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On directional cluster sets

by

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Abstract. Let f be a function from the open upper half plane H into a second countable topological space W. Let $\psi \in (0, \pi)$ be a fixed direction and let $\theta(x)$ denote the set of all directions $\theta \in (0, \pi)$ in which the directional cluster set ψ of f at x in direction θ does not contain the qualitative directional cluster set of f at x in direction. Let $\Delta(x)$ denote the set of all directions $\theta \in (0, \pi)$ in which the directional cluster set of f at x in direction ψ does not contain the qualitative directional cluster set of f at x in direction θ. It is shown that if f has the Baire property, then $\theta(x)$ and $\Delta(x)$ are sets of the first category for every $x \in R$ except a set of the first category.

1. Let H denote the open upper half plane and let z denote a point of H. Let x denote a point on the real line R. For each $x \in R$ and $h>0$ let

$$S(x, h) = \{z\colon\ z \in H, |z-x|<h\}\,.$$

and for each direction θ, $0<\theta<\pi$, let

$$L_\theta(x) = \{z\colon\ z \in H, \arg(z-x) = \theta\}$$

and

$$L_\theta(x, h) = S(x, h) \cap L_\theta(x)\,.$$

Let $E \subset H$. Then a point $x \in R$ is called a *second category point of* E if and only if for every $h>0$ the set $S(x, h) \cap E$ is of the second category in H. A point $x \in R$ is called a *first category point of* E if and only if it is not a second category point of E. The set of all first (second) category points of E will be denoted by E_{I} (E_{II} respectively).

A point $x \in R$ is called a *directional second category point of* E in the direction θ if and only if for every $h>0$ the set $L_\theta(x, h) \cap E$ is of the second category as a linear set. A point $x \in R$ is called a *directional first category point of* E in the direction θ if and only if it is not a directional second category point of E in the direction θ. The set of all directional first (second) category points of E in the direction θ will be denoted by $E_{\mathrm{I}}(\theta)$ ($E_{\mathrm{II}}(\theta)$ respectively).

Let $f\colon H \to W$, where W is a topological space. The directional cluster set $C(f, x, \theta)$ of f at x in the direction θ is the set of all $w \in W$ such that for every open set U in W containing w we have $x \in \overline{f^{-1}(U) \cap L_\theta(x)}$. The essential cluster set $C_e(f, x, \theta)$ of f at x in the direction θ is the set of all $w \in W$ such that for every open set U containing w the set $f^{-1}(U) \cap L_\theta(x)$ has positive upper density at x.

The following definition has been introduced in [3]. The qualitative directional cluster set $C_q(f, x, \theta)$ of f at x in the direction θ is the set of all $w \in W$ such that for every open set U in W containing w we have $x \in [f^{-1}(U)]_{\mathrm{II}}(\theta)$.

2. It is known [1] that if f is a continuous function from H to a topological space with a countable basis W and $\psi \in (0, \pi)$ is a fixed direction, then for every $x \in R$ except a first category set of measure zero on R the set

$$\{\theta:\ 0<\theta<\pi,\ C_e(f, x, \psi) \not\subset C(f, x, \theta)\}$$

is of the first category. If f is measurable and $\psi \in (0, \pi)$ is a fixed direction, then, except a set of measure zero on R, the set

$$\{\theta:\ 0<\theta<\pi,\ C_e(f, x, \theta) \not\subset C(f, x, \psi)\}$$

is of measure zero.

In the present paper we prove in Theorem 1 and Theorem 2 that if f has the Baire property and $\psi \in (0, \pi)$ is a fixed direction, then the sets

$$\{\theta:\ 0<\theta<\pi,\ C_q(f, x, \psi) \not\subset C(f, x, \theta)\},$$
$$\{\theta:\ 0<\theta<\pi,\ C_q(f, x, \theta) \not\subset C(f, x, \psi)\},$$

are of the first category, except for a set of points of the first category in R.

