The number of metrizable spaces

by

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Abstract. The theorems in this paper solve problems of the following sort. Given infinite cardinals $m$ and $n$ with $m < n < m^*$ and a topological property $P$, find the number of topologically distinct metrizable spaces having weight $m$, cardinality $n$, and property $P$. The properties considered include connectedness, local compactness, andČech completeness.

1. Introduction. Let $X$ be an infinite metrizable space of weight $m$. It is well known that $m$ and $|X|$ satisfy the inequality $m < |X| < m^*$. This suggests the following problem. Given infinite cardinals $m$ and $n$ with $m < n < m^*$, find the number of topologically distinct metrizable spaces having weight $m$ and cardinality $n$. The main result in this paper, Theorem 4.11 in § 4, states that the number of such spaces is $2^n$. Lozier and Marty [LM] have proved that for each infinite cardinal $m$ the number of topologically distinct continua of weight $m$ is $2^n$. (A continuum is a compact connected Hausdorff space; it need not be metrizable.) We obtain analogues of this result. For example, we show that for each infinite cardinal $m$ the number of topologically distinct connected metrizable spaces of weight $m$ is $2^{m^*}$ and the number of topologically distinct connected completely metrizable spaces of weight $m$ is $2^{m^*}$. (See § 4 and § 5.) In § 6 we prove that for each infinite cardinal $m$ the number of topologically distinct locally compact locally countable metrizable spaces of cardinality $m$ is $\omega_1 \cdot \varphi(m)$, where $\varphi(m)$ is the number of cardinals $\leq m$. This result can be regarded as an extension to higher cardinals of the classical result due to Mazurkiewicz and Sierpiński [MS] that the number of topologically distinct compact countable metrizable spaces is $\omega_1$. In § 7 we show that the number of topologically distinct locally compact metrizable spaces of cardinality $m$ is $\omega_1 \cdot \varphi(m)$ if $m < 2^n$ and $\varphi(m)^{m^*}$ if $m \geq 2^n$. Finally in § 8 we show that for each $m \geq 2^n$ the number of topologically distinct connected paracompact Hausdorff spaces of cardinality $m$ is the maximum possible, namely $2^{m^*}$.

2. Notation, definitions, and known results. We adopt the following set-theoretic notation: $k$ and $\omega$ denote natural numbers; $\omega$ is the first infinite ordinal and also the set of natural numbers; $\omega_1$ is the first uncountable ordinal; $m$, $n$, and $\rho$ denote cardinal numbers and $\alpha$, $\beta$, and $\gamma$ denote ordinal numbers; $|E|$ is the cardinality of the set $E$; the set of real numbers is denoted by $\mathbb{R}$ and $\mathbb{R}^n$ denotes Euclidean $n$-space. Regarding separation axioms, we assume that regular, paracompact, locally compact, and compact spaces are always Hausdorff.
Let \( m \geq 1 \), let \( X \) be a connected space with more than one point, let \( p \in X \). Then \( p \) is a cut point of \( X \) of order \( m \) if \( X - \{ p \} \) has \( m \) components. If \( p \) is a cut point of \( X \) of order \( m \geq 1 \), then \( p \) is called a cut point of \( X \); otherwise, \( p \) is a noncut point of \( X \). It is clear that if \( p \) is a cut point of \( X \) of order \( m \), and \( A \) is a homeomorphism from \( X \) onto \( Y \), then \( \delta(p) \) is a cut point of \( Y \) of order \( m \).

We shall make frequent use of the sum \( \bigoplus_{s \in S} X_s \) of a collection \( \{ X_s \colon s \in S \} \) of topological spaces. The reader is referred to pp. 103–106 in [E] for the definition and a list of basic properties of \( \bigoplus_{s \in S} X_s \). It is especially useful to remember that if \( \{ X_s \colon s \in S \} \) is a pairwise disjoint collection of open sets which covers a topological space \( X \), then \( X = \bigoplus_{s \in S} X_s \). We adopt the convention that in constructing \( \bigoplus_{s \in S} X_s \), one tacitly assumes that the underlying sets for the spaces \( \{ X_s \colon s \in S \} \) are pairwise disjoint. The discrete space of cardinality \( m \), denoted \( \mathbb{I}_m \), is the sum of \( m \) one-point spaces. Alexandroff [A] has proved that if \( X \) is a locally separable metrizable space, then \( X = \bigoplus_{s \in S} X_s \), where each \( X_s \) is separable. (See p. 359 in [E].) It should be noted that if \( X = \bigoplus_{s \in S} X_s \), then \( X^{(0)} = \bigoplus_{s \in S} X_s^{(0)} \) for all \( \alpha \geq 1 \). (Here \( X^{(0)} \) denotes the derived set of \( X \) of order \( \alpha \).)

We now discuss a technique for constructing a new connected metrizable space from a given collection of connected metrizable spaces. (This construction generalizes the well-known hedgehog with \( m \) spines.) Let \( \{ (X, \mathcal{F}_s) \colon s \in S \} \) be a collection of connected metrizable spaces, each having more than one point. We tacitly assume that \( \{ (X_s, \mathcal{F}_s) \colon s \in S \} \) is a pairwise disjoint collection. For each \( s \in S \) let \( d_s \) be a metric on \( X_s \) compatible with \( \mathcal{F}_s \), let \( p_s \in X_s \), and let \( Y_s = X_s - \{ p_s \} \). Let \( p(S) \) be a point not in \( \bigcup_{s \in S} X_s \) and let \( X(S) = \bigcup_{s \in S} Y_s \cup \{ p(S) \} \). Define \( d : X(S) \times X(S) \to \mathbb{R}^{+} \) as follows: if \( x, y \in Y_s \), \( d(x, y) = d_s(x, y) \); if \( x \in Y_s \), \( y \in Y_t \), and \( s \neq t \), \( d(x, y) = d_s(x, p_s) + d_t(p_s, y) \); if \( x \in X_s \), \( d_s(x, p_s) = d(p(S), x) = d(X_s, X(S)) \); \( d(p(S), p(S)) = 0 \). One can show that \( d \) is a metric on \( X(S) \), that \( X_s \) is open subset of \( X(S) \), and that the function from \( X_s \) into \( X(S) \) which is the identity on \( Y_s \) and takes \( p_s \) to \( p(S) \) is a homeomorphism from \( X_s \) onto the subspace \( Y_s \cup \{ p(S) \} \) of \( X(S) \). We call \( X(S) \) the star-space determined by \( \{ (X_s, d_s) \colon s \in S \} \) and \( \{ p_s \colon s \in S \} \) is called the point \( X \) the star-space point of \( X(S) \). It should be emphasized that \( d_s \) is replaced by an equivalent metric \( d_s \) for infinitely many \( s \in S \) when the two star-spaces obtained need not be homeomorphic. However, in most applications of the star-space construction this is unimportant; in such cases we will omit all mention of the metrics \( d_s \) and refer to \( X(S) \) as a star-space determined by \( \{ X_s \colon s \in S \} \) and \( \{ p_s \colon s \in S \} \).

