to the following algebraic question:

(3.8) Is it true that for every polyhedron $(P, p_0)\subset E^3$ the group $\tilde{K}_0(\pi_1(P, p_0))$ is trivial?

(For definition of $\tilde{K}_0(G)$ see [28]).

Let us note that, if $(P, p_0)\subset E^3$ is a polyhedron and $\mathrm{Sh}(P, p_0)$ dominate $\mathrm{Sh}(X, x_0)$, then (X, x_0) need not be embedded up to shape in E^3 (see [16]).

(3.9) Remark. Recently J. Dydak and H. Hastings have proved that every FANR-set of fundamental dimension 2 is a pointed FANR-set. Therefore all the results of this section are true for unpointed FANR-sets in E^3 (see [9]).

IV. Characterizations of shapes of E^3 pointed subcontinua. Suppose $(X, x_0)\subset E^3$ is a pointed continuum. We denote the bouquet of n-copies of 2-spheres by S_n. Let us define number $s(X, x_0)$ by setting

$$s(X, x_0) = \max\{n\colon \mathrm{Sh}(X, x_0)\geqslant\mathrm{Sh}(S_n, s_0)\}.$$

If such a number does not exist, then we set $s(X, x_0) = \infty$.

If we want to describe fully the shape of the pointed continuum $(X, x_0)\subset E^3$, it is sufficient to know the number $s(X, x_0)$ and the pro-group pro-$\pi_1(X, x_0)$. Namely the following theorem holds:

(4.1) THEOREM. *Let (X, x_0), $(Y, y_0) \subset E^3$ be pointed 1-movable continua. Then the following conditions are equivalent*:

(i) $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$,

(ii) pro-$\pi_1(X, x_0) \approx$ pro-$\pi_1(Y, y_0)$ *and* $s(X, x_0) = s(Y, y_0)$.

Proof. Implication (i)⇒(ii) is obvious. Suppose condition (ii) holds. By Theorem (2.2) there exist continua (X_1, x_0), (X_2, x_0), (Y_1, y_0), (Y_2, y_0) such that

$\mathrm{Sh}(X, x_0) = \mathrm{Sh}(X_1, x_0) + \mathrm{Sh}(X_2, x_0)$,
$\mathrm{Sh}(Y, y_0) = \mathrm{Sh}(Y_1, y_0) + \mathrm{Sh}(Y_2, y_0)$,
(X_1, x_0), (Y_1, y_0) are inverse limits of inverse sequences of aspherical polyhedra,
(X_2, x_0), (Y_2, y_0) are bouquets of 2-spheres.

Since (X_1, x_0) and (Y_1, y_0) are inverse limits of inverse sequences of aspherical polyhedra and

$$\text{pro-}\pi_1(X_1, x_0) = \text{pro-}\pi_1(X, x_0) \approx \text{pro-}\pi_1(Y, y_0) = \text{pro-}\pi_1(Y_1, y_0),$$

the isomorphism of these pro-groups is induced by a shape morphism (see [26], Theorem 9, p. 427). Therefore, by the generalized Whitehead theorem (see [7], Theorem 6.2′, p. 27), we infer that $\mathrm{Sh}(X_1, x_0) = \mathrm{Sh}(Y_1, y_0)$. Since $\mathrm{Sh}(X_1, x_0)$ and $\mathrm{Sh}(Y_1, y_0)$ does not dominate $\mathrm{Sh}(S^2, s_0)$, the condition $s(X, x_0) = s(Y, y_0)$ implies that $\mathrm{Sh}(X_2, x_0) = \mathrm{Sh}(Y_2, y_0)$, and therefore $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$. The proof is finished.

(4.2) Remark. Easy examples show that the condition $s(X, x_0) = s(Y, y_0)$ in the above theorem cannot be replaced by a weaker one where $\check{H}_2(X, Z) \approx \check{H}_2(Y, Z)$.

In [2] K. Borsuk has raised the following problem (Problem 8.2, p. 351):

Is it true that every movable compactum X for which the group $\underline{\pi}_1(X, x_0)$ is finitely generated for every $x_0 \in X$ and all Betti numbers $p_n(X)$ are finite and $\mathrm{Fd}(X) < \infty$ is a FANR-set?

In the case of pointed continua the following theorem gives a partially positive answer:

(4.3) THEOREM. *If $(X, x_0) \subset E^3$ is a pointed 1-movable continuum such that $\underline{\pi}_1(X, x_0)$ is countable and the group $\check{H}_2(X, Z)$ is finitely generated, then (X, x_0) is a pointed* FANR-*set.*

Proof. By Theorem (2.2) we may assume that $(X, x_0) = (Y, y_0) \vee (Z, z_0)$ where (Y, y_0) is the inverse limit of an inverse sequence of aspherical polyhedra in E^3 and (Z, z_0) is a bouquet of 2-spheres. Since the group $\check{H}_2(X, Z)$ is finitely generated, (Z, z_0) is a finite bouquet. Moreover, since $\underline{\pi}_1(X, x_0) \approx \underline{\pi}_1(Y, y_0)$ is countable and pro-$\pi_2(Y, y_0)$ is trivial, Lemma (3.2) implies that (Y, y_0) is of a polyhedral shape. This means that (X, x_0) is of a polyhedral shape and therefore it is a pointed FANR-space. The proof is completed.

Let us note the following corollary:

(4.4) COROLLARY. *A pointed 1-movable continuum $(X, x_0) \subset E^3$ has a trivial shape if and only if the groups $\underline{\pi}_1(X, x_0)$ and $\check{H}_2(X, Z)$ are trivial.*

Let us observe that the above corollary may also easily be obtained from Corollaries (2.8) and (2.10).

