On the continuity and monotonicity of restrictions of connected functions

by

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Abstract. The paper deals with connected real functions defined on a topological space. Some results are given concerning the restrictions of such functions to supersets of the union of their connected levels. In § 1 the above-mentioned problems are considered in connection with continuity, quasi-continuity and the Blumberg sets of those restrictions. In § 2 there is an analysis of different kinds of monotonicity and related properties of restrictions of connected functions. The last part of the paper refers to the same kind of problems as the previous ones but with the additional assumption that the functions considered are open.

§ 0. Introduction and basic definitions and notation. The paper contains results concerning restrictions of real connected functions defined on some topological spaces. The results here presented are extensions of the researche of K. M. Garg [2], Z. Grande [4] and J. S. Lipiński [8]. In particular, the paper contains Theorem 1.1, the corollary of which was Garg's theorem in [3]. Theorem 1.1 is essentially stronger than Garg's theorem, which is shown by Example 1.1. This result is a starting point for proving new theorems such as Theorem 1.4 and Corollary 3.1.

In § 1 the above mentioned problems are considered in connection with continuity, quasi-continuity and Blumberg sets of restrictions to some supersets of the union of connected levels of connected functions. In the second section we consider different kinds of monotonicity and the corresponding properties of restrictions of connected functions. The last part of the paper relates to problems similar to the previous onės with the additional assumption that the functions considered are open.

We use the standard notions and notation, which were used in the monograph of R. Engelking [1] and in the article of K. M. Garg [3]. The notions which were not defined in [1] or [3] and those we define differently from R. Engelking and K. M. Garg (for example an open function) are defined immediately before we use them.

By R we shall denote the set of real numbers with its natural topology.

The symbol E^d denotes the derived set of a set E. This set depends on the space in which it is considered. Since every point x of a subspace A of a space X is an accumulation point of E ($E \subset A$) in A iff x is accumulation point of E in X, then the use of this symbol will not lead to a misunderstanding.

The symbol $A\text{-}\lim\limits_{\sigma\in\Sigma} x_\sigma$ will denote the limit of the M-S sequence $\{x_\sigma\}_{\sigma\in\Sigma}$ in the subspace A of X.

By the bilaterally closed set we mean the set contained in R and containing all of its bilateral accumulation points.

The symbols (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$ and $[\alpha, \beta]$ for $\alpha<\beta$ denote (respectively) open, left-sided open, right-sided open and closed intervals in R. We denote by $f^{-1}(\alpha, \beta)$, $f^{-1}[\alpha, \beta)$, $f^{-1}(\alpha, \beta]$, $f^{-1}[\alpha, \beta]$ the inverse images of those intervals to avoid superfluous brackets.

The closure and interior of a set A we denote by $\bar{A}$ and $\operatorname{Int} A$. According to the notation in [3] we write:

$$Y_c(f) = \{\alpha \in f(X)\colon f^{-1}(\alpha) \text{ is a connected set}\},$$
$$S_c(f) = f^{-1}(Y_c(f)),$$
$$S^c(f) = f^{-1}(\overline{Y_c(f)}),$$

for a function $f\colon Y \to Y$.

The symbol $K(x, r)$ denotes an open ball at the centre in x and with radius r, i.e. $K(x, r) = \{y\colon \varrho(x, y) \leqslant r\}$, where ϱ always denotes a natural metric and $\varrho(x, A)$ denotes the distance of x from a set A.

Finally, it is necessary to settle (according to the terminology of R. Engelking) that compact space is always a Hausdorff space in which every open cover has a finite open subcover.

§ 1. Remarks on continuity, quasi-continuity and Blumberg sets for connected functions.

Definition 1.1 ([5]). We say that a function $f\colon X \to Y$, where X, Y are arbitrary topological spaces, is *weakly connected* if $f(C)$ is a connected set for every open connected set $C \subset X$.

Definition 1.2 ([3]). We say that a function $f\colon X \to Y$, where X, Y are arbitrary topological spaces, is *connected* if $f(C)$ is a connected set for an arbitrary connected set $C \subset X$.

Definition 1.3. We say that a nonvoid set K cuts a topological space X if $X \setminus K = A \cup B$, where A and B are nonvoid open and disjoint sets.

Before we prove the first theorem of this part of the work we prove 5 lemmas, Lemmas 1.1, 1.4 and 1.5 were proved by K. M. Garg [3] in a case of real functions. The generalization of them to arbitrary topological spaces is connected with a possibility of wide application, for a series of ours theorems are true even in a case of consideration arbitrary functions $f\colon X \to Y$. Indispensable (in such a case) changes are contained in undermentioned lemmas.

Lemma 1.1. *Let X be a locally connected space, Y — arbitrary topological space. If $f\colon X \to Y$ is a weakly connected function and K cuts Y into A and B, then $f^{-1}(A)$, $f^{-1}(B)$ are open disjoint sets if and only if $f^{-1}(K)$ is closed. Moreover, if $f^{-1}(A) \neq \emptyset \neq f^{-1}(B)$, then $f^{-1}(K)$ cuts the space X into $f^{-1}(A)$ and $f^{-1}(B)$.*

Proof. Necessity is obvious. We shall prove only that the above condition is sufficient.

Let $x \in f^{-1}(A)$. Since $f^{-1}(K)$ is a closed set, then there is a connected neighbourhood U of x such that $U \cap f^{-1}(K) = \emptyset$. Of course $f(U)$ is a connected set. If $f(U) \cap B \neq \emptyset$ then $f(U) = P \cup Q$, where $P = f(U) \cap A$ and $Q = f(U) \cap B$.

But, in view of an obvious fact, P and Q are closed in a subspace $f(U)$; this contradicts the fact that the set $f(U)$ is connected. So $f(U) \cap (B \cup K) = \emptyset$, and hence $f(U) \subset A$, which implies that $f^{-1}(A)$ is an open set.

In a similar way one can prove that B is an open set too.

Remark that $f^{-1}(A) \cap f^{-1}(B) = \emptyset$, which implies, according to the relation $f^{-1}(A) \neq \emptyset \neq f^{-1}(B)$, that $f^{-1}(K)$ cuts X (into $f^{-1}(A)$ and $f^{-1}(B)$).

Lemma 1.2 ([3]). *Let X be an arbitrary topological space, Y a T_1 space and $f\colon X \to Y$ a connected function. If $K \subset Y$ has closed components (in Y) then $f^{-1}(K)$ has closed components as well.*

The following lemma is a similar consequence to the above ones.

Lemma 1.3. *Let X be a locally connected space and $f\colon X \to R$ a connected function. If $\alpha \in Y_c(f)$, then $f^{-1}(-\infty, \alpha)$, $f^{-1}(\alpha, \infty)$ are disjoint open sets.*

Lemma 1.4. *Let $f\colon X \to Y$ be a connected function, where X is a connected and locally connected space, Y a T_1 space. If*

1^0 the set K cuts Y into sets A and B, and

2^0 $f^{-1}(K)$ is a connected set,

then the sets $f^{-1}(A \cup K)$, $f^{-1}(B \cup K)$ are connected.

Proof. The lemma is true if $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$, for in this case $f^{-1}(A \cup K)$ and $f^{-1}(B \cup K)$ are equal to X or $f^{-1}(K)$.