For future reference we list some basic properties of the star-space construction. (1) The space \( X(S) \) is connected. (2) \( X(S) - \{ p(S) \} = \bigcup_{s \in S} Y_s \). (3) If \( |S| = m \geq 1 \) and each \( p_s \) is a noncut point of \( X_s \), then \( p(S) \) is a cut point of \( X(S) \) of order \( m \). (4) Let \( m \geq 1 \) and let \( x \in Y_s \) for some \( s \in S \). Then \( x \) is a cut point of \( X(S) \) of order \( m \)

if and only if \( x \) is a cut point of \( X_s \) of order \( m \). (5) If \( d_s \) is a complete metric on \( X_s \) for all \( s \in S \), then the star-space determined by \( \{ X_s, d_s \colon s \in S \} \) and \( \{ p_s \colon s \in S \} \) is a complete metric space. (Thus, if each \( X_s \) is compact then every star-space determined by \( \{ X_s, d_s \colon s \in S \} \) and \( \{ p_s \colon s \in S \} \) is a complete metric space.)

Let \( |S| = m \geq 0 \) and for \( s \in S \) let \( X_s = [0, 1] \times \{ p_s \} \) and \( p_s = (0, s) \). Let \( d_s \) be the "natural" metric on \( X_s \) obtained by using the Euclidean metric on \( [0, 1] \times \{ p_s \} \). The star-space determined by \( \{ (X_s, d_s) \colon s \in S \} \) and \( \{ p_s \colon s \in S \} \) is called the hedgehog with \( m \) spines and is denoted \( J(m) \). (See p. 314 in [E] or p. 95 in [N].) Note that \( J(m) \) is a pathwise connected complete metric space of weight \( m \) and cardinality \( m^{2^m} \).

3. Constructing topologically distinct spaces. In this section we give several propositions which will be used in §§ 4 and 5 when constructing large collections of topologically distinct spaces. Most of these results belong to the folklore or generalize well known arguments. For example, Proposition 3.3 generalizes the argument given on p. 263 in [K] that the number of topologically distinct subsets of a separable metrizable space of cardinality \( 2^m \) is \( 2^{2^m} \).

**Proposition 3.1.** Let \( \{ X_s \colon s \in S \} \) be a collection of topologically distinct connected metrizable spaces, each having more than one point, and let \( p_s \) be a noncut point of \( X_s \) for each \( s \in S \). Let \( S_1 \subseteq S \), and for \( k = 1, 2 \) let \( X^{(k)} \) be a star-space determined by \( \{ X_s \colon s \in S_k \} \) and \( \{ p_s \colon s \in S_k \} \). If \( h \) is a homeomorphism from \( X(S_1) \) onto \( X(S_2) \) which takes the adjunction point \( p(S_1) \) of \( X(S_1) \) to the adjunction point \( p(S_2) \) of \( X(S_2) \), then \( S_1 = S_2 \).

**Proposition 3.2.** Let \( A_1 \) and \( A_2 \) be two spaces, each dense in itself, let \( B_1 \) and \( B_2 \) be scattered spaces. If \( A_1 \oplus B_1 \) is homeomorphic to \( A_2 \oplus B_2 \), then \( A_1 \approx A_2 \) and \( B_1 \approx B_2 \).

**Proposition 3.3.** Let \( X \) be a \( T_1 \) space of weight \( m \), let \( A \) be a collection of subsets of \( X \) such that \( |A| > 2^m \). Then there is a subcollection \( A_0 \) of \( A \) such that \( |A_0| = |A| \) and no two distinct elements of \( A_0 \) are homeomorphic.

**Proof.** First note that \( A \approx 2^m \) and every subset of \( X \) has a dense subset of cardinality at most \( m \). Define an equivalence relation \( \sim \) on \( A \) as follows: \( A \sim B \) if and only if \( A \) is homeomorphic to \( B \) for each \( A \in A \) the number of continuous functions from \( A \) into \( X \) is at most \( 2^m \); consequently each equivalence class of \( \sim \) has cardinality at most \( 2^m \). Since \( |A| > 2^m \), it follows that the number of distinct equivalence classes is \( |A| \). The desired subcollection \( A_0 \) of \( A \) is obtained by choosing a representative element from each equivalence class of \( \sim \).

4. The number of metrizable spaces. Let \( m \) and \( n \) satisfy \( 0 < m < n < m^2 \). In this section we show that the number of topologically distinct metrizable spaces having weight \( m \) and cardinality \( n \) is \( 2^m \). It is easy to show that the number of such spaces is at most \( 2^m \) (see Proposition 4.1), and so the problem reduces to constructing the required number of spaces. The solution of this problem naturally factors into two cases: \( m = 2^m \) and \( n = 2^m \). For the case \( m = 2^m \) it is convenient to construct connected metrizable spaces, since connectedness is the key property used in showing the spaces
non-homeomorphic. For the case $\kappa < 2^\omega$ the key tool is the Cantor-Bendixon theorem and the Mazurkiewicz-Sierpiński result on the number of scattered subsets of $\mathcal{R}$.

**Proposition 4.1.** The number of topologically distinct metrizable spaces of cardinality $n$ is at most $2^n$.

**Proof.** Clearly we may assume that $n \geq \omega$. Let $X$ be a set with $|X| = n$. A metric on $X$ is a function from $X \times X$ into $\mathcal{R}$. Hence the total number of metrics on $X$ is at most $(2^\omega)^n = 2^n$, and so there are at most $2^n$ metrizable topologies on $X$.

**Remark.** The technique used by Lozier and Marty also gives a collection of $2^\omega$ topologically distinct spaces satisfying all the conditions in Lemma 4.2. (See p. 273 of [LM].)

**Lemma 4.3.** Let $\omega \leq \kappa \leq 2^\omega$. Then there exist $2^\omega$ topologically distinct connected completely metrizable spaces of weight $\omega$ and cardinality $\kappa^\omega$, each having infinitely many non-cut points.

**Proof.** Let $\{X_\alpha : 0 \leq \alpha < 2^\omega\}$ be a collection of topologically distinct spaces as in Lemma 4.2. We may assume that $\kappa = \omega$. For each $\alpha < 2^\omega$ let $p_\alpha$ be a non-cut point of $X_\alpha$. Let $\mathcal{S} = \{S : S \subseteq 2^\omega, |S| = \kappa\}$, and for each $S \in \mathcal{S}$ let $X(S)$ be a star-space determined by $\{X_\alpha : \alpha \in S\}$ and $\{p_\alpha : \alpha \in \bar{S}\}$. Clearly $X(S)$ is a connected metrizable space of weight $\omega$ and cardinality $2^\omega$ with infinitely many non-cut points, and $X(S)$ is complete since each $X_\alpha$ is compact. Since we are assuming that $\kappa = \omega$, it follows that the adjunction point $p(S)$ is the only point of $X(S)$ which is a cut point of order $\kappa$. Hence by Proposition 3.1, $X(S)$ is not homeomorphic to $X(S')$ whenever $S_1 \neq S_2$. Thus $\{X(S) : S \in \mathcal{S}\}$ is a collection of $2^\omega$ topologically distinct spaces, each having the required properties.

