Some topological consequences of the Product Measure Extension Axiom

by

Heikki J. K. Junnila * (Helsinki)

Abstract. Under the assumption that the Product Measure Extension Axiom holds, the following results obtain:

Every (sub)normal space of weak character below $\underset{\sim}{c}$ is collectionwise (sub)normal.

If X is either of weak character below $\underset{\sim}{c}$ or semi-stratifiable, then every G_δ-additive partition of X has a σ-discrete closed refinement.

The unparenthesized part of the first result answers a question of P. Nyikos.

In [12], P. Nyikos gave a "provisional" solution to one of the outstanding problems of general topology by showing that if the usual set-theory (ZFC) has a model in which the Product Measure Extension Axiom (PMEA) holds, then in that model, every normal Moore space is metrizable. It was already known (see [15]) that the existence of a normal, non-metrizable Moore space is consistent with ZFC; however, Nyikos's result did not quite establish the independence of the Normal Moore Space Conjecture from ZFC, since it is not known whether PMEA is consistent with ZFC.

In this paper, we investigate properties that certain topological spaces have in models of ZFC + PMEA. We answer a question of Nyikos ([12]) by showing that under PMEA, every normal space of weak character below 2^ω is collectionwise normal. Our other results deal with the relationship between subnormal and collectionwise subnormal spaces (see [5]) and with properties of G_δ-additive families of subsets of topological spaces.

1. Preliminaries. Let ω be the set of all finite ordinals, and let $\underset{\sim}{c} = 2^\omega$. The Product Measure Extension Axiom, as defined in [12], asserts the existence, for each cardinal number $\varkappa$, of a $\underset{\sim}{c}$-additive measure defined for all subsets of $\{0,1\}^\varkappa$ such that the measure extends the product measure $\mu^\varkappa$, where μ is the measure on $\{0,1\}$ defined by $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$. We shall use an equivalent formulation of the axiom that is better suited for the purposes of this paper.

The *Product Measure Extension Axiom* (PMEA) is equivalent (in ZFC) with the statement that for any set A, there exists a non-negative real-valued function μ

---

* This paper was written while the author was visiting the University of Pittsburgh as an Andrew Mellon Postdoctoral Fellow.

whose domain of definition is the set consisting of all families of subsets of A and which satisfies the following conditions:

1. If $\mathscr{H}$ is a collection of mutually disjoint families of subsets of A and $|\mathscr{H}|<\mathfrak{c}$, then $\mu(\bigcup\mathscr{H}) = \sum\{\mu(\mathscr{H})|\ \mathscr{H}\in\mathscr{H}\}$.

2. If F and G are mutually disjoint finite subsets of A and $n = |F\cup G|$, then $\mu(\{B\subset A|\ F\subset B \text{ and } G\cap B = \emptyset\}) = 2^{-n}$.

By abuse of language, we call any function with the properties required above a *product measure extension for A*.

For a discussion on the consistency of PMEA relative to that of some other set-theoretic axioms, as well as for other background on PMEA, we refer the reader to [12].

For the meaning of topological concepts used without definition in this paper, we refer the reader to [7]. Let X be a topological space, and for each $x\in X$, let $\mathscr{M}_x$ be a filterbase on X such that $x\in\bigcap\mathscr{M}_x$. The indexed collection $\langle\mathscr{M}_x\colon x\in X\rangle$ is called a *weak neighborhood base assignment* for the space X provided that for every $G\subset X$, the set G is open if, and only if, for each $x\in G$ there exists $M\in\mathscr{M}_x$ such that $M\subset G$. We say that the space X has *weak character below* $\mathfrak{c}$ provided that there exists a weak neighborhood base assignment $\langle\mathscr{M}_x\colon x\in X\rangle$ for X such that for each $x\in X$, we have $|\mathscr{M}_x|<\mathfrak{c}$. For a relationship between weak neighborhood base assignments in topological spaces and neighborhoods in closure spaces, see [4], Section 14.B; we just remark here that a topological space Y has weak character below $\mathfrak{c}$ if, and only if, the topology of Y can be determined by a closure operation so that in the associated closure space, every point has a neighborhood base of cardinality $<\mathfrak{c}$.

The purpose of the above definitions is to provide us with as large as possible a class of spaces for which the conclusion of the following lemma obtains.