Suppose now that $f^{-1}(A) \neq \emptyset \neq f^{-1}(B)$. According to 2^0 and Lemma 1.2, $f^{-1}(K)$ is closed and, in view of Lemma 1.1 and 1^0, $f^{-1}(K)$ cuts X into sets $f^{-1}(A)$ and $f^{-1}(B)$, which means that $f^{-1}(A \cup K)$ and $f^{-1}(B \cup K)$ are closed. Moreover, $f^{-1}(A \cup K) \cup f^{-1}(B \cup K) = X$ is a connected set as well as the set $f^{-1}(K) = f^{-1}(A \cup K) \cap f^{-1}(B \cup K)$. This implies (see [7] p. 133) that $f^{-1}(A \cup K)$ and $f^{-1}(B \cup K)$ are connected sets.

Lemma 1.5. *Let $f\colon X \to Y$ be a connected function, where X is a connected and locally connected space, Y a T_1 space. Moreover, let K cuts Y into sets A and B. If $\{K_n\}$ is a sequence of sets contained in Y fulfilling the conditions*

1^0 K_n cuts Y into A_n and B_n for $n = 1, 2, \ldots$,

2^0 $A_{n-1} \cup K_{n-1} \subset A_n \cup K_n \subset A$ for $n = 2, 3, \ldots$,

3^0 $f^{-1}(K_n)$ is a connected set for $n = 1, 2, \ldots$,

then $f^{-1}\big(\bigcup\limits_{n=1}^{\infty} (A_n \cup K_n)\big) \subset f^{-1}(A \cup K)$.

Proof. In virtue of Lemma 1.4 the sets $f^{-1}(A_n \cup K_n)$ are connected for $n = 1, 2, \ldots$; thus $f^{-1}\big(\bigcup\limits_{n=1}^{\infty} (A_n \cup K_n)\big) = \bigcup\limits_{n=1}^{\infty} f^{-1}(A_n \cup K_n)$ is a connected set, and thus it is contained in some component S of $f^{-1}(A \cup K)$. The set $A \cup K$, as a closed set, has closed components; thus $f^{-1}(A\cup K)$ has closed components (see Lemma 1.2) and in particular

$$\overline{f^{-1}\Big(\bigcup_{n=1}^{\infty}(A_n\cup K_n)\Big)}\subset S\subset f^{-1}(A\cup K)\,.$$

THEOREM 1.1. *Let $f\colon X\to R$ be a connected function, where X is a connected and locally connected space. If $S\subset X$ fulfils the condition*

1^0 $f(S)\subset\overline{Y_c(f)}$,

2^0 $f^{-1}(\alpha)$ is closed in S for every $\alpha\in f(S)$,

then $f_{|S}$ is continuous.

Proof. Denote $Y_c=Y_c(f)$, $A=f(S)$, $g=f_{|S}$, and moreover let $\tilde{P}$ denote the closure of P in S. If $x\in S$ and $y=f(x)$ then, according to 2^0, $f^{-1}(y)$ is closed in S.

Put

$$\alpha=\inf\{r\in\overline{Y_c(f)}\colon\ r>y\}=\inf\{r\in Y_c(f)\colon\ r>y\}$$

and

$$\beta=\sup\{r\in\overline{Y_c(f)}\colon\ r<y\}=\sup\{r\in Y_c(f)\colon\ r<y\}\,.$$

Let ε be an arbitrary positive number. We shall show that

($*$) there exists a set H open in S and such that $x\in H$ and

$$g(H)\subset(y-\varepsilon,y+\varepsilon)=G\,.$$

Consider the following cases:

(i) $\alpha=\infty$, $\beta=-\infty$. Then $g(S)=Y_c=\overline{Y}_c=\{y\}$ and putting $H=S$ we have ($*$).

(ii) $\alpha=\infty$, $\beta=y$. Remark that $\{r\in Y_c\colon r<y\}\neq\emptyset$. Then there exists in $(y-\varepsilon,y)$ a number $z\in Y_c$. Let $H=g^{-1}(z,\infty)=f^{-1}(z,\infty)\cap S$. This set is open in S (Lemma 1.3), contains x and $g(H)\subset(z,y]\subset G$, which means that in this case condition ($*$) is also fulfilled.

(iii) $\alpha=y$, $\beta=-\infty$. Then the proof is analogous to the previous one.

(iv) $\alpha=y$, $\beta=y$. Then there exist numbers $c,d\in Y_c$ such that

$$y-\varepsilon<c<y<d<y+\varepsilon\,.$$

Now let $H=g^{-1}(c,d)=f^{-1}(c,\infty)\cap S\cap f^{-1}(-\infty,d)$. As in case (ii) one can prove ($*$).

(v) $y<\alpha<\infty$. If $\alpha\in Y_c$ then the set $H_1=g^{-1}(-\infty,\alpha)$ is open in S and contains x.

Suppose now that $\alpha\notin Y_c$. Let $\{\alpha_n\}$ be a decreasing sequence contained in Y_c such that $\lim\alpha_n=\alpha$. Notice that all the assumptions of Lemma 1.5 are fulfilled; thus

$$\overline{f^{-1}(\alpha,\infty)}=\overline{f^{-1}\Big(\bigcup_{n=1}^{\infty}[\alpha_n,\infty)\Big)}\subset f^{-1}[\alpha,\infty)\,,$$

and so

$$\text{(1)}\qquad \widetilde{g^{-1}(\alpha,\infty)}=\overline{f^{-1}(\alpha,\infty)}\cap S\subset\big(f^{-1}(\alpha,\infty)\cup f^{-1}(\alpha)\big)\cap S\,.$$

If $\alpha\in A$ then for $H_1=g^{-1}(-\infty,\alpha)=f^{-1}(-\infty,\alpha)\cap S$ we have:

$$\begin{aligned}\widetilde{g^{-1}[\alpha,\infty)}&=\widetilde{g^{-1}(\alpha,\infty)}\cup\widetilde{g^{-1}(\alpha)}\subset\big([f^{-1}(\alpha,\infty)\cup f^{-1}(\alpha)]\cap S\big)\cup g^{-1}(\alpha)\\&=g^{-1}[\alpha,\infty)\,,\end{aligned}$$

which proves that H_1 is open and $x\in H_1$.

Now if $\alpha\notin A$, then according to (1) and the obvious fact $f^{-1}(\alpha)\cap S=\emptyset$, we infer that

$$\widetilde{g^{-1}(\alpha,\infty)}=f^{-1}(\alpha,\infty)\cap S=g^{-1}(\alpha,\infty)\,,$$

and in this case $H_1=g^{-1}(-\infty,\alpha]$ is open in S and $x\in H_1$.

Suppose now that $\beta=-\infty$. Then $H_2=g^{-1}(R)$ is open in S and $x\in H_2$.

If $\beta=y$, then let $H_2=g^{-1}(z,\infty)$, where z is such an element of Y_c that $y-\varepsilon<z<y$. Then $x\in H_2$ and H_2 is open in S.

Finally, if β is a finite number such that $\beta<y$ then, in the same way as we proved for $\alpha>y$, we can prove that the set

$$H_2=\begin{cases}g^{-1}(\beta,\infty) & \text{if } \beta\in A\,,\\ g^{-1}[\beta,\infty) & \text{if } \beta\notin A\end{cases}$$

is open in S and contains x.

Thus in each of the possible cases the set $H=H_1\cap H_2$ is open in S and contains x. Moreover, the constructions of H_1, H_2 imply that

$$g(H)=g(H_1\cap H_2)\subset g(H_1)\cap g(H_2)\subset G\,.$$

So we have just proved ($*$) in this case.

In the last two cases

(vi) $\alpha=\infty$, $-\infty<\beta<y$;

(vii) $\alpha=y$, $-\infty<\beta<y$;

the proofs of ($*$) are very similar to that in case (v).