**Remark.** In § 5 we shall make use of the fact that the spaces constructed in 4.3 are topologically complete.

**Lemma 4.4.** Let $X$ and $Y$ be connected spaces, each having more than one point. Then every point of $X \times Y$ is a non-cut point of $X \times Y$.

**Lemma 4.5.** Let $m$ and $n$ be cardinals with $2^\omega \leq m \leq n \leq m^n$. Then there is a connected metrizable space of weight $m$ and cardinality $n$ with no cut points.

**Proof.** Let $\mathcal{M}(m)$ be the hedgehog with $m$ spines and let $Y$ be the product of $\omega$ copies of $\mathcal{M}(m)$. Then $Y$ is a pathwise connected metrizable space of weight $m$ and cardinality $m^n$. Let $p \in Y$ and let $S$ be a subset of $Y$ with $|S| = n$. For each $x \in S$ let $f_x$ be a continuous function from $[0, 1]$ into $Y$ such that $f_x(0) = p$ and $f_x(1) = x$, and let $A_x = \{f_x(t) : t \in [0, 1]\}$. Note that $A_x$ is connected and $|A_x| < 2^n$. Let $Z = \bigcup_{x \in S} A_x$; clearly $Z$ is a connected metrizable space of weight $\leq m$ and cardinality $n$. Finally, $Z \times \mathcal{M}(m)$ is a connected metrizable space of weight $m$ and cardinality $n$ with no cut points. (See Lemma 4.4.)

**Lemma 4.6.** Let $m$ and $n$ be cardinals with $2^\omega \leq m \leq n \leq m^n$. Then there exist $2^\omega$ topologically distinct connected metrizable spaces of weight $m$ and cardinality $n$, each having infinitely many non-cut points.

**Proof.** The proof is by induction on $m$. The case $m = 2^\omega$ follows from Lemma 4.3. Now let $m = 2^{\geq \omega}$ and assume that the theorem is true for all $p$ with $2^\omega < p < m$. Let $n$ be a fixed cardinal with $m \leq n < m^n$; our objective is to construct $2^\omega$ topologically distinct connected metrizable spaces, each having weight $m$, cardinality $n$, and infinitely many non-cut points. For each $p$ with $2^\omega < p < m$ let $\mathcal{A}_p$ be a collection of $2^\omega$ topologically distinct connected metrizable spaces, each having weight $p$, cardinality $p$, and infinitely many non-cut points. Let $\mathcal{A} = \bigcup_{2^\omega < p < m} \mathcal{A}_p$; note that no two distinct elements of $\mathcal{A}$ are homeomorphic and $|\mathcal{A}| \geq m$. If $|\mathcal{A}| = p < m$, then $|\mathcal{A}_p \subset \mathcal{A}|$ and $|\mathcal{A}_p \subset \mathcal{A}|$ gives a contradiction.) Let $\{X_\alpha : 0 \leq \alpha < m\}$ be a collection of topologically distinct spaces such that $X_\alpha \in \mathcal{A}$ for $\alpha \geq 1$ and $X_0$ is a connected metrizable space with weight $m$, cardinality $n$, and no cut points. For each $\alpha < m$ let $p_\alpha$ be a non-cut point of $X_\alpha$. Let $\mathcal{S} = \{S : S \subseteq \mathcal{A}, \mathcal{A} \subseteq \mathcal{S}\}$, and for each $S \in \mathcal{S}$ let $X(S)$ be a star-space determined by $\{X_\alpha : \alpha \in S\}$ and $\{p_\alpha : \alpha \in \bar{S}\}$. It is clear that each $X(S)$ is a connected metrizable space having weight $m$, cardinality $n$, and infinitely many non-cut points. Moreover, the adjunction point $p(S)$ is the only point of $X(S)$ which is a cut point of order $m$. (Recall that $X_0$ has no cut points and $|X_\alpha | < m$ for $\alpha > 1$.) Hence by Proposition 3.1, $X(S)$ is not homeomorphic to $X(S')$ whenever $S_1 \neq S_2$. Thus $\{X(S) : S \in \mathcal{S}\}$ is a collection of $2^\omega$ topologically distinct spaces, each having the required properties.

**Theorem 4.7.** Let $m$ and $n$ be infinite cardinals with $m \leq n < m^n$ and $m > 2^\omega$. Then the number of topologically distinct connected metrizable spaces of weight $m$ and cardinality $n$ is $2^\omega$.

**Proof.** By Proposition 4.1 the number of such spaces is at most $2^\omega$, and so it remains to prove the existence of $2^\omega$ such spaces. First assume $2^\omega = \omega$. If $m > 2^\omega$ then the only allowable value for $n$ is $2^\omega$ and we are finished by Lemma 4.3. If $m > 2^\omega$ then we are finished by Lemma 4.6.
Now assume $2^\alpha < 2^\beta$. Let $Y$ be a connected metrizable space of weight $m$ and cardinality $\aleph_0$. If $m \geq 2^\alpha$ use Lemma 4.5; if $m < 2^\alpha$ use Lemma 4.4. Let $X = Y \times Y$, let $p$ be a point of $Y$, let $A = \{p\} \times Y$, and for each $y \in Y$ let $A_y = Y \times \{y\}$. Finally, for each non-empty subset $S$ of $Y$ let $X(S) = A \cup \bigcup A_y$. It is clear that $X(S)$ is a connected metrizable space of weight $m$ and cardinality $\aleph_0$, and that $X(S_1) \neq X(S_2)$ whenever $S_1 \neq S_2$. Let $\mathcal{A} = \{X(S) : S \subseteq Y, S \neq \emptyset\}$. Then $|\mathcal{A}| = 2^\alpha$, so by 3.3 there is a subset $\mathcal{A}_\alpha$ of $\mathcal{A}$ with $|\mathcal{A}_\alpha| = 2^\alpha$ such that no two distinct elements of $\mathcal{A}_\alpha$ are homeomorphic. This completes the proof of the case $2^\alpha < 2^\beta$.

**Corollary 4.8.** For each cardinal $m \geq 2^\omega$ the number of topologically distinct connected metrizable spaces of weight $m$ is $2^\omega$. In particular the number of topologically distinct connected metrizable spaces of weight $\omega$ is $2^\omega$.

**Corollary 4.9.** For each cardinal $\aleph_1 \geq 2^\omega$ the number of topologically distinct connected metrizable spaces of cardinality $\aleph_1$ is $2^\omega$.

**Lemma 4.10** (Mazurkiewicz and Sierpiński). The number of topologically distinct scattered subsets of $R$ is $2^\omega$.