LEMMA 1.1. *Let X be a space with weak character below $\mathfrak{c}$, let A be some set, and let $\{G(B)|\ B\subset A\}$ be a family of open subsets of X. For each $x\in X$, let $\mathscr{A}_x = \{B\subset A|\ x\in G(B)\}$. Let μ be a product measure extension for A. Then for all $x\in X$ and $\varepsilon>0$, the set $\{y\in X|\ \mu(\mathscr{A}_x\sim\mathscr{A}_y)<\varepsilon\}$ is open.*

Proof. Let $\langle\mathscr{M}_x\colon x\in X\rangle$ be a weak neighborhood base assignment for X such that for each $x\in X$, $|\mathscr{M}_x|<\mathfrak{c}$. For all $x\in X$ and $\varepsilon>0$, let

$$U_\varepsilon(x) = \{y\in X|\ \mu(\mathscr{A}_x\sim\mathscr{A}_y)<\varepsilon\}:$$

Note that for all $x\in X$ and $\varepsilon>0$, if $y\in U_\varepsilon(x)$, then, setting $\delta = \varepsilon-\mu(\mathscr{A}_x\sim\mathscr{A}_y)$, we have $\delta>0$ and $U_\delta(y)\subset U_\varepsilon(x)$. Consequently, to establish the openness of the sets $U_\varepsilon(x)$, it suffices to show that for all $y\in X$ and $\delta>0$, there exists $M\in\mathscr{M}_y$ such that $M\subset U_\delta(y)$. Let $y\in X$ and $\delta>0$. For every $B\in\mathscr{A}_y$, since the set $G(B)$ is open and $y\in G(B)$, there exists $M(B)\in\mathscr{M}_y$ such that $M(B)\subset G(B)$. For each $M\in\mathscr{M}_y$, let $\mathscr{A}(M) = \{B\in\mathscr{A}_y|\ M(B) = M\}$. We have $\mathscr{A}_y = \bigcup\{\mathscr{A}(M)|\ M\in\mathscr{M}_y\}$ and it follows, since $|\mathscr{M}_y|<\mathfrak{c}$, that $\mu(\mathscr{A}_y) = \sum\{\mu(\mathscr{A}(M))|\ M\in\mathscr{M}_y\}$. Consequently, there exists a finite subfamily $\mathscr{N}$ of $\mathscr{M}_y$ such that $\sum\{\mu(\mathscr{A}(N))|\ N\in\mathscr{N}\}>\mu(\mathscr{A}_y)-\delta$.

Let $M\in\mathscr{M}_y$ be such that $M\subset\bigcap\mathscr{N}$, and let $\mathscr{B} = \{B\subset A|\ M\subset G(B)\}$. We have $\bigcup\{\mathscr{A}(N)|\ N\in\mathscr{N}\}\subset\mathscr{B}$, and it follows that $\mu(\mathscr{B})>\mu(\mathscr{A}_y)-\delta$. For each $z\in M$, we have $\mathscr{B}\subset\mathscr{A}_z$ and hence $\mathscr{A}_y\sim\mathscr{A}_z\subset\mathscr{A}_y\sim\mathscr{B}$ so that $\mu(\mathscr{A}_y\sim\mathscr{A}_z)\leqslant\mu(\mathscr{A}_y)-\mu(\mathscr{B})<\delta$; consequently, $M\subset U_\delta(y)$.

COROLLARY 1.2. *Let X, A, $\{G(B)|\ B\subset A\}$ and μ be as in Lemma 1.1, and let $a\in A$. For each $x\in X$, let*

$$\mathscr{B}_x = \{B\subset A|\ a\in B\Rightarrow x\in G(B)\} \quad\text{and}\quad \mathscr{C}_x = \{B\subset A|\ a\in A\sim B\Rightarrow x\in G(A\sim B)\}.$$

Then for each $r\in R$, the sets $\{x\in X|\ \mu(\mathscr{B}_x)>r\}$ and $\{x\in X|\ \mu(\mathscr{C}_x)>r\}$ are open.

Proof. Define $\mathscr{A}_x$, $x\in X$, as in Lemma 1.1. Openness of the sets $\{x\in X|\ \mu(\mathscr{B}_x)>r\}$ follows using Lemma 1.1 and the observation that $\mu(\mathscr{B}_y)\geqslant\mu(\mathscr{B}_x)-\mu(\mathscr{B}_x\sim\mathscr{B}_y)$ and $\mathscr{B}_x\sim\mathscr{B}_y\subset\mathscr{A}_x\sim\mathscr{A}_y$ for all $x\in X$ and $y\in X$.

For each $x\in X$, let $\mathscr{A}_x^* = \{B\subset A|\ x\in G(A\sim B)\}$. Applying Lemma 1.1 to the family $\{G(A\sim B)|\ B\subset A\}$, we obtain the result that for all $x\in X$ and $\varepsilon>0$, the set $\{y\in X|\ \mu(\mathscr{A}_x^*\sim\mathscr{A}_y^*)<\varepsilon\}$ is open. Openness of the sets $\{x\in X|\ \mu(\mathscr{C}_x)>r\}$ now follows easily, since $\mathscr{C}_x\sim\mathscr{C}_y\subset\mathscr{A}_x^*\sim\mathscr{A}_y^*$ for all $x\in X$ and $y\in X$.