COROLLARY 1.1 ([3]). *If X is a connected and locally connected space and $f\colon X\to R$ a connected function, then $f_{|\overline{S_c(f)}}$ is continuous.*

Now Theorem 1.1 suggests the following problem: Is a restriction to $S^c(f)$ of a connected function f defined on a connected and locally connected space continuous? The answer to this problem is negative, as we can see in the following example:

EXAMPLE 1.1. Let $X=R$ and $f\colon R\to R$ be as follows:

$$f(x)=\begin{cases}-x+1 & \text{for } x\in(-\infty,0]\,,\\ \sin\dfrac{1}{x} & \text{for } x\in\left(0,\dfrac{2}{3\pi}\right]\,,\\ -\dfrac{3\pi}{2}x & \text{for } x\in\left(\dfrac{2}{3\pi},\infty\right)\,.\end{cases}$$

It is easy to see that $S^c(f) = (-\infty, 0) \cup \left\{\frac{2}{(2n+1)\pi}: n = 1, 2, \dots\right\} \cup \left[\frac{2}{3\pi}, \infty\right)$ and $f_{|S^c(f)}$ is not a continuous function.

With Theorem 1.1 another question is connected: What kind of structure has the set P consisting of those elements of $\overline{Y_c(f)}$ for which $f^{-1}(\alpha)$ is not closed in $S^c(f)$? Theorem 1.2 describes this set.

THEOREM 1.2. *If $f\colon X \to R$ is a connected function, where X is a connected and locally connected space, then*

(i) *if α is such a point of $\overline{Y_c(f)}$ that $f^{-1}(\alpha)$ is not closed in $S^c(f)$, then α is the unilateral limit point of $Y_c(f)$;*

(ii) *the set P of all α from $\overline{Y_c(f)}$ for which $f^{-1}(\alpha)$ is not closed in $S^c(f)$ is at most denumerable.*

Proof. Let $\alpha \in P$. In virtue of Lemma 1.2 $\alpha \in \overline{Y_c(f)} \setminus Y_c(f)$, which implies that α is an accumulation point of $Y_c(f)$. Suppose that α is a bilateral limit point of $Y_c(f)$. There exist two sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $Y_c(f)$ such that $\alpha_n \nearrow \alpha$, $\beta_n \searrow \alpha$. Hence

$$\{\alpha\} = \bigcap_{n=1}^{\infty} [\alpha_n, \beta_n]$$

and

$$f^{-1}(\alpha) = \bigcap_{n=1}^{\infty} f^{-1}[\alpha_n, \beta_n],$$

which implies, according to Lemma 1.3, that $f^{-1}(\alpha)$ is closed in X, and so also in $S^c(f)$. This means that $\alpha \notin P$. The contradiction ends the proof of (i).

(ii) can easily be deduced from (i).

DEFINITION 1.4 ([6], [9]). The function $f\colon X \to Y$ is said to be *quasi-continuous at x* if for every neighbourhood W of $f(x)$ and every neighbourhood U of x the set $\mathrm{Int}[U \cap f^{-1}(W)]$ is nonempty.

The function $f\colon X \to Y$ is said to be *quasi-continuous* if it is quasi-continuous at every point $x \in X$.

THEOREM 1.3. *Let $f\colon X \to R$ be a connected function defined on a connected and locally connected space X. Assume that $S \subset X$ fulfils the following conditions:*

1° $S_c(f) \subset S \subset S^c(f)$,

2° *for $x \in S$ if $x \in [S \setminus S_c(f)]^d$ then $x \in [S_c(f) \setminus f^{-1}(f(x))]^d$.*

Then $f_{|S}$ is quasi-continuous.

Proof. Let $g = f_{|S}$ and $h = f_{|\overline{S_c(f)}}$. Let $\alpha = f(x)$ for $x \in S$, and ε be an arbitrary positive number. Put $W = (\alpha - \varepsilon, \alpha + \varepsilon)$ and let U be an arbitrary set open in S and containing x, and U_x a set open (in X) and such that $U = S \cap U_x$. There exists set V_x open (in X) and connected and such that $x \in V_x \subset U_x$. If $V = S \cap V_x$, then $V \subset U$. Suppose now that there is a set V^* open in S, containing x and such that V^* consists only of elements of the level $g^{-1}(\alpha)$. Then the set $V \cap V^*$ is open in S, contains x and is contained in $\mathrm{Int}[U \cap f^{-1}(W)]$.

Now assume to the contrary that an arbitrary set open in S and containing x contains also a point not belonging to $g^{-1}(\alpha)$. This means that $x \in [S \setminus g^{-1}(\alpha)]^d$. Then let $\{x_\sigma\}_{\sigma \in \Sigma}$ be an arbitrary M-S sequence of points from $S \setminus g^{-1}(\alpha)$ such that $x \in \lim_{\sigma \in \Sigma} x_\sigma$ independently of whether $\lim_{\sigma \in \Sigma} x_\sigma$ denotes X-$\lim_{\sigma \in \Sigma} x_\sigma$ or S-$\lim_{\sigma \in \Sigma} x_\sigma$.

Notice that $x \in [S_c(f) \setminus g^{-1}(\alpha)]^d$. This is obvious if we can choose a subsequence $\{x_{\sigma'}\}_{\sigma' \in \Sigma'} \subset S_c(f)$ of $\{x_\sigma\}_{\sigma \in \Sigma}$. In the opposite case $x \in [S \setminus S_c(f)]^d$ and in view of 2° also $x \in [S_c(f) \setminus g^{-1}(\alpha)]^d$.

Let $\{y_\delta\}_{\delta \in \Delta}$ be an arbitrary M-S sequence such that $x \in X$-$\lim_{\delta \in \Delta} y_\delta$ and $\{y_\delta\} \subset (S_c(f) \setminus g^{-1}(\alpha))$. There exists a $\delta_0 \in \Delta$ such that $y_\delta \in V_x$ for every $\delta \geqslant \delta_0$. Hence, according to the inclusion $S_c(f) \subset S$, $y_\delta \in V \subset U$.

Since $x \in \overline{S_c(f)}$, we have, in view of Corollary 1.1, $\alpha = \lim_{\delta \in \Delta} h(y_\delta) = \lim_{\delta \in \Delta} g(y_\delta)$. We now infer that there exists a $\delta'_0 \in \Delta$ such that $g(y_\delta) \in W$ for $\delta \geqslant \delta'_0$, and there exists a δ^* such that $\delta_0 \leqslant \delta^*$, $\delta'_0 \leqslant \delta^*$.

Put $\delta_1 = \delta^*$ and $r_1 = |g(y_{\delta_1}) - \alpha|$. Of course $r_1 > 0$. Since $\alpha = \lim_{\delta \in \Delta} g(y_\delta)$, there exists a $\delta_1^* \in \Delta$ such that $g(y_\delta) \in (\alpha - \frac{1}{2} r_1, \alpha + \frac{1}{2} r_1)$ for every $\delta \geqslant \delta_1^*$. There is an element δ_2 in Δ such that $\delta_1 \leqslant \delta_2$ and $\delta_1^* \leqslant \delta_2$. Then $g(y_{\delta_2}) \in V$ and let $r_2 = |g(y_{\delta_2}) - \alpha|$. There exists a $\delta_2^*, \delta_3 \in \Delta$ such that $g(y_\delta) \in (\alpha - \frac{1}{2} r_2, \alpha + \frac{1}{2} r_2)$ for $\delta \geqslant \delta_2^*$; $\delta_2 \leqslant \delta_3$ and $\delta_2^* \leqslant \delta_3$. Of course $y_{\delta_3} \in V$ and points $g(y_{\delta_1})$, $g(y_{\delta_2})$, $g(y_{\delta_3})$ are pairwise distinct, and so one of them is placed between the other two. Let $g(y_{\delta_1}) < g(y_{\delta_2}) < g(y_{\delta_3})$ for example. If $A = f^{-1}(g(y_{\delta_1}), g(y_{\delta_3}))$, then A is open in X (in virtue of Lemma 1.3), and so $A \cap S$ is open in S and nonempty for $y_{\delta_2} \in A \cap S$. It follows that $A \cap S \cap V$ is nonempty and open (in S), contained in V and U. Since $(g(y_{\delta_1}), g(y_{\delta_3})) \subset W$, we have $A \cap S \cap V \subset g^{-1}(W)$. This proves that $\mathrm{Int}(U \cap g^{-1}(W)) \neq \emptyset$.