**Theorem 4.11.** Let $m$ and $n$ be infinite cardinals with $m \leq n \leq 2^\omega$. Then the number of topologically distinct metrizable spaces of weight $m$ and cardinality $n$ is $2^\omega$.

**Proof.** By 4.1, 2$^\omega$ is an upper bound. By Theorem 4.7 we may assume that $n < 2^\omega$, and by Lemma 4.10 we may assume that $n = \omega$. Recall that every scattered subset of $R$ is countable. In summary we have $m \leq n < 2^\omega$ and $m \leq n$, and we want to construct $2^\omega$ topologically distinct metrizable spaces, each of weight $m$ and cardinality $n$.

We consider two cases, namely $2^\omega = 2^\omega$ and $2^\omega < 2^\omega$.

First suppose $2^\omega = 2^\omega$. We begin by constructing a metrizable space $X$ of weight $m$ and cardinality $n$ which is dense in itself. Let $A \subseteq R$, $|A| = n$. By the Cantor-Bendixson theorem, $A = B \cup C$, where $B$ is dense in itself and $C$ is countable. Since $|A| = n$, $|B| = m$. Let $X$ be the sum of $m$ copies of $B$; clearly $X$ is a metrizable space of weight $m$ and cardinality $n$ which is dense in itself. Now let $X_n = X_0$, let $X_n = X_n \cup C_n$. Clearly each $X_n$ is a metrizable space of weight $m$ and cardinality $n$, and from Proposition 3.2 it follows that $X_n$ is not homeomorphic to $X_{n'}$ for $n \neq n'$.

Now suppose $2^\omega < 2^\omega$. Let $\mathcal{A}$ be all subsets of $R$ of cardinality $n$; $|\mathcal{A}| = 2^\omega$. Let $\mathcal{B}$ be all subsets of $R$ of cardinality $n$ which are dense in themselves, and let $\mathcal{D}$ be all countable subsets of $R$. By the Cantor-Bendixson theorem,

$$|\mathcal{A}| = 2^\omega, \quad |\mathcal{B}| = 2^\omega, \quad |\mathcal{D}| < 2^\omega.$$
having weight $m$, cardinality $n$, and infinitely many noncut points. (The proof for "cardinality $n$" is similar but simpler.) For each $p$ with $2^{n} \leq p < m$ let $\mathcal{A}_{p}$ be a collection of $2^p$ topologically distinct connected completely metrizable spaces, each having weight $p$, cardinality $p$, and infinitely many noncut points, and let $\mathcal{A} = \bigcup_{p} \mathcal{A}_{p}$. Let $\{X_{\alpha}: 0 \leq \alpha < m\}$ be a collection of topologically distinct spaces such that $X_{\alpha} \in \mathcal{A}_{\alpha}$ for $\alpha \geq 1$ and $X_0$ is a connected completely metrizable space with weight $m$, cardinality $m$, and no cut points. For each $\alpha < m$ let $p_{\alpha}$ be a noncut point of $X_{\alpha}$ and let $d_{\alpha}$ be a complete metric on $X_{\alpha}$. Let $\mathcal{A} = \{S: S \subseteq m, 0 \in S \} \cup \{\mathcal{A}_{\alpha}: \alpha \in S\}$ and $\langle p_{\alpha}: \alpha \in S\rangle$. Then $\{X_{\alpha}: S \subseteq m\}$ is the desired collection of $2^m$ topologically distinct spaces.

Remark: Theorem 5.4 and the result of Lozier and Marty [LM] suggest the problem of counting the number of connected locally compact metrizable spaces. It turns out that such spaces always have weight $\omega$ and the total number of such spaces is $2^{\omega}$.

**Theorem 5.5.** The number of topologically distinct connected locally compact metrizable spaces is $2^{\omega}$. Moreover, each such infinite space has weight $\omega$ and cardinality $2^{\omega}$.

**Proof.** Let $X$ be an infinite connected locally compact metrizable space. By Alexandrov's Theorem $X = \bigcup_{\alpha \in S} X_{\alpha}$, where each $X_{\alpha}$ is a locally compact separable metrizable space. Since $X$ is connected, $|S| = 1$ and so $X$ is a locally compact separable metrizable space. Note that $|X| = 2^{\omega}$. By 5.2 the number of topologically distinct locally compact separable metrizable spaces is at most $2^{\omega}$, and the existence of $2^{\omega}$ such spaces which are also connected follows from Lemma 4.2.

6. The number of locally compact locally countable metrizable spaces. In this section we show that the number of topologically distinct locally compact locally countable metrizable spaces of cardinality $m$ is $\omega_{1}(\mathcal{A}(m))$, where $\mathcal{A}(m)$ denotes the number of cardinal numbers $\leq m$. There are two reasons for considering this enumeration result. First, it is a reasonable extension to higher cardinals of the classical result of Mazurkiewicz and Sierpiński [MS] that the number of topologically distinct compact countable metrizable spaces is $\omega_{1}$. Second, the result is used in §7 where we enumerate topologically distinct locally compact metrizable spaces of cardinality $m$.

The following notation is used in this section. The space consisting of a single point is denoted by $W_{0}$, and for $1 \leq \beta < \omega_{1}$, $W_{\beta}$ denotes the space of all ordinals $\leq \beta$ with the order topology. Thus each $W_{\beta}$ is a compact countable metrizable space. See Lemma 4.5 in [MP] for a proof that $W_{\beta}^{\omega} = (\omega^{\beta})^{\omega}$ for $\beta \geq 1$; note that $W_{0}^{\omega} = \emptyset$ for $\beta = 0$. For $0 < \beta < \omega_{1}$ and for any cardinal $m$ we let $K(\beta, m)$ denote the sum of $m$ copies of $W_{\beta}$. Mazurkiewicz and Sierpiński [MS] have proved that every compact countable metrizable space is homeomorphic to $K(\beta, \omega)$ for some $\beta < \omega_{1}$ and some natural number $n$.

**Lemma 6.1.** Let $X$ be an infinite locally countable metrizable space. Then the weight of $X$ and the cardinality of $X$ are the same.

**Proof.** Let the weight of $X$ be $m$; it suffices to show that $|X| \leq m$. Let $\mathcal{B}$ be an open cover of $X$, each element of which is countable. Since the weight of $X$ is $m$, there is a subcollection $\mathcal{B}_{0}$ of $\mathcal{B}$ with $|\mathcal{B}_{0}| \leq m$ such that $\mathcal{B}_{0}$ covers $X$. Hence $|X| \leq m$. Hence $\approx = m$.

**Lemma 6.2.** (Mazurkiewicz and Sierpiński). The number of topologically distinct compact countable metrizable spaces is $\omega_{1}$. Moreover, each such space is homeomorphic to the sum of a finite number of copies of some $W_{\beta} = \bigoplus_{\alpha < \omega_{1}} W_{\beta\alpha}$.