2. PMEA and normalized families. Recall that a family $\mathscr{L}$ of subsets of a topological space is a *normalized family* provided that for every $\mathscr{L}'\subset\mathscr{L}$, there exist open sets U and V such that $\bigcup\mathscr{L}'\subset U$, $\bigcup(\mathscr{L}\sim\mathscr{L}')\subset V$ and $U\cap V = \emptyset$. An *expansion* of a family $\mathscr{L}$ is a family $\{E(L)|\ L\in\mathscr{L}\}$ of sets such that $L\subset E(L)$ for each $L\in\mathscr{L}$; the expansion $\{E(L)|\ L\in\mathscr{L}\}$ is a *separation* of $\mathscr{L}$ if $E(L)\cap E(L') = \emptyset$ whenever $L\neq L'$. A family of subsets of a topological space is said to be *well separated* if the family has an open separation. Every well-separated family is normalized. The converse is not true, since there exist normal spaces that are not collectionwise normal (see [2]), and a space is normal (collectionwise normal) if and only if every discrete family of subsets of the space is normalized (well separated).

Nyikos showed in [12] that under PMEA, every normalized family of subsets in a space of character below $\mathfrak{c}$ is well separated, and he asked whether this result can be extended to spaces with weak character below $\mathfrak{c}$. We now show that the answer to Nyikos's question is in the affirmative.

THEOREM 2.1. (PMEA) *In a space of weak character below $\mathfrak{c}$, every normalized family of subsets of the space is well separated.*

Proof. Let X be a space of weak character below $\mathfrak{c}$, and let $\mathscr{L}$ be a normalized family of subsets of X. Then there exist open subsets $G(\mathscr{L}')$, $\mathscr{L}'\subset\mathscr{L}$ of X such that for each $\mathscr{L}'\subset\mathscr{L}$ we have $\bigcup\mathscr{L}'\subset G(\mathscr{L}')$ and $G(\mathscr{L}')\cap G(\mathscr{L}\sim\mathscr{L}') = \emptyset$. For all $L\in\mathscr{L}$ and $x\in X$, let

$$\mathscr{B}_x(L) = \{\mathscr{L}'\subset\mathscr{L}|\ L\in\mathscr{L}'\Rightarrow x\in G(\mathscr{L}')\}$$

and

$$\mathscr{C}_x(L) = \{\mathscr{L}'\subset\mathscr{L}|\ L\in\mathscr{L}\sim\mathscr{L}'\Rightarrow x\in G(\mathscr{L}\sim\mathscr{L}')\};$$

note that if $x\in L$, then for each $\mathscr{L}'\subset\mathscr{L}$, $\mathscr{L}'\in\mathscr{B}_x(L)\cap\mathscr{C}_x(L)$.

Let μ be a product measure extension for $\mathscr{L}$. For every $L \in \mathscr{L}$, let

$$U(L) = \{x \in X \mid \mu(\mathscr{B}_x(L)) > \tfrac{7}{8} \text{ and } \mu(\mathscr{C}_x(L)) > \tfrac{7}{8}\}.$$

Since $\mu(\{\mathscr{L}' \mid \mathscr{L}' \subset \mathscr{L}\}) = 1$, we see that for each $L \in \mathscr{L}$, $L \subset U(L)$. By Corollary 1.2, the sets $U(L)$, $L \in \mathscr{L}$, are open. Consequently, $\{U(L) \mid L \in \mathscr{L}\}$ is an open expansion of $\mathscr{L}$. We show that the expansion is a separation. Assume on the contrary that there are $L \in \mathscr{L}$ and $K \in \mathscr{L}$ such that $L \neq K$ and $U(L) \cap U(K) \neq \emptyset$. Let $y \in U(L) \cap U(K)$ and $\mathscr{D} = \mathscr{B}_y(L) \cap \mathscr{C}_y(K)$. We have $\mu(\mathscr{B}_y(L)) > \frac{7}{8}$ and $\mu(\mathscr{C}_y(K)) > \frac{7}{8}$, and it follows that $\mu(\mathscr{D}) > \frac{3}{4}$. Let $\mathscr{Q} = \{\mathscr{L}' \subset \mathscr{L} \mid L \in \mathscr{L}' \text{ and } K \notin \mathscr{L}'\}$. By property 2° of product measure extensions, we have $\mu(\mathscr{Q}) = \frac{1}{4}$. It follows that $\mu(\mathscr{D} \cap \mathscr{Q}) > 0$ and hence that $\mathscr{D} \cap \mathscr{Q} \neq \emptyset$. Let $\mathscr{L}' \in \mathscr{D} \cap \mathscr{Q}$. Then $L \in \mathscr{L}'$ and $K \in \mathscr{L} \sim \mathscr{L}'$. Since $\mathscr{L}' \in \mathscr{B}_y(L)$ and $L \in \mathscr{L}'$, we have $y \in G(\mathscr{L}')$; also, since $\mathscr{L}' \in \mathscr{C}_y(K)$ and $K \in \mathscr{L} \sim \mathscr{L}'$, we have $y \in G(\mathscr{L} \sim \mathscr{L}')$. However, $G(\mathscr{L}') \cap G(\mathscr{L} \sim \mathscr{L}') = \emptyset$. This contradiction shows that $\{U(L) \mid L \in \mathscr{L}\}$ is a separation of $\mathscr{L}$.