The next definition and Theorem 1.4 are basic for our considerations concerning the strong sets of Blumberg.

DEFINITION 1.5 ([9]). Let X, Y be topological spaces and $f\colon X \to Y$. We say that $B \subset X$ is a *Blumberg set for f* if B is dense in X and $f_{|B}$ is a continuous function.

THEOREM 1.4. *If $f\colon X \to R$ is a connected function defined on a connected and locally connected space X, then the function $g = f_{|S^c(f)}$ has a Blumberg set.*

Proof. Let P be the set of all $\alpha \in \overline{Y_c(f)}$ for which $g^{-1}(\alpha)$ is not a closed set in $S^c(f)$. In virtue of Theorem 1.2(ii) P is at most denumerable: let it be $\{\alpha_n\}$ (this sequence can be finite). Let A_1 be a set of those accumulation points (in $S^c(f)$) of $g^{-1}(\alpha_1)$ which do not belong to $g^{-1}(\alpha_1)$.

Suppose we have defined sets $A_1, \dots, A_{n-1}$ for $\alpha_1, \dots, \alpha_{n-1} \in P$. Then let A_n be the set of all the limit points (in $S^c(f)$) of $g^{-1}(\alpha_n) \setminus \bigcup_{i=1}^{n-1} A_i$ which that do not belong to $g^{-1}(\alpha_n)$. Continuing this procedure, we can connect a set A_k with every point $\alpha_k \in P$.

Put $B = S^c(f) \setminus \bigcup_i A_i$. We shall show that B is a Blumberg set for g.

Of course, $f(B)\subset \overline{Y_c(f)}$ and, moreover, from the construction of B it follows that for every $\alpha \in \overline{Y_c(f)}$ the level $h^{-1}(\alpha)$ (where $h = g_{|B} = f_{|B}$) is closed in B (it can be the empty set); so according to Theorem 1.1, h is continuous.

Let $x \in S^c(f)\setminus B$ and let U be an arbitrary neighbourhood (in $S^c(f)$) of x, $\alpha = f(x)$. Then there exists an $\alpha_k \in P$ such that x is an accumulation point (in $S^c(f)$) of $g^{-1}(\alpha_k)$. In virtue of Lemma 1.2 $\alpha_k \in \overline{Y_c(f)}\setminus Y_c(f)$.

Denote by $\{\beta_n\}$ a sequence in $Y_c(f)$ converging to α_k. According to Theorem 1.2 (i) all β_n are placed on one side of α_k. Assume, for example, that $\beta_n<\alpha_k$ for $n = 1, 2, \ldots$ If $\alpha<\alpha_k$, then there is an element β_N such that $\alpha<\beta_N<\alpha_k$ and the set $g^{-1}(-\infty, \beta_N) = f^{-1}(-\infty, \beta_N) \cap S^c(f)$ contains x but no element of $g^{-1}(\alpha_k)$. At the same time (Lemma 1.3) $g^{-1}(-\infty, \beta_N)$ is open in $S^c(f)$. This contradicts the assumption that x is an accumulation point of $g^{-1}(\alpha_k)$. So we infer that $\alpha_k<\alpha$.

Now consider two cases:

I. $\alpha \in Y_c(f)$. In this case we shall show that no element of $g^{-1}(\alpha_k)$ is an accumulation point of any other level of g. Suppose, on the contrary, that there is an element $y \in g^{-1}(\alpha_k)$ being an accumulation point of $g^{-1}(\alpha_s)$, $s \neq k$. Then none of the inequalities $\alpha_s<\alpha_k$, $\alpha_s>\alpha$, $\alpha_k<\alpha_s<\alpha$ can hold, for otherwise in each of these cases there would be an element $c \in Y_c(f)$ such that $\alpha_s<c<\alpha_k$ or $\alpha_k<c<\alpha_s$ (in the first case the required element c could be found in a sequence $\{\beta_n\}$, in the second case c could be equal to α, in the last case the existence of such an element would follow from the fact that $\alpha_s \in \overline{Y_c(f)}\setminus Y_c(f)$). Deducing as before, we infer in each possible case a contradiction of the assumption that y is an accumulation point of $g^{-1}(\alpha_s)$ or x is an accumulation point of $g^{-1}(\alpha_k)$.

In this situation $g^{-1}(\alpha_k)\subset B$, and since every neighbourhood of x contains elements of $g^{-1}(\alpha_k)$, we have $U \cap B \neq \emptyset$.

II. $\alpha \in \overline{Y_c(f)}\setminus Y_c(f)$. Then α is a limit point of a sequence $\{\delta_n\}$ contained in $Y_c(f)$ and such that $\alpha<\delta_n$. As in case I, we can notice that $g^{-1}(\alpha)$ contains no accumulation point of any other levels $g^{-1}(\alpha_l)$, $l \neq k$, and $g^{-1}(\alpha_k)$ contains no accumulation point of levels $g^{-1}(\alpha_l)$ distinct from $g^{-1}(\alpha)$.

Now, if $g^{-1}(\alpha_k)$ contains no accumulation point of $g^{-1}(\alpha)$, then $g^{-1}(\alpha_k)\subset B$ and (in the same way as in case I) U meeting with $g^{-1}(\alpha_k)$ has common points with B.

Now, if $g^{-1}(\alpha_k)$ contains some accumulation point of $g^{-1}(\alpha)$, then $\alpha \in P$ and $\alpha = \alpha_m$ for some positive integer m. If

1^0 $k<m$, then let z be an arbitrary element of $U \cap g^{-1}(\alpha_k)$. If $z \in B$ then the theorem is of course fulfilled. If $z \notin B$ then z is an accumulation point of $g^{-1}(\alpha_m)\setminus \bigcup_{i=1}^{m-1} A_i$, and so some element p of $g^{-1}(\alpha_m)\setminus \bigcup_{i=1}^{m-1} A_i$ belongs to U. We can easily see that $p \in B$ because, in view of $p \notin A_k$, p is not an accumulation point of $g^{-1}(\alpha_k)$. Thus $p \notin \bigcup_i A_i$, and this implies that $U \cap B \neq \emptyset$.

2^0 $k>m$. We can remark that $x \in S^c(f)\setminus B$, and so x is an accumulation point of $g^{-1}(\alpha_k)\setminus \bigcup_{i=1}^{k-1} A_i$. Thus U contains some element q of $g^{-1}(\alpha_k)\setminus \bigcup_{i=1}^{k-1} A_i$. In this way $q \notin A_m$ and $q \notin \bigcup_i A_i$, so $q \in B \cap U$.

In this way we have proved that in each of the possible cases $U \cap B \neq \emptyset$, which means (in accordance with the arbitrariness of x and U) that B is dense in $S^c(f)$.