**Lemma 6.3.** The number of topologically distinct locally compact countable metrizable spaces is $\omega_{1}$.

**Proof.** By Lemma 6.2 it suffices to prove that $\omega_{1}$ is an upper bound. Let $\{X_{\alpha}: 0 < \alpha < \omega_{1}\}$ be all topologically distinct compact countable metrizable spaces. For each $\alpha < \omega_{1}$ let $\mathcal{A}_{\alpha}$ be all subsets of $X_{\alpha}$ obtained by removing at most one point from $X_{\alpha}$, and let $\mathcal{A} = \bigcup_{0 < \alpha < \omega_{1}} \mathcal{A}_{\alpha}$; note that $|\mathcal{A}| = \omega_{1}$. Now $X$ is a locally compact countable metrizable space. To complete the proof it suffices to show that $X$ is homeomorphic to some element of $\mathcal{A}$. We may assume that $X$ is not compact. Let $X^{*}$ be the Alexandrov one-point compactification of $X$. Clearly $X^{*}$ is a compact countable metrizable space, so $X^{*} = \bigoplus_{\alpha < \omega_{1}} X_{\alpha}$ for some $\alpha < \omega_{1}$. Hence $X^{*}$ is homeomorphic to a subset of $X_{\alpha}$ obtained by removing one point from $X_{\alpha}$.

**Lemma 6.4.** Let $X$ be a locally compact locally countable metrizable space of cardinality $\omega_{1}$. Then $X$ is homeomorphic to $\bigoplus_{0 < \alpha < \omega_{1}} W_{\beta_{\alpha}}$, where $0 < m_{\alpha} < m$ for all $\beta$ and $\sum_{0 < \alpha < \omega_{1}} m_{\alpha} = m$.

**Proof.** By Alexandrov's Theorem $X = \bigoplus_{\alpha \in S} X_{\alpha}$, where each $X_{\alpha}$ is a separable metrizable space. By Lemma 6.1 each $X_{\alpha}$ is countable. We are going to show that each $X_{\alpha}$ is the sum of a countable number of spaces, each of which is homeomorphic to some $W_{\beta}$. The proof is then completed by taking $m_{\alpha}$ to be the total number of copies of $W_{\beta}$ as $\beta$ ranges over $S$. (Since each $W_{\beta}$ is countable and $m_{\alpha} > 0$, each $\sum_{0 < \alpha < \omega_{1}} m_{\alpha}$ is finite.)

Fix $\alpha \in S$ and let $X_{\alpha} = \{p_{\alpha}: n < \omega\}$. Now $X_{\alpha}$ is zero-dimensional, so each point $p_{\alpha}$ has a neighborhood $V_{\alpha}$ which is both open and closed. We may assume that each $V_{\alpha}$ is also compact. Let $U_{n} = V_{\alpha}$, and for $n > 1$ let $U_{n} = V_{\alpha} - \overline{U}_{n-1}$. Clearly $\{U_{n}: n < \omega\}$ is a pairwise disjoint collection of open sets, and $X_{\alpha} = \bigcup_{n < \omega} U_{n}$. Now each $U_{n}$ is also closed and hence compact. By Lemma 6.2 $U_{n}$ is homeomorphic to the sum of a finite number of copies of some $W_{\beta}$.

**Lemma 6.5.** Let $\beta < \omega_{1}$. Then $W_{\beta} = G[H_{\beta}]$, where $G$ is $W_{\beta}$ and $H_{\beta}$ is $W_{\beta}$.

**Proof.** Since $W_{\beta} = [0, \omega_{1}^{\beta}] \oplus [\omega_{1}^{\beta}, 0]$ it suffices to show that $[\omega_{1}^{\beta}, 0]$ is $W_{\beta}$. Recall that $\omega_{1}^{\beta} = \omega_{1}^{\beta} \leq \omega_{1}^{\beta} \leq \omega_{1}^{\beta} \leq \omega_{1}^{\beta}$. So $[\omega_{1}^{\beta}, 0] = W_{\beta}$. Q.E.D.
Now \((\omega^\alpha, \omega^\beta) = (\omega^\alpha, \omega^\alpha+1) \oplus (\omega^\beta+1, \omega^\beta)\) and \(W_\beta = [0, \omega^\alpha+1] \oplus (\omega^\beta+1, \omega^\beta]\), so it suffices to show that \((\omega^\alpha, \omega^\alpha+1) \oplus [0, \omega^\beta+1]\). This easily follows from the fact that \((\omega, n, \omega^\alpha, (n+1)) \oplus (\omega^\alpha+1, \omega^\beta+2)\) for all \(n < \omega\). (See p. 298 in [S].)

**Lemma 6.6.** Let \(m\) be an infinite cardinal, let \(a\) be an ordinal with \(1 \leq a < \omega\), and let \(X\) be a topological space. Suppose there exists a subset \(U\) of \(X\) such that (1) \(X = U \cup (\bigcup_{\beta < \omega} K(\beta, m))\); (2) \(X \cap U \cong K(\beta, m)\); (3) \(X \cap U \cong \bigoplus_{\beta < \omega} X_\beta\), where \(|\beta| \leq m\) and for each \(\beta < \omega\) there exists \(\beta < \alpha\) such that \(X_\beta \cong W_\beta\). Then \(X \cong K(\beta, m)\).

**Proof.** Since \(U \cong \bigoplus_{\beta < \omega} K(\beta, m)\), \(U \cong \bigoplus_{\beta < \omega} W(\beta, i)\), where \(|\beta| = m\) and \(W(\beta, i) \cong W_\beta\) for each \(i \in T\). For each \(\beta < \omega\) let \(S_\beta = \{x \in S, X_\beta \cong W_\beta\}\); then \(X \setminus U = \bigoplus_{\beta < \omega} X_\beta\). Since \(X = U \cup (\bigcup_{\beta < \omega} X_\beta)\), it easily follows that

\[
X = \bigoplus_{\beta < \omega} ([\bigoplus_{i \in T} W(\beta, i)] \oplus \bigoplus_{\beta < \omega} X_\beta)
\]

and so \(X \cong K(\beta, m)\).

**Lemma 6.7.** Let \(m\) be a cardinal with \(m \geq \omega\), let \(a\) be an ordinal with \(1 \leq a < \omega\), and let \(\{m_\beta : 0 \leq \beta < a\}\) be a sequence of cardinals such that for all \(\beta < a\), \(\sum_{\beta < a} m_\beta = m\). If \(X\) is homeomorphic to \(\bigoplus_{\beta < \omega} K(\beta, m)\), then \(X\) is homeomorphic to \(\bigoplus_{\beta < \omega} K(\beta, m)\).