3. Let $E \subset H$ and $x \in R$. Let

$$\mathcal{O}(E, x) = \{\theta:\ 0<\theta<\pi,\ x \notin \overline{E \cap L_\theta(x)}\}.$$

For a fixed positive integer n and rationals $r, s, 0<r<s<\pi$, we also define

$$\mathcal{O}_n(E, x) = \{\theta:\ 0<\theta<\pi,\ E \cap L_\theta(x, n^{-1}) = \emptyset\}$$

and

$$\mathcal{O}_{nrs}(E, x) = \mathcal{O}_n(E, x) \cap (r, s).$$

Then, clearly

$$\mathcal{O}(E, x) = \bigcup_n \bigcup_r \bigcup_s \mathcal{O}_{nrs}(E, x).$$

We now prove

LEMMA 1. *If $E \subset H$ is open and $\psi \in (0, \pi)$ is a fixed direction, then the set*

$$B = \{x:\ x \in E_{\mathrm{II}}(\psi),\ \mathcal{O}(E, x) \text{ is of the second category in } (0, \pi)\}$$

is of the first category.

Proof. For positive integers n and rationals r, s, $0<r<s<\pi$, let

$$B_{nrs} = \{x:\ x \in E_{\mathrm{II}}(\psi),\ \mathcal{O}_{nrs}(E, x) \text{ is dense in } (r, s)\}.$$

Then, clearly

$$B \subset \bigcup_n \bigcup_r \bigcup_s B_{nrs}. \tag{1}$$

We shall prove that the sets B_{nrs} are nowhere dense in R for all n, r and s. We assume that for some n, r and s the set B_{nrs} is dense in an open interval $I(x_0)$, where $x_0 \in B_{nrs}$ is the centre of $I(x_0)$. Let $x \in B_{nrs}$. Since E is open and $\mathcal{O}_{nrs}(E, x)$ is dense in (r, s), we have $\mathcal{O}_{nrs}(E, x) = (r, s)$. Let $\gamma = \frac{1}{2}(r+s)$. Then, clearly, $L_\gamma(x, n^{-1}) \cap E = \emptyset$ for $x \in B_{nrs}$. Again, B_{nrs} is dense in $I(x_0)$ and E is open; therefore

$$L(x, n^{-1}) \cap E = \emptyset \quad \text{for all } x \in I(x_0). \tag{2}$$

Let

$$C = \bigcup_{x \in I(x_0)} L_\gamma(x, n^{-1}).$$

Then, by (2),

$$C \cap E = \emptyset. \tag{3}$$

Since $x_0 \in B_{nrs}$, we have $x_0 \in E_{\mathrm{II}}(\psi)$, and hence $L_\psi(x_0, h) \cap E \neq \emptyset$ for all $h>0$. Since $x_0 \in I(x_0)$, there is an h_0 such that $L_\psi(x_0, h_0) \cap E \subset C$, which implies $C \cap E \neq \emptyset$. But this contradicts (3). Hence B_{nrs} is nowhere dense in R for all n, r, s, and so, by (1), B is a set of the first category. The proof is complete.

We quote below the Kuratowski–Ulam theorem in polar coordinates.

THEOREM P. *If $E \subset H$ is a plane set of the first category, then for a fixed point $x \in R$ $L_\theta(x) \cap E$ is a linear set of the first category for all directions θ except a set of the first category in $(0, \pi)$.*

LEMMA 2. *If $G \subset H$ is open and $P \subset H$ is a set of the first category, then for every $x \in R$ there exists a set of the first category $Q(x) \subset (0, \pi)$ such that*

$$\mathcal{O}(G \Delta P, x) \subset \mathcal{O}(G, x) \cup Q(x).$$

Proof. Let

$$Q(x) = \{\theta:\ 0<\theta<\pi,\ P \cap L_\theta(x) \text{ is of the second category in } L_\theta(x)\}.$$

Hence, in virtue of Theorem P, it follows that $Q(x)$ is of the first category in $(0, \pi)$. Let $\theta \in \mathcal{O}(G\Delta P, x) \setminus Q(x)$. Then there exists a positive integer n such that

$$L_\theta(x, n^{-1}) \cap (G \Delta P) = \emptyset \tag{4}$$

and

(5) $P \cap L_\theta(x)$ is of the first category in $L_\theta(x)$.