Example 1.2. Let $X = [0, \infty)$, $\{a_n\} = \left\{\frac{1}{n}\right\}$, $\{b_n\} = \left\{\frac{2n+1}{2n^2+2n}\right\}$. Let $f\colon X \to R$ be as follows:

$$f(x) = \begin{cases} 1 & \text{for } x = 0,\\ 1-1/n & \text{for } x = b_n,\ n = 1, 2, \ldots,\\ -1 & \text{for } x = a_n,\ n = 1, 2, \ldots,\\ -x & \text{for } x>1,\\ \text{linear in each of the intervals } [a_{n+1}, b_n],\ [b_n, a_n],\ n = 1, 2, \ldots \end{cases}$$

Without difficulty, we can see that f is a connected function and

$$\overline{Y_c(f)} = (-\infty, -1] \cup \{1\},$$

which means that $S^c(f) = \{0\} \cup \{1/n\}_{n=1,2,\ldots} \cup [1, \infty)$. Hence $g = f_{|S^c(f)}$ is not quasi-continuous at 0 and $B = \{1/n\colon n = 1, 2, \ldots\} \cup [1, \infty)$ is a Blumberg set for g. Remark that if $U = [0, \frac{1}{10}) \cap S^c(f)$, then U is an open set in $S^c(f)$ and $U \cap B = \{1/n\colon n = 11, 12, \ldots\}$. Then $g(U \cap B) = \{-1\}$, and $g(U \cap B)$ is not a dense in $g(U) = \{-1, 1\}$.

The above-mentioned example substantiates the necessity of the next definition.

Definition 1.6 ([9]). Let X, Y be two topological spaces and a function $f\colon X \to Y$. The set $B\subset X$ is said to be a *strong set of Blumberg for* f if B is a Blumberg set for f and, for an arbitrary open set $V\subset X$, the set $f(V \cap B)$ is dense in $f(V)$.

Example 1.2 suggests the following question: What additional assumptions are sufficient for $f_{|S^c(f)}$ to have a strong set of Blumberg. Here is the answer.

Theorem 1.5. *Let X be a connected and locally connected space and $f\colon X \to R$ a connected function. Then the function $g = f_{|S^c(f)}$ has a strong set of Blumberg if and only if it is quasi-continuous.*

Proof. Necessity. Let x be a point of $S^c(f)$, $\beta = g(x)$; moreover, let U be an open set in $S^c(f)$, and ε any positive number.

Then $g(U \cap B)$ is dense in $g(U)$ and, since $\beta \in g(U)$, there exists a $y \in U \cap B$ such that $\gamma = g(y) \in (\beta-\varepsilon, \beta+\varepsilon)$. Let $\delta = \frac{1}{2}\min(\beta+\varepsilon-\gamma, \gamma-\beta+\varepsilon)$. Then $[\gamma-\delta, \gamma+\delta]\subset(\beta-\varepsilon, \beta+\varepsilon)$. Since $g_{|B}$ is continuous, there exists an open set W in B such that $y \in W$, $g(W)\subset(\gamma-\delta, \gamma+\delta)$. There is an open set W^* in S^c such that $W^* \cap B = W$. Since B is a strong set of Blumberg, we have

$$g(W^*)\subset\overline{g(W^* \cap B)} = \overline{g(W)}\subset[\gamma-\delta, \gamma+\delta]\subset(\beta-\varepsilon, \beta+\varepsilon),$$

which implies that g is quasi-continuous at x.

Sufficiency. Now g is quasi-continuous. According to Theorem 1.4 g has a Blumberg set B. Let V be an arbitrary open set in $S^c(f)$, $\alpha \in g(V)$, and ε a positive number. Since g is quasi-continuous, there exists an open set V^* in $S^c(f)$ such that

$$V^* \subset V \cap g^{-1}(\alpha-\varepsilon, \alpha+\varepsilon).$$

It follows that $g(V^*) \subset (\alpha-\varepsilon, \alpha+\varepsilon)$. Of course $B \cap V^* \neq \emptyset$. From the inclusions $g(B \cap V^*) \subset (\alpha-\varepsilon, \alpha+\varepsilon)$, $B \cap V^* \subset B \cap V$ we infer that in an arbitrary neighbourhood of α there are points of $g(B \cap V)$. This ends the proof.

§ 2. Monotonicity of connected functions.

DEFINITION 2.1 ([3]). We say that a function $f\colon X \to Y$, where X and Y are two topological spaces, is *weakly monotone* if every level $f^{-1}(y)$ is a connected set.

DEFINITION 2.2. We say that a function $f\colon X \to Y$, where X and Y are two arbitrary topological spaces, *cuts the space* X if each component of its arbitrary level cuts X.

PROPOSITION 2.1. *A connected function $f\colon R \to R$ cuts the space R if and only if none of its levels contains a halfline.*

THEOREM 2.2. *If $f\colon R^n \to R$ is a connected function cutting R^n, then the set $Y_c(f)$ is bilaterally closed.*

Proof. Suppose, on the contrary, that there exist $\gamma \in R$ and sequences $\{\alpha_n\}, \{\beta_n\} \subset Y_c(f)$ such that $\alpha_n \to \gamma$, $\beta_n \to \gamma$, $\alpha_n < \alpha_{n+1}$, $\beta_n > \beta_{n+1}$ for $n = 1, 2, \ldots$, and $f^{-1}(\gamma)$ is not a connected set.

Consider a component A of $f^{-1}(\gamma)$. In view of our assumption the component A cuts R^n into two open sets G_1 and G_2. We shall first prove that

$$(1) \qquad \bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G_1 \quad \text{or} \quad \bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G_2 .$$

Suppose now that there are n_1, n_2 such that $f^{-1}(\alpha_{n_1}) \subset G_1$ and $f^{-1}(\alpha_{n_2}) \subset G_2$. Then, according to Lemma 3.2(b) of a paper by K. M. Garg [3], the set $f^{-1}[\alpha_{n_1}, \alpha_{n_2}]$ is connected. Simultaneously

$$f^{-1}[\alpha_{n_1}, \alpha_{n_2}] = (G_1 \cap f^{-1}[\alpha_{n_1}, \alpha_{n_2}]) \cup (G_2 \cap f^{-1}[\alpha_{n_1}, \alpha_{n_2}]),$$

where both sets on the right side of this equality are nonempty. This contradicts the connectness of $f^{-1}[\alpha_{n_1}, \alpha_{n_2}]$.

In a similar way we can prove

$$(1') \qquad \bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \subset G_1 \quad \text{or} \quad \bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \subset G_2 .$$

We now prove that

$$(2) \qquad \text{if} \quad \bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G_1 \quad \text{then} \quad \bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \subset G_2 \quad \text{and}$$

$$\text{if} \quad \bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \subset G_1 \quad \text{then} \quad \bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G_2 .$$

Suppose now that $\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G_1$ and $\bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \subset G_1$. Let C be a component of G_2, $q \in \bar{C} \setminus C$. Of course $q \notin G_1 \cup G_2$, and $q \in A$, and $C \cup \{q\}$ is a connected set meeting $f^{-1}(\gamma)$ but containing no element of $\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \cup \bigcup_{n=1}^{\infty} f^{-1}(\beta_n)$. The set $f(C \cup \{q\})$ is a nondegenerated interval containing γ and neither α_n nor β_n ($n = 1, 2, \ldots$). This contradicts $\alpha_n \nearrow \gamma$, $\beta_n \searrow \gamma$. In a similar way we can prove that the relation $\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G_2$ and $\bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \subset G_2$ is impossible. Thus we have proved relation (2).

Let S_1, S_2 be two components of $f^{-1}(\gamma)$ ($f^{-1}(\gamma)$ is not connected). Those components cut R^n into sets G', G'' and H', H'', respectively.