**Proof.** It suffices to construct a subset \(U\) of \(X\) satisfying (1)–(3) in Lemma 6.6. We first consider a special case, namely \(m = \omega\) and \(a = \omega\). Let \(X = \bigoplus_{\beta < \omega} W(\beta, i)\), where \(|\beta| = m\) and \(W(\beta, i) \cong W_\beta\) for all \(i \in T\). Let \(A = \{\beta : 0 \leq \beta < \omega, m_\beta \neq 0\}\). Since \(\sum_{\beta < a} m_\beta = m\) for all \(\beta < a\), \(A\) is uncountable. Let \(\{A_\beta : 0 \leq \beta < a\}\) be a pairwise disjoint collection of subsets of \(A\) such that each \(A_\beta\) has cardinality \(a\) and \(\gamma \in A_\beta\) implies \(\beta \gamma\). Now let \(\beta < a\) be fixed. For each \(\gamma \in A_\beta\) and each \(i \in T\), let \(W(\gamma, t) = G(\gamma, t) \oplus H(\gamma, t)\), where \(G(\gamma, t) \cong W_\beta\) and \(H(\gamma, t) \cong W_\gamma\). (Use Lemma 6.5.5.) Let

\[
X_\beta = \bigcup \{G(\gamma, t) : \gamma \in A_\beta, t \in T\}.
\]

Then \(X_\beta\) is the union of a pairwise disjoint collection of \(A_\beta\) open sets, each of which is homeomorphic to \(W_\beta\), and so \(X_\beta \cong K(\beta, m)\). Now let \(U = \bigcup_{\beta < \omega} X_\beta\); since \(X_\beta : 0 \leq \beta < a\) is a pairwise disjoint collection of open sets, and \(X_\beta \cong K(\beta, m)\) for all \(\beta < a\), \(U \cong \bigoplus_{\beta < \omega} K(\beta, m)\). It is easy to check that \(U\) also satisfies (1) and (3) of Lemma 6.6.

We now construct \(U\) under the assumption that \(a < \omega\) or \(m \geq \omega\). Let \(X = \bigoplus_{\beta < \omega} W(\beta, i)\), where \(|\beta| = m\) and \(W(\beta, i) \cong W_\beta\) for each \(i \in T\). Let \(X = \bigoplus_{\beta < \omega} W(\beta, i)\), and for each \(\gamma \in A\) let \(E(\gamma, t) = [0, \beta \gamma] \subseteq W(\beta, i)\), be a partition of \(E_\beta\) such that \(E(\gamma, \beta) = m_\beta\) for all \(\beta \leq \gamma\). For \(\gamma \in A\), \(0 \leq \gamma \leq \beta < \gamma\), and \(t \in E(\gamma, \beta)\), let \(X(\gamma, t) = G(\gamma, t) \oplus H(\gamma, \gamma, t)\), where \(G(\gamma, t) \cong W_\beta\) and \(H(\gamma, \gamma, t) \cong W_\gamma\). Let \(\beta \in \beta\) be fixed, \(0 < \beta \leq \gamma\). Let

\[
Y_\beta = \bigcup \{G(\gamma, t) : \beta \gamma < \gamma, \gamma \in A, t \in E(\gamma, \beta)\}
\]

and let \(Y_\beta = Y_\beta\) for \(\beta \in A\) and \(X_\beta = Y_\beta \cup \bigcup_{t \in T} W(\beta, i)\) for \(\beta \in \beta\). It is clear that \(X_\beta\) is the union of a pairwise disjoint collection of open sets, each of which is homeomorphic to \(W_\beta\), we now want to show that the number of copies of \(W_\beta\) in this union is \(m\). Note that

\[
\sum_{\beta < \omega} m_\beta + \sum_{\beta < \omega} m_\beta = m.
\]

First suppose \(a = \omega\). Then \(m > \omega\), and

\[
\sum_{\beta < \omega} m_\beta = m_\beta = m.
\]

Next suppose \(a \leq \omega\). Then

\[
\sum_{\beta < \omega} m_\beta = m_\beta = m.
\]

Thus in either case

\[
\sum_{\beta < \omega} m_\beta = m.
\]

and from this it easily follows that the number of copies of \(W_\beta\) is \(m\). Thus \(X_\beta \cong K(\beta, m)\). Now let \(U = \bigcup_{\beta < \omega} X_\beta\); since \(X_\beta : 0 \leq \beta < \omega\) is a pairwise disjoint collection of open sets and \(X_\beta \cong K(\beta, m)\) for all \(\beta < \omega\), \(U \cong \bigoplus_{\beta < \omega} K(\beta, m)\). It is easy to check that \(U\) also satisfies (1) and (3) of Lemma 6.6.

**Theorem 6.8.** For each cardinal \(m \geq \omega\) the number of topologically distinct locally compact locally countable metrizable spaces of cardinality \(m\) is \(\omega(m)\), where \(\omega(m)\) is the number of cardinals \(\leq m\).

**Proof.** We begin by showing that for each \(m \geq \omega\) there exist \(\omega(m)\) topologically distinct locally compact locally countable metrizable spaces of cardinality \(m\). First suppose \(\omega(m) = \omega\). Then \(K(\beta, m) = [0, \beta \omega)\), and for all \(\beta < \omega\), \(K(\beta, m) = [0, \beta \omega)\), is a collection of \(a\) locally compact locally countable metrizable spaces, each of cardinality \(m\). Moreover if \(\beta < \gamma\) then \(K(\beta, m) = \emptyset\) (and \(K(\gamma, m) = \emptyset\)) and so \(K(\beta, m)\) is not homeomorphic to \(K(\gamma, m)\) whenever \(\beta < \gamma\). Now suppose \(\omega(m) = \omega\). Then for each cardinal \(p \leq m\) let \(X(p) = K(W_1, p) \oplus W_\beta\). Then \(X(p) = \omega(p)\) is a collection of \(\omega(m)\) locally compact locally countable metrizable spaces, each of cardinality \(m\). Moreover, if \(p < m\) then \(|X(p)| = p\) and \(|X(m)| = m\) and so \(X(p)\) cannot be homeomorphic to \(X(m)\).
The number of metrizable spaces

To complete the proof we must show that $\alpha_1 \cdot \nu(m)$ is an upper bound. The proof is by induction on $m$. If $m = \omega$ we are finished by Lemma 6.3. Now let $m = \omega$ and assume true for all cardinals $\eta$ with $\omega \leq \eta < m$. Let $S = \{\eta : \omega \leq \eta < m\}$, for each $\eta \in S$ let $\Delta_{\eta}$ be all topologically distinct locally compact metrizable spaces of cardinality $\eta$, and let $\Delta = \bigcup_{\eta \in S} \Delta_{\eta}$. Since $|\Delta| \leq \alpha_1 \cdot \nu(\eta) \leq \alpha_1 \cdot \nu(m)$ for all $\eta$, it follows that $|\Delta| \leq \sum_{\eta \in S} \alpha_1 \cdot \nu(m) = \alpha_1 \cdot \nu(m)$. Let

$$\mathcal{B} = \left\{ \left( \bigoplus_{x \in \alpha_1} \mathbb{K}(x, \beta, m) \right) : 1 \leq x \leq \alpha_1, \beta \in \alpha_2, \alpha \in \Delta \right\}.$$  

It is clear that $|\mathcal{B}| \leq \alpha_1 \cdot \nu(m)$ and that each $B \in \mathcal{B}$ is a locally compact locally countable metrizable space of cardinality $m$.