In virtue of (4) and (5) and using the fact that G is open, we have

$$L_\theta(x, n^{-1}) \cap G = \emptyset.$$

Hence $\theta \in \mathcal{O}(G, x)$ and

$$\mathcal{O}(G\Delta P, x) \setminus Q(x) \subset \mathcal{O}(G, x).$$

LEMMA 3. *If the set $E \subset H$ has the Baire property and $\psi \in (0, \pi)$ is a fixed direction, then the set*

$$B = \{x:\ x \in E_{\mathrm{II}}(\psi),\ \mathcal{O}(E, x) \text{ is of the second category in } (0, \pi)\}$$

is of the first category in R.

Proof. Let $E = G\Delta P$, where G is open and P is of the first category. Clearly

(6) $$E_{\mathrm{II}}(\psi)\subset G_{\mathrm{II}}(\psi)\cup P_{\mathrm{II}}(\psi).$$

From Lemma 2 it follows that for every $x\in R$ there exists a set of the first category $Q(x)\subset(0,\pi)$ such that

(7) $$\mathcal{O}(E,x)\subset\mathcal{O}(G,x)\cup Q(x)$$

Let

$$A = \{x\colon x\in G_{\mathrm{II}}(\psi),\ \mathcal{O}(G,x) \text{ is of the second category in } (0,\pi)\}.$$

Let $x\in B$. Then $\mathcal{O}(E,x)$ is of the second category in $(0,\pi)$. Hence we infer by (7) that $\mathcal{O}(G,x)$ is of the second category. By (6), $x\in G_{\mathrm{II}}(\psi)$ or $x\in P_{\mathrm{II}}(\psi)$. In the first case $x\in A$. Thus

$$B\subset A\cup P_{\mathrm{II}}(\psi).$$

The set A is of the first category by Lemma 1. The set $P_{\mathrm{II}}(\psi)$ is of the first category in virtue of the Kuratowski–Ulam theorem ([2], p. 56). Hence the set B is of the first category.

Let W be a second countable topological space and let $\psi\in(0,\pi)$ be a fixed direction.

THEOREM 1. *If $f\colon H\to W$ has the Baire property, then, for every $x\in R$ except a set of the first category in R, the set*

$$\mathcal{O}(x) = \{\theta\colon 0<\theta<\pi,\ C_q(f,x,\psi)\not\subset C(f,x,\theta)\}$$

is of the first category.

Proof. Let $\{V_n\}$ be a countable basis for the topology of W. Let

$$E_n = f^{-1}(V_n).$$

Every set E_n has the Baire property. Let $\mathcal{O}(E_n,x,\psi) = \mathcal{O}(E_n,x)$ if $x\in E_{n\mathrm{II}}(\psi)$ and $\mathcal{O}(E_n,x,\psi) = \emptyset$ if $x\notin E_{n\mathrm{II}}(\psi)$. Let $\theta_0\in\mathcal{O}(x)$. Then $C_q(f,x,\psi)\not\subset C(f,x,\theta_0)$. So there is a $w\in C_q(f,x,\psi)\setminus C(f,x,\theta_0)$ and hence there exists a V_{n_0} containing a w such that $x\in E_{n_0\mathrm{II}}(\psi)$ and $x\notin\overline{L_{\theta_0}(x)\cap E_{n_0}}$. Clearly $\theta_0\in\mathcal{O}(E_{n_0},x,\psi)$. Thus