Notice that S_2 is contained in one of the sets G', G'' and S_1 is contained in one of the sets H', H''. Assume that $S_2 \subset G'$ and $S_1 \subset H'$. Then $V = G' \cap H'$ is nonempty, for if $z \in \bar{H}' \setminus H'$, then $z \in S_2 \subset G'$, which implies that $K(z, 1/k) \subset G'$ for some k; moreover, $K(z, 1/k) \cap H' \neq \emptyset$.

Two cases are possible

$$1^0 \qquad V \cap \Big[\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \cup \bigcup_{n=1}^{\infty} f^{-1}(\beta_n)\Big] \neq \emptyset,$$

$$2^0 \qquad V \cap \Big[\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \cup \bigcup_{n=1}^{\infty} f^{-1}(\beta_n)\Big] = \emptyset.$$

Consider the first case. Then $G'' \subset H'$ and $H'' \subset G'$. If $G'' \cap H'' \neq \emptyset$; thus there exists a $z_1 \in \overline{G'' \cap H''} \setminus (G'' \cap H'')$. Hence $z_1 \in \bar{H}'' \subset S_2 \cup H''$. According to $G'' \cap S_2 = \emptyset$, we infer that $z_1 \in (S_2 \cup H'') \setminus G''$. On the other hand, in view of $\overline{G'' \cap H''} \subset \bar{G}''$ we can deduced that $z_1 \in \bar{G}'' \subset G'' \cup S_1$, which contradicts the equality $[(S_2 \cup H'') \setminus G''] \cap [S_1 \cup G''] = \emptyset$. Thus $G'' \cap H'' = \emptyset$, and since $G'' \cap S_2 = \emptyset$, we have $G'' \subset H'$, and in a easy way $H'' \subset G'$.

Suppose that for some m $f^{-1}(\alpha_m) \cap V \neq \emptyset$. From (1) it follows that $\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \subset G' \cap H' = V$, which (according to (2)) implies that $\bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \not\subset G''$ and $\bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \not\subset H''$ and also $\bigcup_{n=1}^{\infty} f^{-1}(\beta_n) \not\subset R^n$. But this is impossible because $\{\beta_n\} \subset Y_c(f)$.

In this way we have to consider case 2^0.

Let $x \in V$ and $r = \min(\varrho(x, S_1), \varrho(x, S_2))$. Of course $r > 0$. Now let $S^* = S_1$ if $r = \varrho(x, S_1)$ or $S^* = S_2$ if $r = \varrho(x, S_2)$. Denote by x_0 an element of S^* such that $r = \varrho(x, x_0)$.

If P_x is a closed interval with ends at x and x_0, then P_x contains no element of $\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \cup \bigcup_{n=1}^{\infty} f^{-1}(\beta_n)$, for $P_x \subset V \cup S^*$. Without difficulty we can notice that

$$P_x \setminus f^{-1}(\gamma) \neq \emptyset,$$

for otherwise, in view of $x_0 \in P_x \cap S^*$, $P_x \subset S^*$, which contradicts $V \cap S^* = \emptyset$.

Since $f(P_x)$ is a connected set (in R) containing γ, we have

$$f(P_x) \cap [\bigcup_{n=1}^{\infty} \{\alpha_n\} \cup \bigcup_{n=1}^{\infty} \{\beta_n\}] \neq \emptyset .$$

This contradicts the equality

$$P_x \cap [\bigcup_{n=1}^{\infty} f^{-1}(\alpha_n) \cup \bigcup_{n=1}^{\infty} f^{-1}(\beta_n)] = \emptyset .$$

This contradiction ends the proof.

The above theorem suggests the following question: Is $Y_c(f)$ closed under the assumptions of Theorem 2.2?

Moreover, if the answer is "no" then: Is f weakly monotone on $\overline{S_c(f)}$? These questions are interesting (in view of Garg's theorems [2]) in the case where the domain of f is equal to R^n, $n \geqslant 2$.

The next example solves these problems.

EXAMPLE 2.1. Let

$$A = \{(x, y)\colon (-\infty < x \leqslant -\tfrac{1}{2}\pi) \vee (-\tfrac{1}{2}\pi < x < -\tfrac{1}{4}\pi \wedge y \geqslant \tan x) \vee \vee (-\tfrac{1}{4}\pi \leqslant x < 0 \wedge y \geqslant \tan(2x + \tfrac{1}{2}\pi) - 1)\} ,$$

$$B = \{(x, y)\colon (-\tfrac{1}{2}\pi < x < -\tfrac{1}{4}\pi \wedge y < \tan x) \vee (-\tfrac{1}{4}\pi \leqslant x \leqslant \tfrac{1}{4}\pi \wedge y < -1) \vee (\tfrac{1}{4}\pi < x < \tfrac{1}{2}\pi \wedge y < -\tan x)\} ,$$

$$C = \{(x, y)\colon (-\tfrac{1}{4}\pi < x < 0 \wedge -1 \leqslant y < \tan(2x + \tfrac{1}{2}\pi) - 1) \vee (x = 0 \wedge y \geqslant -1) \vee \vee (0 < x \leqslant \tfrac{1}{4}\pi \wedge -1 \leqslant y < \tan(-2x + \tfrac{1}{2}\pi) - 1)\} ,$$

$$D = \{(x, y)\colon (0 < x \leqslant \tfrac{1}{4}\pi \wedge y \geqslant \tan(-2x + \tfrac{1}{2}\pi) - 1) \vee \vee (\tfrac{1}{4}\pi < x < \tfrac{1}{2}\pi \wedge y \geqslant -\tan x) \vee (\tfrac{1}{2}\pi \leqslant x < \infty)\} .$$

Of course $A \cup B \cup C \cup D = R^2$. Now, let $f\colon R^2 \to R$ be defined as follows:

$$f(x) = \begin{cases} x & \text{for } (x, y) \in A , \\ \arctan y & \text{for } (x, y) \in B , \\ \tfrac{1}{2}\arctan(y+1) - \tfrac{1}{4}\pi & \text{for } (x, y) \in C , \\ -x & \text{for } (x, y) \in D . \end{cases}$$

The function f is continuous and connected as well. Remark that, for $\alpha \in (-\tfrac{1}{2}\pi, 0)$, $f^{-1}(\alpha)$ cuts R^2 and moreover, for $\alpha \in (-\infty, -\tfrac{1}{2}\pi]$, the set $f^{-1}(\alpha)$ is an union of two disjoint straight lines, and also cuts R^2. But $Y_c(f) = (-\tfrac{1}{2}\pi, 0)$ is not closed and $f_{|\overline{S_c(f)}}$ is not weakly monotone, since $f^{-1}(-\tfrac{1}{2}\pi)$ is not connected in spite of $f^{-1}(-\tfrac{1}{2}\pi) \subset \overline{S_c(f)}$.

We can formulate the results of the example in the following theorems:

THEOREM 2.3. *There exists a connected function $f\colon R^n \to R$ ($n = 1, 2, \ldots$) cutting R^n for which $Y_c(f)$ is not closed.*

Proof. If $n = 1$ then the function from Example 1.1 fulfils all our requirements (according to Proposition 2.1).

If $n = 2$ then the function described in the above example fulfils our requirements, and for $n > 2$ one can easily modify this function to obtain an adequate one.

THEOREM 2.4. *There exists a connected function $f\colon R^n \to R$ ($n = 2, 3, \ldots$) cutting R^n for which $f_{|\overline{S_c(f)}}$ is not weakly monotone.*

DEFINITION 2.3. We say that a function $f\colon X \to Y$, where X and Y are two topological spaces, *strongly cuts the space X* if

(i) f cuts the space X,

(ii) if $\alpha \in Y$ and S_1, S_2, S_3 are arbitrary distinct components of $f^{-1}(\alpha)$ and $C_1, C_2, C_3 \subset X$ are connected sets such that $C_i \cap S_j \neq \emptyset$ if $i \neq j$, then $C_i \cap S_i \neq \emptyset$ for some $i = 1, 2, 3$.