Now let $X$ be a locally compact locally countable metrizable space of cardinality $m$. To complete the proof it suffices to show that $X$ is homeomorphic to some $B \in \mathcal{B}$. By Lemma 6.4 $X \cong \bigoplus_{\alpha \in \alpha_1} \mathbb{K}(x, \beta, m)$, where $\sum_{\alpha \in \alpha_1} m_\alpha = m$. We now consider two cases: (1) $\sum_{\alpha \in \alpha_1} m_\alpha = m$ for all $\alpha < \omega_1$; (2) there is an ordinal $\alpha$ such that $\sum_{\alpha \in \alpha_1} m_\alpha = m$ for all $\alpha < \omega_1$ and $m_\omega < m$. If (1) holds, then $X \cong \bigoplus_{\alpha \in \alpha_1} \mathbb{K}(x, \beta, m)$ by Lemma 6.5 and so $X \cong B$ for some $B \in \mathcal{B}$. Suppose (2) holds, and note that $X \cong \bigoplus_{\beta \in \alpha_2} \mathbb{K}(x, \beta, m_\beta)$ by Lemma 6.6. Let $\mathcal{B} = \bigoplus_{\beta \in \alpha_2} \mathbb{K}(x, \beta, m_\beta)$, and since $\sum_{\beta \in \alpha_2} m_\beta < m$ it follows that $\bigoplus_{\alpha \in \alpha_1} \mathbb{K}(x, \beta, m_\beta) \cong B$ for some $B \in \mathcal{B}$. Consequently $X \cong B$ for some $B \in \mathcal{B}$.

**Corollary 6.9.** The number of topologically distinct locally compact locally countable metrizable spaces of cardinality $\omega_1$ is $\alpha_1$.

**Corollary 6.10.** The number of topologically distinct locally compact locally countable metrizable spaces of cardinality $\omega^2$ is $\alpha_1 \cdot \nu(m)$.

**Corollary 6.11.** For each cardinal $m \geq \omega$ the number of topologically distinct locally compact locally countable metrizable spaces of weight $m$ is $\alpha_1 \cdot \nu(m)$.

**Remark.** There exist cardinals $m$ for which $\nu(m) = m$. Indeed, if $m$ is a fixed point of the Aleph function (i.e., $\aleph_0 = m$), then $\nu(m) = m$.

7. The number of locally compact metrizable spaces. In this section we find the number of locally compact metrizable spaces of cardinality $m$ and also the number of locally compact metrizable spaces of weight $m$. The solution is obtained by considering two cases, namely $m \geq 2^\omega$ and $m < 2^\omega$. Recall that $\nu(m)$ is the number of cardinals $\leq m$.

**Theorem 7.1.** Let $m$ be an infinite cardinal. The number of topologically distinct locally compact metrizable spaces of cardinality $m$ is $\alpha_1 \cdot \nu(m)$ if $m < 2^\omega$ and $\nu(m)^{2^\omega}$ if $m \geq 2^\omega$.

**Proof.** First suppose $m < 2^\omega$. By Theorem 6.8 it suffices to show that every locally compact metrizable space $X$ of cardinality $m$ (where $m < 2^\omega$) is locally countable. Let $p \in X$, let $K$ be a compact neighborhood of $p$. Then $|K| \leq \omega$ or $|K| = 2^\omega$. Since $|X| = m < 2^\omega$, $K$ must be countable.

Now suppose $m \geq 2^\omega$. Let $\mathcal{F}$ be the collection of all functions from $2^\omega$ into the set of all cardinal numbers $\leq m$. Note that $|\mathcal{F}| = \nu(m)^{2^\omega}$. First we prove the existence of $\nu(m)^{2^\omega}$ topologically distinct locally compact metrizable spaces, each of cardinality $m$. Let $\mathcal{X}_m = \{X : 0 < \omega < 2^\omega\}$ be a collection of $2^\omega$ topologically distinct compact connected metrizable spaces, each of cardinality $m$. (Use Lemma 4.2.) Let $f \in \mathcal{F}$, for each $x \in X_m$ let $X(f, x)$ be the sum of $|f(x)|$ copies of $X_m$, and let $X(f) = \bigoplus_{x \in X_m} X(f, x)$.

$$X(f) = \left( \bigoplus_{x \in X_m} X(f, x) \right) \oplus D_m,$$

where $D_m$ is the discrete space of cardinality $m$. Clearly each $X(f) = \mathbb{K}(x, \beta, m)$ is a locally compact metrizable space of cardinality $m$. Moreover, if $f \neq g$ then $X(f) \neq X(g)$. Let $f \neq g$, say $f(a) \neq g(a)$ for some $a \in 2^\omega$, and suppose $X(f) \cong X(g)$. Each copy of $X_m$ in the sum defining $X(f)$ must map homeomorphically onto a copy of $X_m$ in the sum defining $X(g)$. Since $f(a) \neq g(a)$ this is impossible.

Next we show that $\nu(m)^{2^\omega}$ is an upper bound. Let $\mathcal{X}_m = \{X : 0 < \omega < 2^\omega\}$ be all topologically distinct locally compact separable metrizable spaces. (Use Proposition 5.2.) For each $f \in \mathcal{F}$ let $X(f) = \bigoplus_{x \in X_m} X(f, x)$, where $X(f, x)$ is the sum of $|f(x)|$ copies of $X_m$. The cardinality of $\{X(f) : f \in \mathcal{F}\} = \nu(m)^{2^\omega}$. Now let $X$ be a locally compact metrizable space of cardinality $m$. To prove the proof, it suffices to show that $X$ is homeomorphic to some $X(f)$. By Alexandroff's Theorem $X = \bigoplus_{x \in X_m} X$, where each $X$ is a separable metrizable space. Note that $X$ is also locally compact and $|X| \leq m$. For each $x \in X_m$ let $S_x = \{x \in S : x \in X_x\}$ and let $f$ be the element of $\mathcal{F}$ defined by $f(a) = |S_a|$. Then $X = \bigoplus_{x \in X_m} X_x$.

**Corollary 7.2.** The number of topologically distinct locally compact metrizable spaces of cardinality $\omega^2$ is $2^\omega$.

**Corollary 7.3.** The number of topologically distinct locally compact metrizable spaces of cardinality $\omega^\omega$ is $2^\omega$.

**Lemma 7.4.** Let $X$ be an locally compact metrizable space of weight $m$. If $m \geq 2^\omega$ then $|X| = m$ and if $m < 2^\omega$ then $|X| = m$ or $|X| = 2^\omega$.