(8) $$\mathcal{O}(x)\subset\bigcup_n\mathcal{O}(E_n,x,\psi).$$

Let

$$B_n = \{x\colon x\in E_{n\mathrm{II}}(\psi),\ \mathcal{O}(E_n,x) \text{ is of the second category in } (0,\pi)\}$$

and

$$D = \{x\colon \mathcal{O}(x) \text{ is of the second category in } (0,\pi)\}.$$

So, if $x_0\in D$, then by (8) there is at last one n_0 such that $\mathcal{O}(E_{n_0},x,\psi)$ is a second category set, and so, by the definition of $\mathcal{O}(E_{n_0},x_0,\psi)$ $x_0\in E_{n_0\mathrm{II}}(\psi)$ and the set $\mathcal{O}(E_{n_0},x_0)$ is of the second category. Therefore $x_0\in B_{n_0}$. Hence

$$D\subset\bigcup_n B_n.$$

Since the set B_n is of the first category for all n by Lemma 3, the set D is of the first category, which proves the theorem.

Let $E\subset H$ and $\psi\in(0,\pi)$. Let

$$\Gamma(E,\psi) = \{x\in R\colon x\notin\overline{L_\psi(x)\cap E}\}.$$

LEMMA 4. *If $G\subset H$ is open and $P\subset H$ is of the first category, then for a fixed direction $\psi\in(0,\pi)$ there exists a set of the first category $Q(\psi)\subset R$ such that*

(9) $$\Gamma(G\Delta P,\psi)\subset\Gamma(G,\psi)\cup Q(\psi).$$

Proof. Let

$$Q(\psi) = \{x\in R\colon L_\psi(x)\cap P \text{ is of the second category in } L_\psi(x)\}.$$

Hence in virtue of the Kuratowski–Ulam theorem ([2], p. 56) it follows that $Q(\psi)$ is of the first category in R. Let $x\in\Gamma(G\Delta P,\psi)\setminus Q(\psi)$. Then there exists a positive integer n such that

(10) $$L_\psi(x,n^{-1})\cap(G\Delta P) = \emptyset$$

and

(11) $P\cap L_\psi(x)$ is of the first category in $L_\psi(x)$.

In virtue of (10) and (11) and using the fact that G is open, we have

$$L_\psi(x,n^{-1})\cap G = \emptyset.$$

Hence

$$x\in\Gamma(G,\psi)$$

and

$$\Gamma(G\Delta P,\psi)\setminus Q(\psi)\subset\Gamma(G,\psi).$$

Let $E\subset H$ and $x\in R$. Let

$$\mathcal{K}(E,x) = \{\theta\colon 0<\theta<\pi,\ x\in E_{\mathrm{II}}(\theta)\}.$$

LEMMA 5. *If the set $E\subset H$ has the Baire property and $\psi\in(0,\pi)$ is a fixed direction, then the set*

$$S = \{x\colon x\notin\overline{L_\psi(x)\cap E},\ \mathcal{K}(E,x) \text{ is of the second category in } (0,\pi)\}$$

is of the first category in R.

Proof. Let $E = G\Delta P$, where G is open and P is of the first category. We will consider two cases:

Case I: $P = \emptyset$. For a positive integer m, let

$$S_m = \{x\colon L_\psi(x,m^{-1})\cap E = \emptyset,\ \mathcal{K}(E,x) \text{ is of the second category in } (0,\pi)\}.$$

Then, clearly

(12) $$S\subset\bigcup_m S_m.$$

We shall prove that the sets S_m are nowhere dense in R for all m. If it is possible, for a certain m let the set S_m be dense in an open interval $I(x_0)$, where $x_0 \in S_m$ is the centre of $I(x_0)$. Since for $x \in S_m$

(13) $$L_\psi(x, m^{-1}) \cap E = \emptyset$$

S_m is dense in $I(x_0)$ and E is open, property (13) holds for all $x \in I(x_0)$. Let

$$A = \bigcup_{x \in I(x_0)} L_\psi(x, m^{-1}).$$

Then

(14) $$A \cap E = \emptyset.$$

So, there is an $h_0 > 0$ such that $S(x_0, h_0) \subset A$. Therefore $\mathscr{K}(E, x_0) = \emptyset$. But this contradicts the definitions of S_m. Hence S_m is nowhere dense in R for all m, and so, by (12), S is of the first category.