Another characterization of pointed continua of trivial shape in E^3 is given in the following theorem:

(4.5) THEOREM. *A pointed continuum $(X, x_0) \subset E^3$ has a trivial shape if and only if it is pointed 1-movable and the groups $\check{H}_1(X, Z)$ and $\check{H}_2(X, Z)$ are trivial.*

Proof. By Theorem (2.2) we may assume that $(X, x_0) = (Y, y_0) \vee (Z, z_0)$ where $(Y, y_0) = \bigcap_{n=1}^{\infty} (Y_n, y_0)$ and $(Y_{n+1}, y_0) \subset (Y_n, y_0)$ is an aspherical polyhedron in E^3 for $n = 1, 2, \ldots$ and (Z, z_0) is a bouquet of 2-spheres. The condition $\check{H}_2(X, Z) = 0$ implies that $\mathrm{Sh}(Z, z_0)$ is trivial. Suppose $\mathrm{Sh}(Y, y_0)$ is non-trivial. Since (Y, y_0) is pointed 1-movable, pro-$\pi_1(Y, y_0)$ satisfies the Mittage–Leffler condition. Without loss of generality we may assume that for every n and $m > 1$ the following condition holds:

$$(i_n^{n+1})_\#(\pi_1(Y_{n+1}, y_0)) = (i_n^{n+m})_\#(\pi_1(Y_{n+m}, y_0)).$$

Let us denote the group $(i_n^{n+1})_\#(\pi_1(Y_{n+1}, y_0))$ by A_n. It is clear that the pro-group $\{A_n, i_n^{n+1}\}$ is isomorphic to pro-$\pi_1(Y, y_0)$. Moreover, the homomorphism

$$(i_n^{n+1})_\#\colon A_{n+1} \to A_n$$

is an epimorphism for every $n = 1, 2, \ldots$

By Lemma (1.23) we can find aspherical PL-3-manifold P_n such that $\pi_1(P_n, p_0) \approx A_n$ for every $n = 1, 2, \ldots$ Since (P_n, p_0) is aspherical, we can find a continuous map

$$g_n^{n+1}\colon (P_{n+1}, p_0) \to (P_n, p_0)$$

such that

$$(g_n^{n+1})_\# = (i_n^{n+1})_\# | A_{n+1}.$$

Therefore

$$\mathrm{Sh}(Y, y_0) = \mathrm{Sh}(\underleftarrow{\lim}\{(P_n, p_0), g_n^{n+1}\}).$$

From our construction it follows that $(g_n^{n+1})_\#$ is an epimorphism for every $n = 1, 2, \ldots$ Since P_n is aspherical and not contractible, we have $P_n = \hat{P}_n$. Moreover, $(X, x_0) \subset E^3$ implies $\partial P_n \neq \emptyset$ for every $n = 1, 2, \ldots$ Therefore, by Lemma (1.24) we infer that $\check{H}_1(P_n, Z)$ is infinite and

$$(g_n^{n+1})_*\colon \check{H}_1(P_{n+1}, Z) \to \check{H}_1(P_n, Z)$$

is an epimorphism. Therefore the group

$$\check{H}_1(Y, Z) = \check{H}_1(\underleftarrow{\lim}\{(P_n, p_0), g_n^{n+1}\}, Z)$$

is non-trivial, which contradicts the assumption $\check{H}_1(Y, Z) = \check{H}_1(X, Z) = 0$. The inverse implication is obvious. The proof is completed.

Recently J. Krasinkiewicz and P. Minc have proved that arcwise connected continua are pointed 1-movable (see [18]). Combining this result with Theorem (4.5), one can obtain the following:

(4.6) COROLLARY. *Let $X \subset E^3$ be an arcwise connected continuum. If $\check{H}_1(X, Z)$ and $\check{H}_2(X, Z)$ are trivial, then for every $x_0 \in X$ the shape $\mathrm{Sh}(X, x_0)$ is trivial.*

As a special case we have obtained the following corollary (see [3], Problem 6.1, p. 216):

(4.7) COROLLARY. *A locally contractible continuum $X \subset E^3$ is an AR-set if and only if X is acyclic (i.e. $\check{H}_i(X, Z) = 0$ for $i = 1, 2, \ldots$).*

V. Pointed continua in E^3 and suspension. We denote by ΣX the suspension of a continuum X. Let us prove the following theorem:

(5.1) THEOREM. *Suppose $(X, x_0) \subset E^3$ is a pointed continuum. If $E^3 \setminus X$ is connected, then $(\Sigma X, x_0)$ has the shape of the inverse limit of an inverse sequence of finite bouquets of 2-spheres. If, moreover, pro-$\check{H}_1(X, Z)$ satisfies the Mittag-Leffler condition, then $(\Sigma X, x_0)$ has the shape of a bouquet of 2-spheres.*

Proof. Since $E^3 \setminus X$ is connected, we infer by the well known Alexander duality theorem ([26], Theorem 16, p. 295) that $\check{H}_2(X, Z) = 0$. Since $\mathrm{Fd}(X) \leqslant 2$, $\check{H}^n(X, Z)$ is trivial for $n > 2$. We recall that

$$c(X) = \max\{n\colon \check{H}^n(X, Z) \neq 0\},$$

and if X is approximatively 1-connected and has a finite fundamental dimension, then $\mathrm{Fd}(X) = c(X)$ (see [23]). Therefore we have

$$c(\Sigma X) \leqslant c(X) + 1 \leqslant 2$$

and hence $\mathrm{Fd}(\Sigma X) \leqslant 2$. On the other, hand, if pro-$\check{H}_1(X, Z)$ is non-trivial, then pro-$\check{H}_2(\Sigma X, Z)$ is non-trivial and hence $\mathrm{Fd}(\Sigma X) = 2$. In [27] S. Spież has proved that a continuum of fundamental dimension n which is approximatively k-connected for $k = 1, 2, \ldots, n-1$ has the shape of the inverse limit of an inverse sequence of finite bouquets of n-dimensional spheres. If pro-$\check{H}_1(X, Z)$ is trivial, then of course $\mathrm{Sh}(\Sigma X, x_0)$ is trivial. If pro-$\check{H}_1(X, Z)$ satisfies the Mittag-Leffler condition, then $(\Sigma X, x_0)$ is movable and hence has the shape of a bouquet of 2-spheres.

Let us formulate the following corollary:

(5.2) COROLLARY. *Suppose $(X, x_0) \subset E^3$ is an arcwise connected continuum such that the set $E^3 \setminus X$ is connected. Then there is a continuum $(Y, y_0) \subset E^3$ such that $\mathrm{Sh}(\Sigma X, x_0) = \mathrm{Sh}(Y, y_0)$.*

Proof. Since X is arcwise connected, it is pointed 1-movable (see [18]). Hence Corollary (5.2) follows from Theorem (5.1).