PROPOSITION 2.5. *For a connected function $f\colon R \to R$ the following conditions are equivalent:*

(1) f cuts R,

(2) f strongly cuts R,

(3) no level of f contains a half-line.

THEOREM 2.6. *Let X be a regular, connected and locally connected space, and $f\colon X \to R$ a connected function strongly cutting X. If α is a unilateral accumulation point of $Y_c(f)$, then the set $\overline{S_c(f)}$ meets only at most two components of $f^{-1}(\alpha)$.*

Proof. Suppose, on the contrary, that there are $\alpha \in R$, and a sequence $\{\alpha_n\} \subset Y_c(f)$ such that α is the (unilateral) limit point of $\{\alpha_n\}$ and $\overline{S_c(f)}$ has common points with three components of $f^{-1}(\alpha)$. For example, let $\alpha_n > \alpha$. Notice that $\alpha \notin Y_c(f)$ and an element of $f^{-1}(\alpha)$ can be a limit of those elements of $S_c(f)$ whose images are greater than α, since $f_{|\overline{S_c(f)}}$ is continuous and α is an unilateral accumulation point of $Y_c(f)$.

Let S_1, S_2, S_3 be three distinct components of $f^{-1}(\alpha)$ meeting $\overline{S_c(f)}$, and let x, y, z be three points of $\overline{S_c(f)}$ such that $x \in S_1$, $y \in S_2$, $z \in S_3$. Of course they are accumulation points of the set $S_c(f)$, and there exist M-S sequences $\{x_\sigma\}_{\sigma \in \Sigma}$, $\{y_\delta\}_{\delta \in \Delta}$, $\{z_\xi\}_{\xi \in \Xi} \subset S_c(f)$ converging to x, y, z, respectively.

Denote by U_y (U_z) an open connected set containing y (z), disjoint with S_1 (S_1 is a closed set according to Lemma 1.2) and containing no element of $S_c(f)$ whose image is less than α. Then there are elements y_{δ_1}, z_{ξ_1}, $\delta_1 \in \Delta$, $\xi_1 \in \Xi$ such that $y_{\delta_1} \in U_y$, $z_{\xi_1} \in U_z$. $f(U_y)$ and $f(U_z)$ are nondegenerated intervals such that

$$f(U_y) \cap (\alpha, \infty) \neq \emptyset \neq f(U_z) \cap (\alpha, \infty) .$$

There exists an N_1 such that $\alpha_{N_1} \in f(U_y) \cap f(U_z)$. It follows that

$$U_y \cap f^{-1}(\alpha_{N_1}) \neq \emptyset \neq U_z \cap f^{-1}(\alpha_{N_1}) .$$

Let $C_1 = U_y \cup f^{-1}(\alpha_{N_1}) \cup U_z$. It can be seen that $C_1 \cap S_2 \neq \emptyset \neq C_1 \cap S_3$ and C_1 is a connected set.

In an analogous way we can prove that there are connected sets C_2, C_3 such that $C_2 \cap S_1 \neq \emptyset \neq C_2 \cap S_3$, $C_3 \cap S_1 \neq \emptyset \neq C_3 \cap S_2$, $f^{-1}(\alpha_{N_2}) \subset C_2$, $f^{-1}(\alpha_{N_3}) \subset C_3$, $[C_2 \setminus f^{-1}(\alpha_{N_2})] \cap S_2 = \emptyset$ and $[C_3 \setminus f^{-1}(\alpha_{N_3})] \cap S_3 = \emptyset$, where N_2, N_3 are some positive integers.

Since f strongly cuts X, either $C_1 \cap S_1 \neq \emptyset$ or $C_2 \cap S_2 \neq \emptyset$ or $C_3 \cap S_3 \neq \emptyset$. Then for this $i = 1, 2, 3$ $f^{-1}(\alpha_{N_i}) \cap S_i \neq \emptyset$, which is impossible. In this way we proved the theorem.

Now we can formulate and prove two theorems giving an answer to the problem of K. M. Garg [3] (Problem 3.11 p. 27). Partial resolutions are contained in [4] and [11]. Our results are a little more general with regard to the range of the functions considered. At the same time (Example 2.2) the assumptions of these theorems cannot be weakened by inquirements as in Grande's analogous theorems for real functions. Before proving the theorems we give two definitions.

DEFINITION 2.4 [3] The function $f: X \to Y$, where X, Y are two topological spaces, is said to be *Morrey monotone* if each of its levels is a continuum.

DEFINITION 2.5. We say that a topological space X is *connectedly embedded in a space* Y if there is a connected injection $f: X \to Y$.

Remark that every discrete space of the power of the continuum is connectedly embedded in R as well as separable continuum containing exactly two non-cut points (for R. L. Moore proved that such a space is homeomorphic to $[0, 1]$).

THEOREM 2.7. *If X is a locally connected continuum, Y — connectedly embedded in R then a connected function $f: X \to Y$ is Morrey monotone on the set $\overline{S_c(f)}$.*

Proof. Let g be a connected injection mapping Y into R and $h = g \circ f$. Then $h: X \to R$ is a real connected function, and so $h_{|\overline{S_c(h)}}$ is Morrey monotone (see [4], [11]), and since g is one to one $f_{|\overline{S_c(f)}}$ is Morrey monotone.

EXAMPLE 2.2. Let $Y = \{(x, y) \in R^2: 0 < x \leqslant \frac{2}{3}\pi,\ y = \sin(1/x)\} \cup \{(0, 1)\}$ with a topology induced by the natural topology of R^2. Then Y is of course connectedly embedded in R. Let

$$A = \{(x, y) \in R^2: 0 \leqslant x \leqslant \tfrac{2}{3}\pi \wedge y = 0\},$$
$$B = \{(x, y) \in R^2: 0 \leqslant x \leqslant \tfrac{2}{3}\pi \wedge y = -1\},$$
$$J_n = \left\{(x, y) \in R^2: x = \frac{2}{(2n+1)\pi} \wedge -1 \leqslant y \leqslant 0\right\} \text{ for } n = 1, 3, \ldots, 2n+1, \ldots$$

If $X = A \cup B \cup \bigcup_{n=1,3,\ldots} J_n$ is a topological space with a topology induced by the natural topology of R^2, then X is a connected and locally connected Hausdorff space. Let $f: X \to R^2$ be defined in the following way:

$$f((x, y)) = \begin{cases} (x, \sin(1/x)) & \text{if } x \in (0, \frac{2}{3}\pi], \\ (0, 1) & \text{if } x = 0. \end{cases}$$

The function f maps X onto Y and is connected and proper (it means that the inverse image of a compact set is compact, see [3] Def. 3.1), but $f_{|\overline{S_c(f)}}$ is not weakly monotone since $f^{-1}((0, 1))$ is not connected.

Z. Grande proved that for real functions the assumption of the compactness of X can be omitted provided f is a proper function. The above example shows that in our case this is impossible. The work [11] contains a proof of the fact that a function fulfilling all the assumptions of Theorem 2.7 need not be monotone (of course its restriction to $\overline{S_c(f)}$).

However, we shall prove that $f_{|\overline{S_c(f)}}$ is quasi-monotone.

DEFINITION 2.6. The function $f: X \to Y$, where X, Y are two topological spaces, is said to be *quasi-monotone* if $f^{-1}(C)$ is connected for every connected set $C \subset f(X)$.

