Characteristic functions freely generate measurable functions

by

Denis Higgs (Waterloo)

Abstract. It is shown that if S is a σ-ring of subsets of a set X and $M(S)$ is the magnitude module of $[0, \infty]$-valued functions on X which are measurable with respect to S then $M(S)$ is freely generated by the characteristic functions of the sets in S, subject to the obvious relations.

§ 1. **Introduction.** Let S be a σ-ring of subsets of a set X and let $M(S)$ be the set of all $[0, \infty]$-valued functions on X which are measurable with respect to S. We prove that $M(S)$ is freely generated as a magnitude module by the characteristic functions χ_a of the members a of S subject to the relations

$$\chi_a = \chi_{a_1}+\chi_{a_2}+\ldots, \quad \chi_{\emptyset} = 0,$$

where a is the disjoint union of $a_1, a_2, \ldots$ (Magnitude modules, introduced in [2] and defined here in the next section, constitute an equational class of universal algebras of a certain type.) This answers a question asked in [2]. The proof hinges on the observation that if $\lambda, \lambda_1, \lambda_2, \ldots$ are reciprocals of powers of 2 such that $\lambda_1, \lambda_2, \ldots \leqslant \lambda < \lambda_1+\lambda_2+\ldots$ then λ equals the sum of finitely many terms of $\lambda_1+\lambda_2+\ldots$

An immediate consequence of the result is the following basic fact of measure/integration theory: if μ: $S \to [0, \infty]$ is a measure then there is a unique integral $\bar{\mu}$: $M(S) \to [0, \infty]$ such that $\bar{\mu}(\chi_a) = \mu(a)$ for all a in S (note that measures and integrals are understood to be countably additive). Moreover, this basic fact is now seen to be true for measures and integrals taking their values not only in $[0, \infty]$ but in any magnitude module; this should be helpful in defining various vector-valued integrals.

The paper concludes with some remarks relating to the use of real-valued functions instead of $[0, \infty]$-valued functions.

§ 2. **Magnitude modules and $M(S)$.** A *magnitude module* is a set M together with an ω-ary operation $\sum$, a nullary operation 0, and a unary operation h satisfying the following identities:

(i) $\sum(\sum(x_{11}, x_{12}, \ldots), \sum(x_{21}, x_{22}, \ldots), \ldots)$
$= \sum(\sum(x_{11}, x_{21}, \ldots), \sum(x_{12}, x_{22}, \ldots), \ldots),$

(ii) $\sum(0, \ldots, 0, x, 0, \ldots) = x$,

(iii) $h(\sum(x_1, x_2, \ldots)) = \sum(h(x_1), h(x_2), \ldots)$,

(iv) $\sum(h(x), h^2(x), \ldots) = x$.

As shown in [2], a magnitude module may also be defined as a set M with operations $\sum$ and 0 satisfying (i) and (ii) along with a scalar multiplication $[0, \infty] \times M \to M$ such that

$$(\lambda_1+\lambda_2+\ldots)x = \lambda_1 x+\lambda_2 x+\ldots, \quad \lambda(x_1+x_2+\ldots) = \lambda x_1+\lambda x_2+\ldots,$$
$$(\lambda\mu)x = \lambda(\mu x), \quad 1x = x, \quad 0x = 0.$$

(Here and elsewhere we write $x_1+x_2+\ldots$, or $\sum(x_n: n \in N)$ where $N = \{1, 2, \ldots\}$, in place of $\sum(x_1, x_2, \ldots)$; we also write $\frac{1}{2}x$ in place of $h(x)$, etc.).

Let S be a σ-ring of subsets of a set X, so that $\emptyset$ is in S and S is closed under countable unions and differences (we do not require X to be in S necessarily). A function f from X to $[0, \infty]$ is *measurable* with respect to S if and only if $\{x \in X: f(x) > \alpha\}$ is in S for all α in $[0, \infty]$. It is well-known that the set $M(S)$ of all such functions is closed under countable addition and scalar multiplication by $[0, \infty]$, both pointwise (see, for example, Berberian [1], Sec. 14, Theorem 3 and Sec. 13, Theorem 1); $M(S)$ thus forms a magnitude module. The characteristic functions χ_a of the sets a in S are evidently in $M(S)$ and satisfy the relations

(R) $\chi_a = \chi_{a_1}+\chi_{a_2}+\ldots, \ \chi_\emptyset = 0$,

where a is the disjoint union of $a_1, a_2, \ldots$ Furthermore, the χ_a's generate $M(S)$ as a magnitude module, this being an immediate consequence of the fact that every $[0, \infty]$-valued measurable function is the pointwise supremum of an increasing sequence of $[0, \infty]$-valued simple functions ([1], Sec. 16, Theorem 4).