COROLLARY 2.2. (PMEA) *Every normal space of weak character below $\mathfrak{c}$ is collectionwise normal.*

In particular, every normal symmetrizable (see [1]) space is collectionwise normal under PMEA.

Next, we consider a generalization of normalized families, and we prove an analogue of the result of Theorem 2.1 for these families.

A family $\mathscr{L}$ of subsets of a topological space X is a *δ-normalized family* provided that for every $\mathscr{L}' \subset \mathscr{L}$, there exist G_δ-sets P and Q such that $\bigcup \mathscr{L}' \subset P$, $\bigcup (\mathscr{L} \sim \mathscr{L}') \subset Q$ and $P \cap Q = \emptyset$. A *δ-separation* of $\mathscr{L}$ is a sequence

$$\langle \{E_n(L) \mid L \in \mathscr{L}\} \rangle_{n=0}^{\infty}$$

of expansions of $\mathscr{L}$ such that for each $x \in X$, there exists $k \in \omega$ such that the family $\{L \in \mathscr{L} \mid x \in E_k(L)\}$ has at most one member. We say that $\mathscr{L}$ is *well δ-separated* if $\mathscr{L}$ has a δ-separation consisting of open expansions of $\mathscr{L}$. Note that if $\langle \{E_n(L) \mid L \in \mathscr{L}\} \rangle_{n=0}^{\infty}$ is a δ-separation of $\mathscr{L}$, then $\{\bigcap_{n \in \omega} E_n(L) \mid L \in \mathscr{L}\}$ is a separation of $\mathscr{L}$; consequently, if $\mathscr{L}$ is well δ-separated, then $\mathscr{L}$ has a G_δ-separation.

THEOREM 2.3. (PMEA) *In a space of weak character below $\mathfrak{c}$, every δ-normalized family of subsets of the space is well δ-separated.*

Proof. Let X be a space of weak character below $\mathfrak{c}$ and let $\mathscr{L}$ be a δ-normalized family of subsets of X. Then there exist G_δ-sets $P(\mathscr{L}')$, $\mathscr{L}' \subset \mathscr{L}$, such that for each $\mathscr{L}' \subset \mathscr{L}$, $\bigcup \mathscr{L}' \subset P(\mathscr{L}')$ and $P(\mathscr{L}') \cap P(\mathscr{L} \sim \mathscr{L}') = \emptyset$. For every $\mathscr{L}' \subset \mathscr{L}$, let $\langle G_n(\mathscr{L}') \rangle_{n=0}^{\infty}$ be a sequence of open sets such that $\bigcap_{n \in \omega} G_n(\mathscr{L}') = P(\mathscr{L}')$ and for each $n \in \omega$, $G_{n+1}(\mathscr{L}') \subset G_n(\mathscr{L}')$. For all $L \in \mathscr{L}$, $x \in X$ and $n \in \omega$, let

$$\mathscr{B}_{x,n}(L) = \{\mathscr{L}' \subset \mathscr{L} \mid L \in \mathscr{L}' \Rightarrow x \in G_n(\mathscr{L}')\}$$

and

$$\mathscr{C}_{x,n}(L) = \{\mathscr{L}' \subset \mathscr{L} \mid L \in \mathscr{L} \sim \mathscr{L}' \Rightarrow x \in G_n(\mathscr{L} \sim \mathscr{L}')\}.$$

Let μ be a product measure extension for $\mathscr{L}$. For all $L \in \mathscr{L}$ and $n \in \omega$, let

$$U_n(L) = \{x \in X \mid \mu(\mathscr{B}_{x,n}(L)) > \tfrac{11}{12} \text{ and } \mu(\mathscr{C}_{x,n}(L)) > \tfrac{11}{12}\}.$$

For each $n \in \omega$, the family $\{U_n(L) \mid L \in \mathscr{L}\}$ is an open expansion of $\mathscr{L}$ (see the proof of Theorem 2.1). To complete the proof, let $x \in X$. For each $n \in \omega$, let $\mathscr{E}_n = \{\mathscr{L}' \subset \mathscr{L} \mid x \notin G_n(\mathscr{L}') \cap G_n(\mathscr{L} \sim \mathscr{L}')\}$. Then $\bigcup_{n \in \omega} \mathscr{E}_n = \{\mathscr{L}' \mid \mathscr{L}' \subset \mathscr{L}\}$ and for each $n \in \omega$, $\mathscr{E}_n \subset \mathscr{E}_{n+1}$; consequently, there exists $k \in \omega$ such that $\mu(\mathscr{E}_k) > \frac{11}{12}$. We show that there exists at most one $L \in \mathscr{L}$ such that $x \in U_k(L)$. Assume on the contrary that there are two distinct members L and K of $\mathscr{L}$ such that $x \in U_k(L) \cap U_k(K)$. Let $\mathscr{D} = \mathscr{B}_{x,k}(L) \cap \mathscr{C}_{x,k}(K) \cap \mathscr{E}_k$. Since each one of the three sets used in the definition of $\mathscr{D}$ has μ-measure exceeding $\frac{11}{12}$, we have $\mu(\mathscr{D}) > \frac{3}{4}$. Let $\mathscr{Q} = \{\mathscr{L}' \subset \mathscr{L} \mid L \in \mathscr{L}' \text{ and } K \notin \mathscr{L}'\}$. Then $\mu(\mathscr{Q}) = \frac{1}{4}$, and it follows that $\mu(\mathscr{D} \cap \mathscr{Q}) > 0$; hence, $\mathscr{D} \cap \mathscr{Q} \neq \emptyset$. Let $\mathscr{L}' \in \mathscr{D} \cap \mathscr{Q}$. Similarly as in the proof of Theorem 2.1, we see that $x \in G_k(\mathscr{L}') \cap G_k(\mathscr{L} \sim \mathscr{L}')$; this is a contradiction, since $\mathscr{L}' \in \mathscr{D} \subset \mathscr{E}_k$.