THEOREM 2.8. *If X is a locally connected continuum, Y a topological space connectedly embedded in R and $f: X \to Y$ a connected function, then $f_{|\overline{S_c(f)}}$ is quasi-monotone.*

Proof. Let the notation be as in the proof of Theorem 2.7. Notice that $\overline{S_c(h)}$ is compact and $\hat{h} = h_{|\overline{S_c(h)}}$ is continuous (in virtue of Corollary 1.1), and so $\hat{h}$ is a closed function (see [1] p. 167). According to Theorem 2.7 and Theorem 6.1.11 in the monograph of R. Engelking [1] the function $\hat{h}$ is quasi-monotone.

Let C be a connected subset of $f(\overline{S_c(f)})$. Then $g(C)$ is connected in R and hence it is an interval contained in $h(X)$. Since $\hat{h}$ is quasi-monotone, $\hat{h}^{-1}(g(C))$ is a connected set in $\overline{S_c(h)}$ and so in $\overline{S_c(f)}$. Since g is injective,

$$\hat{h}^{-1}(g(C)) = \hat{f}^{-1}(C),$$

where $\hat{f} = f_{|\overline{S_c(f)}}$. This implies that $\hat{f}$ is quasi-monotone.

§ 3. Properties of open and quasi-open connected functions.

DEFINITION 3.1. We say that a function $f: X \to Y$ (X, Y are topological spaces) is *open* (*quasi-open*) if for every open set $U \subset X$ the set $f(U)$ is open (has a non-empty interior).

DEFINITION 3.2 (see [3] Def. 1.5). The function $f: X \to Y$, where X and Y are topological spaces, is said to be *nowhere constant* if it assumes at least two distinct values on each open subset of X.

PROPOSITION 3.1. *The weakly connected function $f: X \to R$, where X is a locally connected space, is quasi-open if and only if f is nowhere constant.*

THEOREM 3.2. *Let $f: X \to R$ be an open connected function on a connected and locally connected space X. Then the set $f^{-1}(\alpha)$ is boundary and closed for every $\alpha \in \overline{Y_c(f)}$.*

Proof. With regard to Proposition 3.1 the sets $f^{-1}(\alpha)$ are boundary. We shall show that for $\alpha \in \overline{Y_c(f)}$, $f^{-1}(\alpha)$ is closed.

If $\alpha \in Y_c(f)$ then $f^{-1}(\alpha)$ is closed according to Lemma 1.2.

Now let $\alpha \in \overline{Y_c(f)} \setminus Y_c(f)$ and suppose that $f^{-1}(\alpha)$ is not a closed set. Then there exists a $y \in \overline{f^{-1}(\alpha)} \setminus f^{-1}(\alpha)$ and let $\beta = f(y)$. Assume that $\alpha < \beta$ for example.

One can easily see that $[\alpha, \beta) \cap Y_c(f) = \emptyset$. It follows that there is an increasing sequence $\{\alpha_n\} \subset Y_c(f)$ such that $\lim_{n\to\infty} \alpha_n = \alpha$.

Let U be an arbitrary connected open subset of X containing y. Then $U \cap f^{-1}(\alpha) \neq \emptyset$ and $f(U)$ is an open connected set containing α; so $f(U)$ contains an element $\alpha_{n_0} \in Y_c(f)$ for some n_0. Thus U contains some element of $f^{-1}(-\infty, \alpha)$. This implies that $y \in \overline{f^{-1}(-\infty, \alpha)}$. On the other hand, according to Lemma 1.5

$$\overline{f^{-1}(-\infty, \alpha)} = \overline{f^{-1}(\bigcup_{n=1}^{\infty}(-\infty, \alpha_n])} \subset f^{-1}(-\infty, \alpha].$$

Hence $y \notin f^{-1}(-\infty, \alpha)$. The contradiction ends the proof.

We infer hence:

COROLLARY 3.1. *Let X be a connected and locally connected space. If f: $X \to R$ is an open connected function, then $f_{|S^c(f)}$ is continuous.*

THEOREM 3.3. *Let X be a compact space, and Y a Hausdorff space. If f: $X \to Y$ is open and continuous on $\overline{S_c(f)}$, then f is Morrey monotone on the set $\overline{S^c(f)}$.*

Proof. We shall prove first that $S^c(f) = \overline{S_c(f)}$. Let $x \in S^c(f)$ and $\lambda = f(x)$. If U is an arbitrary neighbourhood of x, then $f(U)$, as an open set, contains some element of $Y_c(f)$; this means that U contains some element of $S_c(f)$. In this way $x \in \overline{S_c(f)}$, and so $S^c(f) \subset \overline{S_c(f)}$. Since $f_{|\overline{S_c(f)}}$ is a continuous function, $\overline{S_c(f)} \subset S^c(f)$.

Suppose now that there exists an $\alpha \in \overline{Y_c(f)} \setminus Y_c(f)$ such that $g^{-1}(\alpha)$ (where $g = f_{|S^c(f)}$) is not a connected subset of $S^c(f)$. This means that $g^{-1}(\alpha) = A \cup B$, where A, B are disjoint nonvoid closed subsets of $S^c(f)$. X is a normal space and then there exist two open sets G and H such that $A \subset G$, $B \subset H$ and $G \cap H = \emptyset$.

Let $\{\alpha_\sigma\}_{\sigma\in\Sigma} \subset Y_c(f)$ be any M-S sequence such that $\alpha = \lim_{\sigma\in\Sigma} \alpha_\sigma$. Since f is an open function, $f(G) \cap f(H)$ is a nonvoid open set. There exists a $\sigma^* \in \Sigma$ such that $\alpha_\sigma \in f(G) \cap f(H)$ for $\sigma \geqslant \sigma^*$. Consider M-S sequences $\{\alpha_\sigma\}_{\sigma\geqslant\sigma^*}$ and $\{f^{-1}(\alpha_\sigma)\}_{\sigma\geqslant\sigma^*}$. For every $\sigma \geqslant \sigma^*$ the level $g^{-1}(\alpha_\sigma)$ meets G and H. Because of the connectedness of $g^{-1}(\alpha_\sigma)$ there exists an $x_\sigma \in g^{-1}(\alpha_\sigma) \setminus (G \cup H)$ for every $\sigma \geqslant \sigma^*$. We form in this way an M-S sequence $\{x_\sigma\}_{\sigma\geqslant\sigma^*} \subset S_c(f)$. Since X is compact, this sequence has an accumulation point x'. Let $\{x_{\sigma'}\}$ be such a subsequence of $\{x_\sigma\}_{\sigma\geqslant\sigma^*}$ that $\lim x_{\sigma'} = x'$. From the construction of $\{x_{\sigma'}\}$ it follows that $\lim f(x_{\sigma'}) = \alpha$. Since g is continuous, we have $\lim g(x_{\sigma'}) = \lim f(x_{\sigma'}) = g(x') = f(x')$, and thus $f(x') = \alpha$.

In this way $x' \in g^{-1}(\alpha)$ but $x' \notin G \cup H$ and so $x' \notin A \cup B$, which contradicts $g^{-1}(\alpha) \subset S^c(f)$ and $g^{-1}(\alpha) = A \cup B$.

Hence the set $g^{-1}(\alpha)$ is connected. By the continuity of g and the compactness of X we infer that $g^{-1}(\alpha)$ is continuum.

According to Corollary 1.1 we have:

COROLLARY 3.2. *If X is a locally connected continuum and f: $X \to R$ is an open connected function, then f is Morrey monotone on the set $\overline{S^c(f)}$.*

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