Let us prove the following theorem:

(5.3) THEOREM. *Let $(X, x_0) \subset E^3$ be a pointed continuum. If there is a continuum $(Y, y_0) \subset E^3$ such that $\mathrm{Sh}(\Sigma X, x_0) = \mathrm{Sh}(Y, y_0)$, then $E^3 \setminus X$ is connected and there is a continuum $(Z, z_0) \subset E^2$ such that $\mathrm{Sh}(\Sigma X, x_0) = \mathrm{Sh}(\Sigma Z, z_0)$.*

Proof. If $E^3 \setminus X$ is not connected, then by the Alexander duality theorem we infer that $\check{H}^2(X, Z) \neq 0$ and hence $H^3(\Sigma X, Z) \neq 0$, which implies $\mathrm{Fd}(\Sigma X) \geqslant 3$. Since $(\Sigma X, x_0)$ is approximatively 1-connected, (Y, y_0) is also approximatively 1-connected. Therefore by Corollary (2.10) it has the shape of a bouquet of 2-spheres. The existence of the required plane continuum (Z, z_0) is now obvious.

(5.4) Remark. It is not true that if $\mathrm{Sh}(\Sigma X, x_0) = \mathrm{Sh}(Y, y_0)$ where $(Y, y_0) \subset E^3$, then (X, x_0) is pointed 1-movable.

In particular, if X is a Case-Chamberlain acyclic curve, then $\mathrm{Sh}(\Sigma X, x_0)$ is trivial.

(5.5) Remark. Theorems (5.1), (5.3) and Corollary (5.2) are true also in the unpointed version.

VI. Decomposition of the shape of pointed continua in E^3 into a one-point union. We recall that a shape $\mathrm{Sh}(X, x_0)$ is called *simple* if the equation $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0) + \mathrm{Sh}(Z, z_0)$ implies that either $\mathrm{Sh}(Y, y_0)$ or $\mathrm{Sh}(Z, z_0)$ is trivial. If both these shapes are non-trivial, then we call them constituents of the shape $\mathrm{Sh}(X, x_0)$.

Let us prove the following theorem (which, for finite bouquets has been proved by K. Borsuk in [2], Theorem 6.1, p. 136):

(6.1) THEOREM. *Let $\bigvee_{i=1}^{\infty}(X_i, x_0)$ and $\bigvee_{j=1}^{\infty}(Y_j, y_0)$ be bouquets. If there is a bijection $\sigma\colon N \to N$ such that $\mathrm{Sh}(X_i, x_0) = \mathrm{Sh}(Y_{\sigma(i)}, y_0)$ for every $i = 1, 2, \ldots$, then $\mathrm{Sh}\big(\bigvee_{i=1}^{\infty}(X_i, x_0)\big) = \mathrm{Sh}\big(\bigvee_{j=1}^{\infty}(Y_j, y_0)\big)$.*

Proof. Since the bouquets $\bigvee_{i=1}^{\infty}(X_i, x_0)$ and $\bigvee_{j=1}^{\infty}(X_{\sigma(i)}, x_0)$ are homeomorphic (cf. [12], Theorem 3.5, p. 167) then we may assume that $\sigma(i) = i$ for every $i = 1, 2, \ldots$ Let

$$(\alpha^i, a_n^i)\colon \{(X_n^i, x_0), f_{i,n}^{n+1}\} \to \{(Y_n^i, y_0), h_{i,n}^{n+1}\}$$

and

$$(\beta^i, b_n^i)\colon \{(Y_n^i, y_0), h_{i,n}^{n+1}\} \to \{(X_n^i, x_0), f_{i,n}^{n+1}\}$$

be shape morphisms such that

$$(\alpha^i, a_n^i)(\beta^i, b_n^i) \simeq \mathrm{id}_{Y_i} \quad \text{and} \quad (\beta^i, b_n^i)(\alpha^i, a_n^i) \simeq \mathrm{id}_{X_i}$$

for every $i = 1, 2, \ldots$ Let

$$\alpha(n) = \max\{\alpha^1(n), \alpha^2(n), \ldots, \alpha^n(n)\}.$$

We define the map

$$a_n\colon (X_{\alpha(n)}, x_0) \to (Y_n, y_0)$$

by setting

$$a_n | X_{\alpha(n)}^i = a_n^i f_{i,\alpha^i(n)}^{\alpha(n)} \quad \text{for} \quad i = 1, 2, \ldots, n$$

$$a_n(X_{\alpha(n)}^i) = \{y_0\} \quad \text{for} \quad i = n+1, n+2, \ldots, \alpha(n).$$

Similarly, we define

$$\beta(n) = \max\{\beta^1(n), \beta^2(n), \ldots, \beta^n(n)\}$$

and the map

$$b_n\colon \{Y_{\beta(n)}, y_0\} \to (X_n, x_0)$$

by setting

$$b_n| Y^i_{\beta(n)} = b^i_n g^{\beta(n)}_{i,\beta^i(n)} \quad \text{for} \quad i = 1, 2, \ldots, n,$$
$$b_n(Y^i_{\beta(n)}) = \{x_0\} \quad \text{for} \quad i = n+1, n+2, \ldots, \beta(n).$$

Let us observe that for every $i = 1, 2, \ldots$ we have

$$f^{n+1}_{i,n} a_{n+1}| Y^i_{\beta(n+1)} = f^{n+1}_{i,n} a^i_{n+1} g^{\beta(n+1)}_{i,\beta^i(n+1)} = f^{n+1}_{i,n} a^i_{n+1} g^{\beta(n+1)}_{i,\beta^i(n+1)}$$
$$\simeq a^i_n g^{\beta^i(n+1)}_{i,\beta^i(n)} g^{\beta(n+1)}_{i,\beta^i(n+1)} \simeq a^i_n g^{\beta(n+1)}_{i,\beta^i(n)} = a_n g^{\beta(n+1)}_{\beta(n)} | Y^i_{\beta(n+1)},$$

which means that (α, a_n) is a well-defined shape morphism. Through analogical calculation we check that (β, b_n) is a well-defined shape morphism.

Now let us consider the composition

$$a_n b_{\alpha(n)}| Y^i_{\beta\alpha(n)} = a^i_n f^{\alpha(n)}_{i,\alpha^i(n)} b^i_{\alpha^i(n)} g^{\beta\alpha(n)}_{i,\beta^i\alpha^i(n)}$$
$$\simeq g^{\beta^i\alpha^i(n)}_{i,n} g^{\beta\alpha(n)}_{i,\beta^i\alpha^i(n)} \simeq g^{\beta\alpha(n)}_n.$$

Through analogical calculation we check that

$$b_n a_{\beta(n)} \simeq f^{\alpha\beta(n)}_n,$$

which means that (α, a_n) and (β, b_n) are shape equivalences. The proof is finished.