Theorem 2.3 has a corollary analogous to that obtained above for Theorem 2.1. A topological space X is *subnormal* ([11] and [5]) provided that whenever S and F are disjoint closed subsets of X, there exist disjoint G_δ-sets P and Q such that $S \subset P$ and $F \subset Q$. The space X is *collectionwise subnormal* ([5]) provided that every discrete family of subsets of X is well δ-separated.

Observing that a space is subnormal if and only if every discrete family of subsets of the space is δ-normalized, we get the following result.

COROLLARY 2.4. (PMEA) *Every subnormal space of weak character below $\mathfrak{c}$ is collectionwise subnormal.*

In [5], it is shown that a topological space is subparacompact if, and only if, the space is submetacompact and collectionwise subnormal (for the definitions of subparacompact spaces and submetacompact spaces, see [3] and [9]; submetacompact spaces are often called θ-refineable [16]). Hence the following result obtains.

THEOREM 2.5. (PMEA) *Every subnormal, submetacompact space of weak character below $\mathfrak{c}$ is subparacompact.*

3. PMEA and G_δ-additive families.

A family $\mathscr{L}$ of subsets of a topological space is *G_δ-additive* (*F_σ-additive*) provided that for each $\mathscr{L}' \subset \mathscr{L}$, $\bigcup \mathscr{L}'$ is a G_δ-set (an F_σ-set) in the space. Every family of open sets is G_δ-additive, and every σ-closure-preserving family of closed sets is F_σ-additive. A partition (i.e., a disjoint cover) of a space is G_δ-additive if and only if the partition is F_σ-additive.

To apply the result of Theorem 2.3 to the study of G_δ-additive partitions, we need the following lemma.

LEMMA 3.1. *Let $\mathscr{L}$ be a partition of a topological space X.*

(i) *$\mathscr{L}$ is G_δ-additive if and only if $\mathscr{L}$ is δ-normalized.*

(ii) *$\mathscr{L}$ has a σ-discrete closed refinement if and only if $\mathscr{L}$ is well δ-separated.*

Proof. (i) Trivial. (ii) If $\langle \{U_n(L) \mid L \in \mathscr{L}\} \rangle_{n=0}^{\infty}$ is a δ-separation of $\mathscr{L}$ consisting of open expansions of $\mathscr{L}$, and if we set $F_n(L) = X \sim \bigcup\{U_n(K) \mid K \in \mathscr{L} \sim \{L\}\}$ for all $L \in \mathscr{L}$ and $n \in \omega$, then each of the families $\{F_n(L) \mid L \in \mathscr{L}\}$, $n \in \omega$, is closed and discrete, and the union of these families is a refinement of $\mathscr{L}$. Conversely, if $\mathscr{F} = \bigcup_{n \in \omega} \mathscr{F}_n$ is a closed refinement of $\mathscr{L}$ such that each of the families $\mathscr{F}_n$, $n \in \omega$, is discrete, and if we set $U_n(L) = X \sim U\{F \in \mathscr{F}_n \mid F \cap L = \emptyset\}$ for all $L \in \mathscr{L}$ and $n \in \omega$, then the sequence $\langle\{U_n(L) \mid L \in \mathscr{L}\}\rangle_{n=0}^{\infty}$ is a δ-separation of $\mathscr{L}$ by open expansions of $\mathscr{L}$.

Theorem 2.3 and Lemma 3.1 yield the following result.

THEOREM 3.2. (PMEA) *In a space of weak character below $\mathfrak{c}$, every G_δ-additive partition of the space has a σ-discrete closed refinement.*

In particular, if X has weak character below $\mathfrak{c}$ and if every subset of X is an F_σ-set, then X is σ-discrete, provided that PMEA holds. Note that for subsets of the real line, this last-mentioned result already follows from the result of Nyikos that under PMEA, every normal Moore space is metrizable and an earlier result of R. H. Bing ([2]) that if there is an uncountable subspace of $\boldsymbol{R}$ such that every subset of the subspace is an F_σ-set in the relative topology, then there is a normal, non-metrizable Moore space.