**Proof.** First assume $m \geq 2^\omega$. Suppose $X$ is not a compact cover of $X$ such that for each $G \in G$, $G$ is compact. Note that $|G| \leq 2^\omega$. Since the weight of $X$ is $m$, there is a subcollection $\mathcal{G}$ of $G$ with $|\mathcal{G}| \leq m$ which covers $X$. Hence $|X| \leq m \cdot 2^\omega = m$. Now suppose $m < 2^\omega$. By Proposition 5.1, $|X| = m$ or $|X| = m^\omega$. Since $\omega \leq m < 2^\omega$, $m^\omega = 2^\omega$.

**Theorem 7.5.** Let $m$ be an infinite cardinal. If $m > 2^\omega$ the number of topologically distinct locally compact metrizable spaces of weight $m$ is $\nu(m)^{2^\omega}$, and each such space has cardinality $m$. If $m < 2^\omega$ the number of topologically distinct locally compact metrizable spaces of weight $m$ is $2^\omega$, and each such infinite space has cardinality $m^\omega$. Moreover the number of cardinality $m$ is $\alpha_1 \cdot \nu(m)$ and the number of cardinality $2^\omega$ is $2^\omega$.

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* Fundamenta Mathematicae CXV
Proof. First suppose \( m \geq 2^n \). By 7.4 every locally compact metrizable space of weight \( m \geq 2^n \) has cardinality \( m \), and so we are finished by 7.1.

Now assume \( m < 2^n \). It follows from 5.2 that \( 2^n \) is an upper bound for the number of topologically distinct locally compact metrizable spaces of weight \( m \). Moreover by 7.4 it follows that each such infinite space has cardinality \( m \) or \( 2^n \), and it follows from 7.1 that the number of such spaces of cardinality \( m \) is \( \omega \cdot \omega(m) \). Consequently the proof is complete if we can construct \( 2^n \) topologically distinct locally compact metrizable spaces, each of weight \( m \) and cardinality \( 2^n \). Let \( X'(s) \) be a collection of \( 2^n \) topologically distinct compact connected metrizable spaces, each of cardinality \( 2^n \). Let \( S = \{ s : S \in 2^n, |S| = m \} \), and for each \( S \in S \) let

\[ X(S) = \biguplus_{s \in S} X'(s) \]

Clearly \( \{ X(S) : S \in S \} \) is a collection of \( 2^n \) locally compact metrizable spaces, each of weight \( m \) and cardinality \( 2^n \), and \( X(S) \) is not homeomorphic to \( X(S') \) whenever \( S \neq S' \).

8. The number of connected paracompact spaces. For each cardinal \( m \geq 2^n \) the number of connected compact spaces of cardinality \( m \) is \( 2^n \) and the number of connected metrizable spaces of cardinality \( m \) is \( 2^n \). (See [LM] and § 4.) Each of these classes of spaces is contained in the class of connected paracompact spaces of cardinality \( m \). Is this latter class larger? Yes. More precisely, we show that the number of connected spaces of this class is the maximum possible, namely \( 2^n \). The proof makes use of the following facts about ultrafilters. Let \( S \) be a set with \( |S| = m \), let \( \mathcal{F} \) be the family of all free ultrafilters on \( S \). Then \( \mathcal{F} \) is the family of all free ultrafilters on \( S \). Then \( \mathcal{F} \) is the family of all free ultrafilters on \( S \).

The following result, which is an easy consequence of Lemma 1 in [M], will be useful.

**Lemma 8.1.** Let \( X \) be a regular space such that \( X = K \cup M \), where \( K \) is compact and \( M \) is metrizable. Then \( X \) is paracompact.

**Theorem 8.2.** For each cardinal \( m \geq 2^n \) the number of topologically distinct connected paracompact spaces of cardinality \( m \) is the maximum possible, namely \( 2^n \).

Proof. It suffices to construct \( 2^n \) such spaces. Let \( S \) be a set with \( |S| = m \). For each \( s \in S \) let \( X_{s} = \{ 0, 1 \} \times \{ s \} \), let \( d_{s} \) be the Euclidean metric on \( X_{s} \), let \( q_{s} = (1, s) \), and let \( W_{s} \) be a fundamental system of open neighborhoods of \( q_{s} \), none of which contains \( (0, s) \). Let \( X(S) \) be the star-space determined by \( \{ (X_{s}, d_{s}) : s \in S \} \) and \( \{ s \} : s \in S \) \}. Let \( \{ p_{k} : 0 < a < 2^{2m} \} \) be a collection of \( 2^{2m} \) free ultrafilters on \( S \), no two of which are the same type. Now let \( a < 2^{2m} \) be fixed; we are going to construct a connected paracompact space \( X_{a} \) of cardinality \( m \). Let \( X_{a} = X(S) \cup \{ a \} \), and take as a base for \( X_{a} \) the collection of all open subsets of \( X(S) \) together with all sets of the form \( \{ a \} \cup \bigcup_{k} W_{s} \), where \( a \in p_{k} \) and \( W_{s} \in W_{s} \). Note that the subspace \( \{ a \} \cup \{ q_{s} : s \in S \} \) of \( X_{a} \) is homeomorphic to the subspace \( \{ p_{k} \} \cup \{ s \} \) of \( \beta S \). It is not
difficult to check that \( X_{a} \) is \( T_{1} \), regular, connected, and has cardinality \( m \); that \( X_{a} \) is paracompact follows easily from 8.1.

Now suppose \( a \) is a homeomorphism from \( X_{a} \) onto \( X_{a} \). Then \( a \) must take \( a \) to \( b \) and \( \{ q_{s} : s \in S \} \) to \( \{ q_{s} : s \in S \} \). (Note that \( \{ q_{s} : s \in S \} \) is the set of noncut points of \( X_{a} \) and \( a \) is the only point of \( X_{a} \) which does not have a countable local base.) Hence \( \{ q_{s} : s \in S \} \neq \{ \beta \} \cup \{ q_{s} : s \in S \} \) and so \( p_{k} \) and \( p_{l} \) are of the same type and \( a = b \). Consequently \( \{ X_{a} : 0 < a < 2^{2m} \} \) is a collection of \( 2^{m} \) topologically distinct spaces, each having the desired properties.

9. Concluding remarks. The results in this paper can be viewed as giving a rough measure of the "niceness" of a class of topological spaces. For example, the number of compact manifolds, with or without boundary, is \( \omega \) (see [CK]); the number of locally compact connected (separable) metrizable spaces is \( 2^{m} \); the number of connected separable metrizable spaces is \( 2^{m} \); the number of connected paracompact spaces is \( 2^{m} \). Similarly, for each cardinal \( m \geq 2^n \) the number of continuas of cardinality \( m \) is \( 2^{m} \) and the number of connected metrizable spaces of cardinality \( m \) is \( 2^{m} \) but the number of connected paracompact spaces of cardinality \( m \) is \( 2^{m} \).

References


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