In general, the converse theorem is not true even in the case of finite bouquets whose leaves are pointed FANR-sets of simple shapes. In [6] M. J. Dunwoody has constructed two polyhedra, X and Y, of dimension 2 such that $X \not\simeq Y$ but $X \vee S^2 \simeq Y \vee S^2$. If we now express X and Y as a finite sum of FANR's of simple shapes (see [14], Cor. 24), we obtain two different bouquets of the same shape.

The main result of this section states that in E^3 such an example cannot be constructed even for infinite bouquets. Namely, the following theorem holds:

(6.2). THEOREM. *If* $\mathrm{Sh}(\bigvee_{i=1}^{\infty}(X_i, x_0)) = \mathrm{Sh}(\bigvee_{j=1}^{\infty}(Y_j, y_0))$ *where* (X_i, x_0), $(Y_j, y_0) \subset E^3$ *are pointed compact connected* FANR*'s of simple shapes for every* $i, j = 1, 2, \ldots$, *then there is a bijection* $\sigma\colon N \to N$ *such that* $\mathrm{Sh}(X_i, x_0) = \mathrm{Sh}(Y_{\sigma(i)}, y_0)$ *for every* $i = 1, 2, \ldots$

In order to prove Theorem (6.2) we need two lemmas.

(6.3) LEMMA. *Suppose* (X, x_0) *is a pointed* FANR*-set which is approximatively* n*-connected for* $n \geqslant 2$. $\mathrm{Sh}(X, x_0)$ *is simple if and only if* $\underline{\pi}_1(X, x_0)$ *is indecomposable into a free product.*

Proof. Suppose $\underline{\pi}_1(X, x_0) \approx G_1 * G_2$. Let (W, w_0) be a CW-complex of the same shape as (X, x_0) (see [4] or [11]). Since (X, x_0) is approximatively n-connected for $n \geqslant 2$, then (W, w_0) is aspherical. Let (W_i, w_0) be a pointed CW-complex such that $\pi_1(W_i, w_0) \approx G_i$ for $i = 1, 2$. Then $(W, w_0) \simeq (W_1, w_0) \vee (W_2, w_0)$. Let (P, p_0) be a polyhedron such that $\mathrm{Sh}(P, p_0) \geqslant \mathrm{Sh}(X, x_0)$; then there are maps

$$f\colon (P, p_0) \to (W, w_0), \quad g\colon (W, w_0) \to (P, p_0),$$
$$f_i\colon (P, p_0) \to (W_i, w_0), \quad g_i\colon (W_i, w_0) \to (P, p_0)$$

such that

$$fg \simeq \mathrm{id}_W \operatorname{rel} w_0 \quad \text{and} \quad f_i g_i \simeq \mathrm{id}_{W_i} \operatorname{rel} w_0 \quad \text{for} \quad i = 1, 2.$$

Let us consider the inverse sequences

$$\{(P, p_0), gf\}, \quad \{(P, p_0), g_1 f_1\}, \quad \{(P, p_0), g_2 f_2\}.$$

We have (see [11], Prop. 3.1):

$$\mathrm{Sh}(X, x_0) = \mathrm{Sh}(\underline{\lim}\{(P, p_0), gf\}),$$
$$\mathrm{Sh}(X_i, x_0) = \mathrm{Sh}(\underline{\lim}\{(P, p_0), g_i f_i\})$$

for $i = 1, 2$. It is clear that $\mathrm{Sh}(X_i, x_0) = \mathrm{Sh}(W_i, w_0)$ and (X_i, x_0) is a pointed FANR-set for $i = 1, 2$ and hence

$$\mathrm{Sh}(X, x_0) = \mathrm{Sh}(X_1, x_0) + \mathrm{Sh}(X_2, x_0).$$

Since $\underline{\pi}_1(X_i, x_0) \approx G_i$, the shapes $\mathrm{Sh}(X_i, x_0)$ are non-trivial for $i = 1, 2$.

Now suppose $\underline{\pi}_1(X, x_0)$ is indecomposable into a free product, and let $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(X_1, x_0) + \mathrm{Sh}(X_2, x_0)$. Let (W_i, w_0) be CW-complexes such that $\mathrm{Sh}(X_i, x_0) = \mathrm{Sh}(W_i, w_0)$ for $i = 1, 2$ (see [4] or [11]). Since (X, x_0) is approximatively n-connected for $n \geqslant 2$, (W_i, w_0) are aspherical for $i = 1, 2$. Since $\pi_1(X, x_0)$ is indecomposable, one of the groups $\pi_1(W_i, w_0)$ is trivial. Suppose $\pi_1(W_1, w_0)$ is trivial; then (W_1, w_0) is contractible and hence the shape $\mathrm{Sh}(X_1, x_0)$ is trivial. The proof is finished.

The second lemma is of a purely algebraic character.

(6.4) LEMMA. *Suppose* A, B, C, D *are abelian groups. Let* $r_A\colon A \oplus B \to A$, $r_B\colon A \oplus B \to B$, $r_C\colon C \oplus D \to C$ *and* $r_D\colon C \oplus D \to D$ *be projections defined by the formulas* $r_A(a+b) = a$, $r_B(a+b) = b$, $r_C(c+d) = c$ *and* $r_D(c+d) = d$ *for every* $a \in A$, $b \in B$, $c \in C$, $d \in D$. *If there are isomorphisms* $\varphi\colon A \oplus B \to C \oplus D$ *and* $\psi\colon C \oplus D \to A \oplus B$ *such that* $\psi\varphi = \mathrm{id}_{A \oplus B}$, $(r_C\varphi|A) \circ (r_A\psi|C) = \mathrm{id}_C$ and $(r_A\psi|C) \circ (r_C\varphi|A) = \mathrm{id}_A$, *then* $r_D\varphi|B\colon B \to D$ *is an isomorphism.*

Proof. Let us show that $r_D\varphi|B$ is an epimorphism. Suppose $d^* \in D$. Then there is an element $a+b \in A \oplus B$ such that

$$\varphi(a+b) = \varphi(a) + \varphi(b) = c+d+c'+d' = d+d' = d^*.$$

Let $\psi(c) = a_1 + b_1$. Then $r_A\psi(c) = a_1$. Since $r_C\varphi(a) = c$ then $a = a_1$. Let us consider the element $b - b_1 \in B$. We have

$$r_D\varphi(b-b_1) = r_D\varphi(a+b-a-b_1) = r_D(\varphi(a) + \varphi(b) - \varphi(a+b_1))$$
$$= r_D(c+d+c'+d'-c) = r_D(c'+d') = d^*$$

and hence $r_D\varphi|B$ is an epimorphism.