Next we show that the conclusion of Theorem 3.2 holds also for semi-stratifiable spaces. We use the following characterization of semi-stratifiable spaces (see [8]): a space X is semi-stratifiable if and only if there exists a sequence $\langle V_n\rangle_{n=0}^{\infty}$ of binary relations on X such that for all $x \in X$ and $n \in \omega$, the set $V_n\{x\}$ is a neighborhood of x, and for every $y \in X$, whenever G is a neighborhood of y, there exists $n \in \omega$ such that $V_n^{-1}\{y\} \subset G$.

THEOREM 3.3. (PMEA) *In a semi-stratifiable space, every G_δ-additive partition of the space has a σ-discrete closed refinement.*

Proof. Let (Z, τ) be a semi-stratifiable space, and let $\mathscr{L}$ be a G_δ-additive partition of this space. For each $x \in Z$, let L_x be that member of $\mathscr{L}$ which contains x. Let $\langle V_n\rangle_{n=0}^{\infty}$ be a sequence of relations on Z with the properties stated above. We may assume that for each $n \in \omega$, $V_{n+1} \subset V_n$. For every $x \in Z$, let

$$\mathscr{M}_x = \{V_n^{-1}\{x\} \mid n \in \omega\}.$$

Then there is a topology π on Z such that the collection $\langle \mathscr{M}_x\colon x \in Z\rangle$ is a weak neighborhood base assignment for the space (Z, π). We have $\tau \subset \pi$ and it follows that $\mathscr{L}$ is G_δ-additive with respect to π. Since the space (Z, π) has weak character below $\mathfrak{c}$, it follows from Lemma 3.1 and Theorem 2.3 that there exists a sequence $\langle\{U_n(L) \mid L \in \mathscr{L}\}\rangle_{n=0}^{\infty}$ of π-open expansions of $\mathscr{L}$ such that for every $x \in Z$, there exists $k(x) \in \omega$ such that $\{L \in \mathscr{L} \mid x \in U_{k(x)}(L)\} = \{L_x\}$. For each $x \in Z$, let $n(x) \in \omega$ be such that $V_{n(x)}^{-1}\{x\} \subset U_{k(x)}(L_x)$. For all $k \in \omega$ and $n \in \omega$, let $H_{k,n} = \{x \in Z \mid k(x) = k$ and $n(x) = n\}$, and let $\mathscr{H}_{k,n} = \{L \cap H_{k,n} \mid L \in \mathscr{L}\}$; note that the family $\mathscr{H}_{k,n}$ is discrete in the subspace $H_{k,n}$ of (Z, τ) since for each $x \in H_{k,n}$, if $z \in H_{k,n} \cap V_n\{x\}$, then $x \in V_n^{-1}\{z\} \subset U_k(L_z)$ and hence $L_z = L_x$. Since (Z, τ) is semi-stratifiable, every closed subset of (Z, τ) is a G_δ-set ([10], [6]). It follows that for all $k \in \omega$ and $n \in \omega$, the discrete-in-itself family $\mathscr{H}_{k,n}$ has a refinement $\mathscr{N}_{k,n}$ such that $\mathscr{N}_{k,n}$ is σ-discrete in (Z, τ). The family $\mathscr{N} = \bigcup\{\mathscr{N}_{k,n} \mid k \in \omega$ and $n \in \omega\}$ is a σ-discrete refinement of $\mathscr{L}$. That $\mathscr{L}$ has a σ-discrete closed refinement now follows easily using the fact that every member of $\mathscr{L}$ is an F_σ-set.

So far, the results in this paper have dealt with the existence of separations and refinements of disjoint families of sets; in our last theorem, we generalize the result of Theorem 3.2 to show that PMEA is useful also in connection with certain point-finite families.

THEOREM 3.4. (PMEA) *Let X be a space of weak character below $\mathfrak{c}$, and let $\mathscr{L}$ be a G_δ-additive family of subsets of X. Then there exists a sequence $\langle\{U_n(L) \mid L \in \mathscr{L}\}\rangle_{n=0}^{\infty}$ of open expansions of $\mathscr{L}$ such that for each $x \in X$, if the family $(\mathscr{L})_x = \{L \in \mathscr{L} \mid x \in L\}$ is finite, then there exists $k \in \omega$ such that $\{L \in \mathscr{L} \mid x \in U_k(L)\} = (\mathscr{L})_x$.*

Proof. For every $\mathscr{L}' \subset \mathscr{L}$, let $\langle G_n(\mathscr{L}')\rangle_{n=0}^{\infty}$ be a sequence of open subsets of X such that $\bigcap_{n \in \omega} G_n(\mathscr{L}') = \bigcup \mathscr{L}'$ and for each $n \in \omega$, $G_{n+1}(\mathscr{L}') \subset G_n(\mathscr{L}')$. For all $L \in \mathscr{L}$, $x \in X$ and $n \in \omega$, let $\mathscr{B}_{x,n}(L) = \{\mathscr{L}' \subset \mathscr{L} \mid L \in \mathscr{L}' \Rightarrow x \in G_n(\mathscr{L}')\}$.