§ 3. THEOREM. *$M(S)$ is freely generated as a magnitude module by the χ_a's, a in S, subject to the relations* (R).

Proof. We have to show that if we are given two expressions, each built up from the χ_a's using $\sum$ and h, and these two expressions denote the same function in $M(S)$ then this equality can be proved using the magnitude module identities and the relations (R) alone; in such case let us say that the two expressions are *equivalent*.

As already noted in [2], every magnitude module expression in variables x_i, $i \in I$, is equivalent either to 0 or to an expression of the form $\sum((\frac{1}{2})^{k_n} x_{i_n}: n \in N)$. In the latter case we may suppose further that

(A) each x_i which occurs, occurs infinitely often, that is, $\{n \in N: i_n = i\}$ is either empty or infinite for each i in I: this may be achieved using magnitude module identity (iv),

(B) a given $(\frac{1}{2})^k$ occurs only finitely many times, that is, $\{n \in N: k_n = k\}$ is finite for each k in N: if the nth coefficient is not already $(\frac{1}{2})^k$ with $k \geq n$ we may split it into a sum of $(\frac{1}{2})^n$'s, splitting the corresponding term accordingly.

(Note. Such transformations are justified by the fact, remarked in [2], that identities (i) and (ii) already allow us to operate with finite and infinite sums in a magnitude module as freely as we may with series of non-negative terms, and by the properties of h, which permit replacing x by $\frac{1}{2}x+(\frac{1}{2})^2x+\ldots$, $\frac{1}{2}x$ by $(\frac{1}{2})^2x+(\frac{1}{2})^2x$, etc.)

It follows from these remarks that without loss of generality we may suppose our given expressions to be the two series

(S) $\sum(\lambda_n a_n: n \in N, \varepsilon(n) = 1)$ and $\sum(\lambda_n a_n: n \in N, \varepsilon(n) = 2)$

where

(C) $\varepsilon: N \to \{1, 2\}$ is a function which enables us to use a single notation for the two series, helpful in view of the back-and-forth nature of the inductive procedure described below,

(D) each a_n denotes the corresponding χ_{a_n}, a_n being a set in S (the convention whereby the characteristic function of a set is denoted by that set is used below repeatedly),

(E) if the common value of the two series at some x in X is non-zero then infinitely many terms of each series are non-zero at x (this is true on account of (A)).

(F) each λ_n is of the form $(\frac{1}{2})^k$ for some k in N, and $\lambda_1 \geq \lambda_2 \geq \ldots$ (this is possible by (B)).

(Note that the case of an expression equivalent to 0 is subsumed under the above using the relation $\chi_\emptyset = 0$ and the magnitude module identities).

The idea behind the proof that the two series (S) are equivalent is to successively split the terms $\lambda_n a_n$ in such a way that the resulting two series are the same. The procedure can best be illustrated by describing the first stage $k = 1$. Suppose that $\lambda_1 a_1$ occurs in the first series ($\varepsilon(1) = 1$). Then for each $x \in a_1$, the sum $\sum(\lambda_n: \varepsilon(n) = 2, x \in a_n)$ of the λ_n's in the second series for which $x \in a_n$ will be $> \lambda_1$; this is because the two series have the same value at x and (E) holds for the first series. On account of (F), it follows by the observation made in § 1 above that $\lambda_1 = \sum(\lambda_n: n \in \pi)$, $x \in \bigcap(a_n: n \in \pi)$ for some finite subset π of $\{n \in N: \varepsilon(n) = 2\}$. We may enumerate these sets π as π_1^t, $t \in N$, and successively cut off from a_1 the intersections $c_1^t = \bigcap(a_n: n \in \pi_1^t)$, thereby expressing a_1 as a disjoint union $a_1 = \bigcup(d_1^t: t \in N)$ where $d_1^t \subseteq c_1^t \subseteq a_n$ for $n \in \pi_1^t$. Thus the equation

$$\lambda_1 a_1 = \sum(\lambda_1 d_1^t: t \in N) = \sum(\lambda_n d_1^t: n \in \pi_1^t, t \in N)$$

expresses the term $\lambda_1 a_1$ from the first series as a sum of portions $\lambda_n d_1^t$ of the terms $\lambda_n a_n$ from the second series. Removing these portions from both series, we next deal similarly with the term $\lambda_2 a_2$ (or rather, if it occurs in the second series, with what is now left of it) and then one-by-one with the subsequent $\lambda_n a_n$'s. The precise argument follows.