Let us show that $r_D\varphi|B$ is also a monomorphism. Suppose $r_D\varphi(b) = r_D\varphi(b')$ and $b, b' \in B$ and $b \neq b'$. We have

$$\varphi(b) = c_1+d \quad \text{and} \quad \varphi(b') = c_2+d \quad \text{and} \quad c_1 \neq c_2 .$$

Let

$$\psi(c_1) = a_1+b_1 \quad \text{and} \quad \psi(c_2) = a_2+b_2 .$$

Since $r_A\psi|C$ is an isomorphism, we have $a_1 \neq a_2$. On the other hand, we have

$$b = \psi\varphi(b) = \psi(c_1+d) = a_1+b_1+\psi(d) = a_1+b_1+x+y ,$$
$$b' = \psi\varphi(b') = \psi(c_2+d) = a_2+b_2+\psi(d) = a_2+b_2+x+y .$$

It follows that $-x = a_1 = a_2$, which contradicts $a_1 \neq a_2$. Hence $r_D\varphi|B$ is an isomorphism. The proof is finished.

Proof of Theorem (6.2). If (X, x_0) is a pointed FANR-set in E^3 and $\mathrm{Sh}(X, x_0)$ is simple, then by Theorem (2.2) we infer that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(S^2, s_0)$ or there is an aspherical polyhedron (P, p_0) such that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(P, p_0)$ and the group $\pi_1(P, p_0)$ is indecomposable into a free product (Lemma 6.3).

Let $\bigvee_{i=1}^{\infty}(\hat{X}_i, x_0)$ and $\bigvee_{j=1}^{\infty}(\hat{Y}_j, y_0)$ be bouquets which satisfy the assumption of Theorem (6.2). We may assume that $(\hat{X}_i, x_0)$ is a 2-sphere or an aspherical polyhedron with an indecomposable fundamental group for every $i = 1, 2, \ldots$ We assume the same about $(\hat{Y}_j, y_0)$ for $j = 1, 2, \ldots$

Let us express $\bigvee_{i=1}^{\infty}(\hat{X}_i, x_0)$ and $\bigvee_{j=1}^{\infty}(\hat{Y}_j, y_0)$ as one-point unions of two bouquets, $\bigvee_{i=1}^{\infty}(X_i, x_0) \vee \bigvee_{i=1}^{\infty}(X'_i, x_0)$ and $\bigvee_{j=1}^{\infty}(Y_j, y_0) \vee \bigvee_{j=1}^{\infty}(Y'_j, y_0)$ respectively, such that (X_i, x_0) and (Y_j, y_0) are aspherical polyhedra for every $i, j = 1, 2, \ldots$ and $\mathrm{Sh}(X'_i, x_0) = \mathrm{Sh}(Y'_j, y_0) = \mathrm{Sh}(S^2, s_0)$ or one or both of the shapes $\mathrm{Sh}(X'_i, x_0)$, $\mathrm{Sh}(Y'_j, y_0)$ are trivial (Theorem 6.1).

Since $\mathrm{Sh}(\bigvee_{i=1}^{\infty}(X_i, x_0))$ does not dominate $\mathrm{Sh}(S^2, s_0)$ and $\mathrm{Sh}(\bigvee_{j=1}^{\infty}(Y_j, y_0))$ does not dominate $\mathrm{Sh}(S^2, s_0)$, we infer by Theorem (4.1) that the two bouquets $\bigvee_{i=1}^{\infty}(X'_i, x_0)$, and $\bigvee_{j=1}^{\infty}(Y'_j, y_0)$ contain the same number of leaves of non-trivial shapes. Therefore it is sufficient to prove (Theorem 4.1) that

$$\mathrm{Sh}(\bigvee_{i=1}^{\infty}(X_i, x_0)) = \mathrm{Sh}(\bigvee_{j=1}^{\infty}(Y_j, y_0)) .$$

Let us assume that for odd indexes the leaves (X_i, x_0) and (Y_i, y_0) have fundamental dimension two or zero, and for even indexes these leaves have fundamental dimension one or zero (Theorem 6.1). The continuum of fundamental dimension zero is of course of a trivial shape. It follows that the leaves with even indexes have the shape of a 1-sphere or a point.

Let $A^i = \pi_1(X_i, x_0)$ and $B^i = \pi_1(Y_i, y_0)$. We have

$$\text{pro-}\pi_1(\bigvee_{i=1}^{\infty}(X_i, x_0)) = \{A_n, \varphi_n^{n+1}\} ,$$
$$\text{pro-}\pi_1(\bigvee_{i=1}^{\infty}(Y_i, y_0)) = \{B_n, \psi_n^{n+1}\}$$

where

$$A_n = A^1 * A^2 * \ldots * A^n , \qquad B_n = B^1 * B^2 * \ldots * B^n ,$$
$$\varphi_n^{n+1}(a) = a \text{ if } a \in A^1 * \ldots * A^n \quad \text{and} \quad \varphi_n^{n+1}(a) = 1 \text{ if } a \in A^{n+1} ,$$
$$\psi_n^{n+1}(b) = b \text{ if } b \in B^1 * \ldots * B^n \quad \text{and} \quad \psi_n^{n+1}(b) = 1 \text{ if } b \in B^{n+1} .$$

In our notation A^{2n-1} and B^{2n-1} are trivial or indecomposable into a free product and are not infinite cyclic groups, and A^{2n}, B^{2n} are trivial or infinite cyclic groups for every $n = 1, 2, \ldots$

Suppose that

$$(\alpha, f_n)\colon \{A_n, \varphi_n^{n+1}\} \to \{B_n, \psi_n^{n+1}\} ,$$
$$(\beta, g_n)\colon \{B_n, \psi_n^{n+1}\} \to \{A_n, \varphi_n^{n+1}\}$$

are isomorphism of pro-groups such that

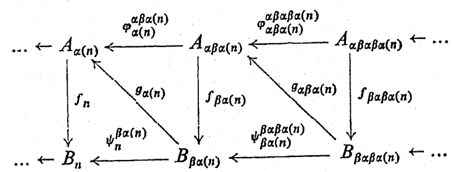
$$(\alpha, f_n)(\beta, g_n) = (\beta\alpha, \psi_{\beta\alpha(n)}^{\beta\alpha(n+1)})$$

and

$$(\beta, g_n)(\alpha, f_n) = (\alpha\beta, \varphi_{\alpha\beta(n)}^{\alpha\beta(n+1)}) .$$