Let μ be a product measure extension for $\mathscr{L}$. For all $L \in \mathscr{L}$ and $n \in \omega$, let

$$U_n(L) = \{x \in X \mid \mu(\mathscr{B}_{x,n}(L)) > 1 - 2^{-n}\},$$

and note that $L \subset U_n(L)$ and that by Corollary 1.2, the set $U_n(L)$ is open. To complete the proof, let $x \in X$ be such that the family $(\mathscr{L})_x$ is finite. Let $m = |(\mathscr{L})_x| + 2$. For each $n \in \omega$, let $\mathscr{E}_n = \{\mathscr{L}' \subset \mathscr{L} \mid x \notin \bigcup \mathscr{L}' \Rightarrow x \notin G_n(\mathscr{L}')\}$. Then there exists $h \in \omega$ such that for each $j \geqslant h$, $\mu(\mathscr{E}_j) > 1 - 2^{-m}$. Let $k = \max(m, h)$. We show that for every $L \in \mathscr{L}$, if $x \notin L$, then $x \notin U_k(L)$. Let $L \in \mathscr{L} \sim (\mathscr{L})_x$. Let $\mathscr{Q} = \{\mathscr{L}' \subset \mathscr{L} \mid L \in \mathscr{L}'$ and $(\mathscr{L})_x \cap \mathscr{L}' = \emptyset\}$. Note that $\mu(\mathscr{Q}) = 2^{-m+1}$. We show that $(\mathscr{Q} \cap \mathscr{E}_k) \cap \mathscr{B}_{x,k}(L) = \emptyset$. Let $\mathscr{L}' \in \mathscr{Q} \cap \mathscr{E}_k$. Since $\mathscr{L}' \in \mathscr{Q}$, we have $L \in \mathscr{L}'$ and $x \notin \bigcup \mathscr{L}'$. Since $x \notin \bigcup \mathscr{L}'$ and $\mathscr{L}' \in \mathscr{E}_k$, we have $x \notin G_k(L)$. Finally, since $L \in \mathscr{L}'$ and $x \notin G_k(L)$, we have $\mathscr{L}' \notin \mathscr{B}_{x,k}(L)$. We have shown that $(\mathscr{Q} \cap \mathscr{E}_k) \cap \mathscr{B}_{x,k}(L) = \emptyset$. It follows that $\mu(\mathscr{B}_{x,k}(L)) \leqslant 1 - \mu(\mathscr{Q} \cap \mathscr{E}_k)$. We have $\mu(\mathscr{Q}) = 2^{-m+1}$ and $\mu(\mathscr{E}_k) > 1 - 2^{-m}$, and hence $\mu(\mathscr{Q} \cap \mathscr{E}_k) > 2^{-m}$. Consequently, $\mu(\mathscr{B}_{x,k}(L)) \leqslant 1 - 2^{-m}$; since $m \leqslant k$, we have $\mu(\mathscr{B}_{x,k}(L)) \leqslant 1 - 2^{-k}$, in other words, $x \notin U_k(L)$. We have shown that for every $L \in \mathscr{L}$, if $x \notin L$, then $x \notin U_k(L)$; it follows that $\{L \in \mathscr{L} \mid x \in G_k(L)\} = (\mathscr{L})_x$.

Note that if a family $\mathscr{L}$ of subsets of a space X has a sequence $\langle\{U_n(L) \mid L \in \mathscr{L}\}\rangle_{n=1}^{\infty}$ of open expansions such that for each $x \in X$, there exists $k \in \omega$ such that $\{L \in \mathscr{L} \mid x \in U_k(L)\} = (\mathscr{L})_x$, then $\mathscr{L}$ is G_δ-additive; hence Theorem 3.4 gives a characterization, under PMEA, of G_δ-additivity for point-finite families of subsets of spaces with weak character below $\mathfrak{c}$. Another consequence of the theorem is given below.

COROLLARY 3.5. (PMEA) *Let X be a space of weak character below $\mathfrak{c}$, and let $\mathscr{L}$ be a G_δ-additive family of subsets of X. Then the set $\{x \in X \mid (\mathscr{L})_x$ is finite$\}$ is an F_σ-subset of X.*

Proof. Use Theorem 3.4 and the observation that for any open expansion $\{U(L)|\ L\in\mathscr{L}\}$ of $\mathscr{L}$, the set $\{x\in X|\ \{L\in\mathscr{L}|\ x\in U(L)\}$ is finite$\}$ is an F_σ-set.

We close this paper with a question related to the results of Theorems 3.2 and 3.3 above.

QUESTION 3.6. Does there exist, in ZFC, a topological space that is not σ-discrete but whose every subset is a G_δ-set?

Under MA+$\neg$CH, such spaces do exist, and they can even be metrizable (in fact, subspaces of $\boldsymbol{R}$; see [15] or [14]). By a result in [13], no space giving an affirmative answer to the above question could be normal and first countable.