Case II: $P \neq \emptyset$. Clearly

(15) $$\mathscr{K}(G \Delta P, x) \subset \mathscr{K}(G, x) \cup \mathscr{K}(P, x).$$

In virtue of Lemma 4 it follows that there exists a set of the first category $Q(\psi) \subset R$ such that

(16) $$\Gamma(E, \psi) \subset \Gamma(G, \psi) \cup Q(\psi).$$

Let

$$K = \{x\colon x \notin \overline{L_\psi(x) \cap G},\ \mathscr{K}(G, x) \text{ is of the second category in } (0, \pi)\}.$$

In virtue of Theorem P, $\mathscr{K}(P, x)$ is of the first category. Then, by (15), $\mathscr{K}(G, x)$ is of the second category for $x \in S$. Let $x_0 \in S$. Then $x_0 \notin \overline{L_\psi(x_0) \cap E}$. By (16), $x_0 \in \Gamma(G, \psi)$ or $x_0 \in Q(\psi)$.

In the first case $x_0 \in K$. Therefore

$$S \subset K \cup Q(\psi).$$

In virtue of case I the set K is of the first category. The set $Q(\psi)$ is of the first category, and therefore the set S is of the first category.

Let W be a second countable topological space and let $\psi \in (0, \pi)$ be a fixed direction.

Theorem 2. *If $f\colon H \to W$ has the Baire property, then for every $x \in R$ except a set of the first category in R the set*

$$\Delta(x) = \{\theta\colon 0 < \theta < \pi,\ C_q(f, x, \theta) \not\subset C(f, x, \psi)\}$$

is of the first category.

Proof. Let $\{V_n\}$ be countable basis for the topology of W. Let $E_n = f^{-1}(V_n)$. Every set E_n has the Baire property. Let $\mathscr{K}(E_n, x, \psi) = \mathscr{K}(E_n, x)$ if $x \notin \overline{L_\psi(x) \cap E_n}$ and $\mathscr{K}(E_n, x, \psi) = \emptyset$ if $x \in \overline{L_\psi(x) \cap E_n}$. Let $\theta_0 \in \Delta(x)$. Then $C_q(f, x, \theta_0) \not\subset C(f, x, \psi)$. So there is a $w \in C_q(f, x, \theta_0) \setminus C(f, x, \psi)$ and hence there is a V_{n_0} containing w such that $x \in E_{n_0 \mathrm{II}}(\theta_0)$ and $x \notin \overline{L_\psi(x) \cap E_{n_0}}$. Clearly $\theta_0 \in \mathscr{K}(E_{n_0}, x, \psi)$. Thus

(17) $$\Delta(x) \subset \bigcup_n \mathscr{K}(E_n, x, \psi).$$

Let

$$S_n = \{x\colon x \notin \overline{L_\psi(x) \cap E_n},\ \mathscr{K}(E_n, x) \text{ is of the second category in } (0, \pi)\}$$

and

$$T = \{x\colon \Delta(x) \text{ is of the second category in } (0, \pi)\}.$$

So, if $x_0 \in T$ then $\Delta(x_0)$ is of the second category in $(0, \pi)$. Hence, by (17), there is at least one n_0 such that $\mathscr{K}(E_{n_0}, x_0, \psi)$ is of the second category, and so, by the definition of $\mathscr{K}(E_{n_0}, x_0, \psi)$, $x_0 \notin \overline{L_\psi(x_0) \cap E_{n_0}}$ and $\mathscr{K}(E_{n_0}, x_0)$ is of the second category. Therefore $x_0 \in S_{n_0}$. Hence

$$T \subset \bigcup_n S_n.$$

Since the sets S_n are of the first category for all n by Lemma 5, the set T is of the first category. This proves the theorem.

References

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