Let n be a natural number. Consider the following commutative diagram of groups and homomorphisms:

(6.5)



Let i be an odd natural number less than $\alpha(n)$. Then

$$\varphi_m^k|A_k^i = \mathrm{id}_{A^i} \quad \text{for every } k \geqslant m \geqslant \alpha(n) .$$

Consider the group $f_{\beta\alpha(n)}(A^i)$. By the well-known Kurosh subgroup theorem (see [19], p. 211) we know that

$$f_{\beta\alpha(n)}(A^i) \approx F * H_1 * H_2 * \ldots * H_l$$

where F is a free group and H_i is a group conjugated with some subgroup of B_j for some index j. Since

$$g_{\alpha(n)} f_{\beta\alpha(n)}|A^i = \mathrm{id}_{A^i} ,$$

$f_{\beta\alpha(n)}|A^i$ is a monomorphism. Moreover, since A^i is indecomposable into a free product and is not an infinite cyclic, there is exactly one odd index j such that

$$f_{\beta\alpha(n)}(A^i) = H$$

where H is conjugated with some subgroup of B_j. This means that there is an element $\varrho \in B_{\beta\alpha(n)}$ such that

$$f_{\beta\alpha(n)}(A^i) = \varrho G \varrho^{-1}$$

where $G \subset B_j$. By similar arguments and the commutativity of (6.5) we infer that there are $\gamma \in A_{\alpha\beta\alpha(n)}$ and $G' \subset A^i$ such that

$$g_{\alpha\beta\alpha(n)}(B^j) = \gamma G' \gamma^{-1}.$$

Since

$$f_{\beta\alpha(n)} g_{\alpha\beta\alpha(n)}(B^j) = B^j,$$

we have

$$f_{\beta\alpha(n)}(\gamma G' \gamma^{-1}) = B^j.$$

Therefore we have

$$B^j = f_{\beta\alpha(n)}(\gamma) f_{\beta\alpha(n)}(G') f_{\beta\alpha(n)}(\gamma^{-1}) \subset f_{\beta\alpha(n)}(\gamma) \varrho G \varrho^{-1} f_{\beta\alpha(n)}(\gamma^{-1}).$$

This implies that $f_{\beta\alpha(n)}(\gamma)\varrho \in B^j$ and hence $G = B^j$. Hence we infer that the group A^i and B^j are isomorphic. In this way for every odd index i we can find exactly one odd index $\sigma(i)$ such that A^i and $B^{\sigma(i)}$ are isomorphic. It is easy to check that σ: {odd numbers} $\to$ {odd numbers} so defined is a bijection.

Let

$$(X, x_0) = \bigvee_{i=1}^{\infty} (X_{2i}, x_0), \quad (X', x_0) = \bigvee_{i=1}^{\infty} (X_{2n-1}, x_0),$$

$$(Y, y_0) = \bigvee_{i=1}^{\infty} (Y_{2i}, y_0), \quad \text{and} \quad (Y', y_0) = \bigvee_{i=2}^{\infty} (Y_{2i-1}, y_0).$$

Suppose that morphisms (α, f_n), (β, g_n) are induced by

$$\underline{f}\colon (X, x_0) \vee (X', x_0) \to (Y, y_0) \vee (Y', y_0)$$

and

$$\underline{g}\colon (Y, y_0) \vee (Y', y_0) \to (X, x_0) \vee (X', x_0),$$

respectively, where $\underline{f}$ and $\underline{g}$ are shape equivalences. By Theorem (6.1) we infer that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$. It remains to show that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$.

Let $r_X\colon (X, x_0) \vee (X', x_0) \to (X, x_0)$ be a retraction such that $r_X(X') = \{x_0\}$. Similarly we define a retraction $r_{X'}$, r_Y, $r_{Y'}$. Since for every odd index i

$$(\underline{f})_{\#}(\pi_1(X_i, x_0)) = \varrho \pi_1(Y_{\sigma(i)}, y_0) \varrho^{-1},$$

we infer that

$$(\underline{f})_*\colon \check{H}_1(X_i, Z) \to \check{H}_1(Y_{\sigma(i)}, Z)$$

is an isomorphism, which implies

$$(r_{Y'}\underline{f})_*\colon \check{H}_1(X', Z) \to \check{H}_1(Y', Z)$$

is an isomorphism and

$$(r_{X'} \underline{g} r_{Y'} \underline{f})_* | \check{H}_1(X', Z) = \mathrm{id}.$$

Therefore, by Lemma (6.4) we infer that

$$(r_Y \underline{f})_*\colon \check{H}_1(X, Z) \to \check{H}_1(Y, Z)$$

is an isomorphism and hence $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$. The proof is completed.

Let us formulate the following corollary:

(6.6) Corollary. *If $(X, x_0) \subset E^3$ is a pointed connected compact* FANR-*set, then the family of constituents of the shape* $\mathrm{Sh}(X, x_0)$ *which may be embedded in E^3 is finite.*

Proof. Since (X, x_0) is a FANR-set, there is a finite family of simple shapes such that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(X_1, x_0) + \ldots + \mathrm{Sh}(X_n, x_0)$ (see [14], Cor. 24). By Theorem (6.2) there exists only one such family.

(6.7) Remark. The above corollary may also be proved directly from Theorems (2.2) and (3.1) and Lemma (6.2).

(6.8) Remark. The assumption (X_i, x_0), $(Y_j, y_0) \subset E^3$ in Theorem (6.2) is essential. Namely, it is easy to check that the polyhedron X from M. J. Dunwoody's example ([6]) has the homotopy type of some polyhedron in E^3 (see [16]). Hence $X \vee S^2$ has two different decompositions into simple shapes in E^3.

Let $\mathscr{M}$ denote the class of compact connected closed manifolds of dimension less than or equal to 2. One can prove the following theorem in the same way as Theorem (6.3):

(6.9) Theorem. *Suppose $X_i, Y_i \in \mathscr{M}$ for every $i = 1, 2, \ldots$ If* $\mathrm{Sh}(\bigvee_{i=1}^{\infty} (X_i, x_0)) = \mathrm{Sh}(\bigvee_{j=1}^{\infty} (Y_j, y_0))$, *then there is a bijection $\sigma\colon N \to N$ such that* $\mathrm{Sh}(X_i, x_0) = \mathrm{Sh}(Y_{\sigma(i)}, y_0)$ *for every $i = 1, 2, \ldots$*

It is known that the family of shapes of plane continua is countable. In E^3 the situation is not so good, even in the case of locally connected and movable continua. Let us prove the following corollary:

(6.10) Corollary. *In E^3 there exists an uncountable family $\{(X_\lambda, x_0)\}_{\lambda \in \Lambda}$ of locally connected and pointed movable continua such that, for $\lambda \neq \mu$,* $\mathrm{Sh}(X_\lambda, x_0) \neq \mathrm{Sh}(X_\mu, x_0)$.