For each k in N, let $(\pi_k^t: t \in N)$ be an enumeration of the finite subsets π of $\{n \in N: n > k, \varepsilon(n) \neq \varepsilon(k)\}$ such that $\sum(\lambda_n: n \in \pi) = \lambda_k$; if for a given k there are only a finite number of such π, let $\pi_k^t = \emptyset$ for $t >$ this number.

Define sets $b_{n,k}$, c_k^t, d_k^t in S, where k, n, t are in N and $n \geqslant k$, by induction on k as follows:

$$b_{n,1} = a_n,$$
$$c_k^t = \bigcap (b_{n,k}\colon\ n \in \pi_k^t) \text{ if } \pi_k^t \neq \emptyset, \qquad c_k^t = \emptyset \text{ if } \pi_k^t = \emptyset,$$
$$d_k^t = b_{k,k} \cap (c_k^t \setminus (c_k^1 \cup \ldots \cup c_k^{t-1})),$$
$$b_{n,k+1} = b_{n,k} \setminus \bigcup (d_k^t\colon\ t \in N, n \in \pi_k^t) \text{ (here } n \geqslant k+1).$$

We show that the following conditions (k.1), (k.2) hold for all k:

(k.1) $\sum (\lambda_n b_{n,k}\colon\ n \in N, n \geqslant k, \varepsilon(n) = 1)$ and $\sum (\lambda_n b_{n,k}\colon\ n \in N, n \geqslant k, \varepsilon(n) = 2)$ denote the same function in $M(S)$ and if the common value of the two series at some x in X is non-zero then infinitely many terms of each series are non-zero at x;

(k.2) $b_{k,k} = \bigcup (d_k^t\colon\ t \in N)$, a disjoint union.

For $k = 1$, (k.1) is true by hypothesis and (E). Hence it is sufficient to show that (k.1) implies (k.2) and that (k.1)+(k.2) implies (k+1.1).

Suppose that (k.1) holds and that $\varepsilon(k) = 1$ say. Let x be in $b_{k,k}$ and evaluate the two series in (k.1) at x. Then if in the first series we retain only the term with $n = k$ and note that by (k.1) some subsequent term is non-zero at x, we obtain

$$\lambda_k < \sum (\lambda_n\colon\ n \in N, n \geqslant k, \varepsilon(n) = 2, x \in b_{n,k})$$

where we may replace $n \geqslant k$ by $n > k$ since $\varepsilon(k) = 1$. By the observation made in § 1, which is applicable on account of (F), there exists a finite subset π of $\{n \in N\colon\ n > k, \varepsilon(n) = 2\}$ such that $\lambda_k = \sum (\lambda_n\colon\ n \in \pi)$, $x \in \bigcap (b_{n,k}\colon\ n \in \pi)$. But then $\pi = \pi_k^t$ for some t so that $x \in c_k^t$. Thus $b_{k,k} \subseteq \bigcup (c_k^t\colon\ t \in N)$ and (k.2) follows using the definition of the d_k^t's.

Now for all $n \geqslant k+1$ we have

(k.3) $b_{n,k} = b_{n,k+1} \cup \bigcup (d_k^t\colon\ t \in N, n \in \pi_k^t)$, a disjoint union.

(This follows from the definition of $b_{n,k+1}$ and the fact that $d_k^t \subseteq c_k^t \subseteq b_{n,k}$ for $n \in \pi_k^t$).