Added in proof. Independently of the author, Z. Balogh has shown that, under PMEA, a space of character less than $\underline{c}$ is σ-discrete, if every subset of the space is an F_σ-set.

References

[1] A. V. Arhangel'skiĭ, *Mappings and spaces*, Russian Math. Surveys, 21 (1966), pp. 115–162.
[2] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), pp. 175–186.
[3] D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969). pp. 655–663.
[4] E. Čech, *Topological Spaces*, Publ. House of the Czechoslovak Acad. Sci., Prague; Interscience Publ., New York 1976.
[5] J. Chaber, *On subparacompactness and related properties*, Gen. Top. Appl. 10 (1979), pp. 13–17.
[6] G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. 32 (1970), pp. 47–54.
[7] R. Engelking, *General Topology*, PWN, Warszawa 1977.
[8] H. J. K. Junnila, *Neighbornets*, Pacific J. Math. 76 (1978), pp. 83–108.
[9] — *On submetacompactness*, Topology Proc. 3 (1978), pp. 375–405.
[10] J. A. Kofner, *On a new class of spaces and some problems of symmetrizability theory*, Soviet Math. Dokl. 10 (1969), pp. 845–848.
[11] T. R. Kramer, *A note on countably subparacompact spaces*, Pacific J. Math. 46 (1973), pp. 209–213.
[12] P. J. Nyikos, *A provisional solution of the Normal Moore Space Problem*, Proc. Amer. Math. Soc. 78 (1980), pp. 429–435.
[13] G. M. Reed, *On normality and countable paracompactness*, Fund. Math. 110 (1980), pp. 145–152.
[14] M. E. Rudin, *Lectures on set-theoretic topology*, Amer. Math. Soc. Regional Confer. Ser. in Math. 23 (1975).
[15] F. D. Tall, *Set-theoretic consistency results and topological theorems concerning the Normal Moore Space Conjecture and related topics*, Dissert. Math. 148 (1977).
[16] J. M. Worrell Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), pp. 820–830.

Accepté par la Rédaction le 14. 1. 1980

Retracts and homotopies for multi-maps

by

A. Suszycki (Warszawa)

Abstract. By means of upper semi-continuous multi-functions defined on compacta and with values of trivial shape we introduce the notions of multi-retracts and multi-homotopies. We give some characterizations of absolute multi-retracts and absolute neighborhood multi-retracts and apply the notion of multi-homotopy to the construction of groups, called multi-homotopy groups. In particular, we show that if $y\in Y\in$ ANR, then the nth multi-homotopy group of the space (Y, y) is isomorphic to the nth shape group of this space.

1. Introduction. Throughout this paper all spaces are compact and metric. By a multi-function φ from a space X to a space Y ($\varphi\colon X\to Y$) we mean one that assigns to every point $x\in X$ a closed non-empty subset $\varphi(x)$ of Y. The upper semi-continuity (shortly u.s.c.) of $\varphi\colon X\to Y$ means that the *graph* Φ of φ defined as

$$\Phi=\{(x,y)\in X\times Y|\ y\in\varphi(x)\}$$

is closed in $X\times Y$. A *map* denotes, as usual, a continuous function. The notion of shape is understood in the sense of Borsuk [3]. A u.s.c. multi-function $\varphi\colon X\to Y$ is called a *multi-map* of X into Y if $\varphi(x)$ is a set of trivial shape for every $x\in X$. By an *extension* of a u.s.c. multi-function $\varphi\colon X\to Y$ onto $M\supset X$ we mean a u.s.c. multi-function $\varphi'\colon M\to Y$ such that $\varphi'|X=\varphi$ and $\varphi'(x)$ has the shape of a point for every $x\in M\smallsetminus X$. We say that a map f of Y onto X is a *cellular map* (compare [12]) if $f^{-1}(x)$ has trivial shape for every $x\in X$. Let us note that, if f is a cellular map of Y onto X, then the multi-function $\varphi\colon X\to Y$, defined by the formula $\varphi(x)=f^{-1}(x)\subset Y$ is a multi-map. Let us call such a multi-function an *inverse* of the map f. We say that X is *countable-dimensional* if it is the union of a countable family of finite-dimensional subspaces.

In the sequel we will need the following theorems:

1.1. THEOREM (Kozlowski [9] thms 9 and 12). *Let f be a cellular map of a space Y onto X such that the set $\{x\in X|\ f^{-1}(x)$ is a nondegenerate set$\}$ is contained in a compact and countable-dimensional subset of X. Then for every closed subset A of X the map $f|f^{-1}(A)\colon f^{-1}(A)\colon f^{-1}(A)\to A$ is a shape equivalence. Moreover, if $Y\in$ ANR, then $X\in$ ANR.*

1.2. THEOREM ([14], [15]). *Let φ be a multi-map of a space X into $Y\in$ ANR, where $X\subset M$. If $\dim(M\smallsetminus X)<\infty$ or X is countable-dimensional, then there exists a neighborhood U of X in M such that φ has an extension onto U. Moreover, if $Y\in$ AR, then φ has an extension onto M.*