Proof. Let

$$B_i = \left\{(x, y, z) \in E^3 : \frac{1}{i+1} \leqslant x \leqslant \frac{1}{i}, \ \frac{-1}{i+1} \leqslant y \leqslant \frac{1}{i+1}, \ 0 \leqslant z \leqslant \frac{1}{i+1}\right\}.$$

Then $\bigcup_{i=1}^{\infty} B_i \cup \{0,0,0\}$ is a continuum. Let

$$I = \{(x, y, z) \in E^3 \colon 0 \leqslant x \leqslant 1,\, y = z = 0\}.$$

Let (P_n, p_0) be a 2-dimensional oriented surface of genus $2n$, and let

$$h_n\colon (P_n, P_0) \to (B_n, B_n \cap I)$$

be a PL-embedding such that $h_n(P_n \setminus \{p_0\}) \subset \operatorname{int} B_n$.

Let Λ be a set of all sequences $(n_1, n_2, \ldots)$ such that $n_i = 0$ or 1 for every $i = 1, 2, \ldots$ Let $\lambda \in \Lambda$. We define a continuum (X_λ, x_0) by setting:

$$(X_\lambda, x_0) = I \cup \{0,0,0\} \cup \bigcup_{i \in \{i\colon n_i \in \lambda \wedge n_i = 1\}} \big(h_i(P_i), \{0,0,0\}\big).$$

Then (X_λ, x_0) is locally connected. By Theorem (6.9) we infer that if $\lambda \neq \mu$, then $\operatorname{Sh}(X_\lambda, x_0) \neq \operatorname{Sh}(X_\mu, x_0)$. The proof that (X_λ, x_0) is pointed movable for every $\lambda \in \Lambda$ is left to the reader as a simple exercise.

The following problem remains open:

(6.11) PROBLEM. Is it true that for every pointed continuum $(X, x_0) \subset E^3$ there is a bouquet of the same shape with leaves of simple shapes? Is such a decomposition unique?

If we consider bouquets in the sense of A. Gmurczyk [12], then the second question has a negative answer. In order to see that, let us consider the following example:

(6.12) EXAMPLE. Let $p_1, p_2, \ldots$ be a sequence of prime numbers sucht that $p_i < p_{i+1}$ for every $i = 1, 2, \ldots$ We denote by (X_k, x_0) the bouquet

$$(S_1^1 \vee S_2^1 \vee \ldots \vee S_{2^k}^1, s_0)$$

of 2^k copies of 1-spheres. We denote by γ_i^k the generator of $\pi_1(X_k, x_0)$ which is represented by

$$(S^1, s_0) \xrightarrow{\text{id}} (S_i^1, s_0) \xrightarrow{i} (X_k, x_0)$$

where i is an inclusion. We define the map

$$f_k^{k+1}\colon (X_{k+1}, x_0) \to (X_k, x_0)$$

by setting generators

$$f_k^{k+1}(\gamma_{2i}^{k+1}) = (\gamma_i^k)^{p_{2^k+2i}}, \quad f_k^{k+1}(\gamma_{2i-1}^{k+1}) = (\gamma_i^k)^{p_{2^k+2i-1}}$$

for every $i = 1, 2, \ldots$

Let $(X, x_0) = \varprojlim\{(X_k, x_0), f_k^{k+1}\}$. It is clear that (X, x_0) is a bouquet in the sense of A. Gmurczyk. Moreover, each leaf of (X, x_0) is a solenoid and hence has a simple shape. On the other hand, $(f_k^{k+1})_\#\colon \pi_1(X_k, x_0) \to \pi_1(X_k, x_0)$ is an epimorphism for every $k = 1, 2, \ldots$ and hence (X, x_0) is pointed 1-movable. Since $\operatorname{Fd}(X) = 1$, (X, x_0) is pointed movable and hence it has the shape of a certain plane continuum. But each plane continuum has the shape of a bouquet of 1-spheres.

This example also shows that the following proposition holds:

(6.13) PROPOSITION. *The pointed movability of a bouquet (in the sense of A. Gmurczyk) does not imply the existence of a movable leaf.*

The following proposition is an immediate consequence of Theorem (2.2):

(6.14) PROPOSITION. *If $(X, x_0) \subset E^3$ is a pointed 1-movable continuum such that $\operatorname{Sh}(X, x_0) \geqslant \operatorname{Sh}(S^2, s_0)$, then $\operatorname{Sh}(X, x_0)$ has a simple constituent.*

This gives a partial positive answer to Borsuk's problem ([2], Problem 6.12, p. 139).

If, in the above proposition, we additionally assume that $\operatorname{Sh}(X, x_0)$ is simple, then we obtain the following corollary:

(6.15) COROLLARY. *Suppose $(X, x_0) \subset E^3$ is a pointed 1-movable continuum of simple shape. If $\operatorname{Sh}(X, x_0) \geqslant \operatorname{Sh}(S^2, s_0)$, then $\operatorname{Sh}(X, x_0) = \operatorname{Sh}(S^2, s_0)$.*

VII. Decomposition of pointed shapes of a subcontinuum in E^3 into a cartesian product. Let us recall that the shape $\operatorname{Sh}(X, x_0)$ is prime if the equation $\operatorname{Sh}(X, x_0) = \operatorname{Sh}\big((Y, y_0) \times (Z, z_0)\big)$ implies that $\operatorname{Sh}(Y, y_0)$ or $\operatorname{Sh}(Z, z_0)$ is trivial.

The following theorem holds:

(7.1) THEOREM. *Let $(X, x_0) \subset E^3$ be a pointed 1-movable continuum. If $\operatorname{Sh}(X, x_0)$ dominates the shape $\operatorname{Sh}(S^2, s_0)$, then $\operatorname{Sh}(X, x_0)$ is prime.*

In order to prove this theorem we need the following lemma:

(7.2) LEMMA. *Suppose X, Y are compacta such that $\check{H}^{2n}(X \times Y, Q) = 0$. If $\operatorname{Sh}(X \times Y) \geqslant \operatorname{Sh}(S^n)$, then $\operatorname{Sh}(X) \geqslant \operatorname{Sh}(S^n)$ or $\operatorname{Sh}(Y) \geqslant \operatorname{Sh}(S^n)$.*

Proof. Let $X = \varprojlim\{X_n, f_n^{n+1}\}$, $Y = \varprojlim\{Y_n, g_n^{n+1}\}$ and $S^n = \varprojlim\{S_k^n, \text{id}\}$ where $S_k^n = S^n$ for $k = 1, 2, \ldots$ Hence $X \times Y = \varprojlim\{X_n \times Y_n, f_n^{n+1} \times g_n^{n+1}\}$ and there are shape morphisms

$$\underline{h} = (\alpha, h_k)\colon S^n \to X \times Y \quad \text{and} \quad \underline{r} = (\beta, r_k)\colon X \times Y \to S^n$$

such that $\underline{r}\underline{h} \simeq \text{id}_{S^n}$. Consider the following commutative diagram:

$$\begin{array}{ccc} X_m \times Y_m & \xleftarrow{f_{\beta(k)}^m \times g_{\beta(k)}^m} & X_{\beta(k)} \times Y_{\beta(k)} \\ \uparrow{\scriptstyle h_m} & & \downarrow{\scriptstyle r_k} \\ S_{\alpha(m)}^n & \xleftarrow{\text{id}} & S_k^n \end{array}$$

If m is sufficiently large, then we may assume that

$$(f_{\beta(k)}^m \times g_{\beta(k)}^m)^*_{,}2n\colon \check{H}^{2n}(X_m \times Y_m, Q) \to \check{H}^{2n}(X_{\beta(k)} \times Y_{\beta(k)}, Q)$$

is a trivial homomorphism. Let $p\colon X_l \times Y_l \to X_l$ be a projection onto X_l, $q\colon X_l \times Y_l \to Y_l$ a projection onto Y_l, and $i\colon X_l \times \{y_0\} \to X_l \times Y_l$ and $j\colon \{x_0\} \times Y_l \to X_l \times Y_l$ inclusions for every $l = 1, 2, \ldots$ Let $\gamma \in H^n(S^n, Q)$. Consider an element

$$(f_{\beta(k)}^m \times g_{\beta(k)}^m)^* r_k^*(\gamma) = (f_{\beta(k)}^m \times g_{\beta(k)}^m)^* \Big(\sum_{i=1}^{s} p^*(a_i) \cup q^*(b_i)\Big)$$

where $a_i \in \check{H}^*(X_{\beta(k)}, Q)$, $b_i \in H^*(Y_{\beta(k)}, Q)$ and $\dim a_i + \dim b_i = n$ (see [5], Prop. 12.16, p. 182 and Prop. 8.18, p. 222). We have

$$(7.3)\qquad (f^m_{\beta(k)} \times g^m_{\beta(k)})^*\Big(\sum_{i=1}^{s} p^*(a_i) \cup q^*(b_i)\Big) = \sum_{i=1}^{s} p^*(f^m_{\beta(k)})^*(a_i) \cup q^*(g^m_{\beta(k)})^*(b_i)\,.$$

Since $(f^m_{\beta(k)} \times g^m_{\beta(k)})^*_{2n}$ is trivial, we have for $\dim a_i = \dim b_i = n$, $p^*(f^m_{\beta(k)})^*(a_i) \cup q^*(g^m_{\beta(k)})^*(b_i) = 0$ and hence $(f^m_{\beta(k)}p)^*(a_i) = 0$ or $(g^m_{\beta(k)}q)^*(b_i) = 0$. Therefore we may assume that $\dim b_i \neq n$ for every b_i in (7.3). Hence

$$h_m\Big(\sum_{i=1}^{s} (f^m_{\beta(k)}p)^*(a_i) \cup (g^m_{\beta(k)}q)^*(b_i)\Big) = \sum_{j=1}^{t} h^*_m(f^m_{\beta(k)}p)^*(a_{i_j})$$

where a_{i_j} are all elements in (7.3) with $\dim a_{i_j} = n$. Let us consider morphisms $\underline{ph}$ and $\underline{ri}$. Let us observe that

$$h^*_m p^*(f^m_{\beta(k)})^* i^* r^*_k(\gamma) = h^*_m p^*(f^m_{\beta(k)})^* i^*\Big(\sum_{i=1}^{s} p^*(a_i) \cup q^*(b_i)\Big)$$
$$= h^*_m(f^m_{\beta(k)}p)^*\Big(\sum_{j=1}^{t} a_{i_j}\Big) = \sum_{j=1}^{t} h^*_m(f^m_{\beta(k)}p)^*(a_{i_j}) = \gamma\,.$$

Therefore by the universal coefficient formula ([26], Theorem 10, p. 246) we infer that $\underline{ri}\,\underline{ph} \simeq \mathrm{id}_{S^n}$ and hence $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(S^n)$. The proof is finished.

From this lemma we infer that if $\mathrm{Sh}(X) = \mathrm{Sh}(Z \times Y)$ and $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(S^2)$ then $\mathrm{Sh}(Z \times Y) \geqslant \mathrm{Sh}(S^2 \times Y)$. But this is impossible because $\mathrm{Fd}(S^2 \times Y) \geqslant 3$ for every continuum Y with a non-trivial shape (see [17]). The proof of Theorem (7.1) is finished.

In [15] the present author has proved the following theorem:

THEOREM. *Let us suppose that the pointed continuum (X, x_0) is approximatively n-connected for $n \geqslant 2$, and $\mathrm{Fd}(X) < \infty$ or (X, x_0) is pointed movable. If $\mathrm{Sh}(X, x_0)$ has a non-trivial factor (constituent), then $\mathrm{Sh}(X, x_0)$ is simple (prime).*

Combining this result with Theorem (7.1) we have the following

(7.4) THEOREM. *Let $(X, x_0) \subset E^3$ be a pointed 1-movable continuum. If $\mathrm{Sh}(X, x_0)$ has a non-trivial constituent (factor), then $\mathrm{Sh}(X, x_0)$ is prime (simple).*

(7.5) Remark. In space E^n for $n \geqslant 5$ Theorem (7.4) fails. A suitable example may be found in [15].

Added in proof. After this paper has been send to the publisher Mr R. Ząbek has found a counter-example to Lemma (1.11). This lemma needs some additional assumption. Correction will be submitted to Fundamenta Mathematicae.

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