Suppose that (k.1) and (k.2) hold and let $\varepsilon(k) = 1$ again. Then in the first series of (k.1), $b_{n,k} = b_{n,k+1}$ for $n > k$ (in the definition of $b_{n,k+1}$, $n \in \pi_k^t$ is impossible if $\varepsilon(n) = 1$) and hence, using (k.3) in the second series, we obtain (purely as equality of functions)

$$\lambda_k b_{k,k} + \sum (\lambda_n b_{n,k+1}\colon\ n \in N, n \geqslant k+1, \varepsilon(n) = 1)$$
$$= \sum (\lambda_n b_{n,k+1}\colon\ n \in N, n \geqslant k+1, \varepsilon(n) = 2) + \sum (\lambda_n d_k^t\colon\ n, t \in N, n \in \pi_k^t).$$

The second sum on the right-hand side here equals

$$\sum (\sum (\lambda_n\colon\ n \in \pi_k^t) d_k^t\colon\ t \in N\} = \sum (\lambda_k d_k^t\colon\ t \in N) = \lambda_k b_{k,k}$$

by virtue of (k.2) (notice that if $\pi_k^t = \emptyset$ for some t then $c_k^t = \emptyset$ and hence $d_k^t = \emptyset$ so that we still have $\sum (\lambda_n\colon\ n \in \pi_k^t) d_k^t = \lambda_k d_k^t$). Cancelling $\lambda_k b_{k,k}$ gives the first part

of $(k+1.1)$. The second part of (k+1.1) is obvious for the first series since the terms are unchanged from (k.1) and it may be seen to hold for the second series as follows. Let x be in X — then $x \in b_{n,k}$ implies $x \in b_{n,k+1}$ except when x is in d_k^t and n is in π_k^t. But x is in d_k^t for at most one t since the d_k^t's are disjoint (for k fixed), and π_k^t is finite. Thus x in $b_{n,k}$ for infinitely many n with $\varepsilon(n) = 2$ gives the same with $k+1$ in place of k, as required.

We are now in a position to show that our two series (S) are equivalent (using the magnitude module identities and (R)). From $a_n = b_{n,1}$, (k.3), and (k.2) (with $k = n$), we have

$$a_n = \bigcup (d_{n'}^t\colon\ n', t \in N, n \in \pi_{n'}^t) \cup \bigcup (d_n^t\colon\ t \in N)$$

for all n, and this is a disjoint union. Thus our first series is equivalent to

$$\sum (\lambda_n d_{n'}^t\colon\ n, n', t \in N, n \in \pi_{n'}^t, \varepsilon(n) = 1) + \sum (\lambda_n d_n^t\colon\ n, t \in N, \varepsilon(n) = 1).$$

In the first sum here we may replace $\varepsilon(n) = 1$ by $\varepsilon(n') = 2$ because of the condition $n \in \pi_{n'}^t$; thus it is equivalent to

$$\sum (\sum (\lambda_n\colon\ n \in \pi_{n'}^t) d_{n'}^t\colon\ n', t \in N, \varepsilon(n') = 2),$$

that is, to $\sum (\lambda_{n'} d_{n'}^t\colon\ n', t \in N, \varepsilon(n') = 2)$ (as before, $\pi_{n'}^t = \emptyset$ cause no trouble). Our first series itself is therefore equivalent to

$$\sum (\lambda_{n'} d_{n'}^t\colon\ n', t \in N, \varepsilon(n') = 2) + \sum (\lambda_n d_n^t\colon\ n, t \in N, \varepsilon(n) = 1).$$

The symmetry of this expression shows that our second series is also equivalent to it. Thus our two series are equivalent to each other and the proof is finished.

§ 4. **Remarks.** It follows by standard arguments of universal algebra that the above theorem may be expressed as a universal property of $M(S)$, namely that for any magnitude module M and any M-valued measure μ on S (so that $\mu\colon S \to M$ satisfies $\mu(a) = \sum \mu(a_n)$ for a the disjoint union of $a_1, a_2, \ldots$ in S, $\mu(\emptyset) = 0$), there is a unique magnitude module morphism $\bar{\mu}\colon M(S) \to M$ such that

$$\begin{array}{ccc} S & \xrightarrow{\ \chi\ } & M(S) \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\bar{\mu}} & \\ & M & \end{array}$$

commutes. We emphasize that $\bar{\mu}$ being a morphism means that $\bar{\mu}(\sum f_n) = \sum \bar{\mu}(f_n)$ and $\bar{\mu}(\alpha f) = \alpha \bar{\mu}(f)$ for all $f_1, f_2, \ldots, f \in M(S)$, $\alpha \in [0, \infty]$; the first of these conditions is just the Monotone Convergence Theorem for the M-valued integral $\bar{\mu}$.

Using results of Linton [3], it is not difficult to extend the theorem to the case of an abstract σ-ring S, $M(S)$ then being defined to consist of all σ-ring morphisms from the σ-ring of Borel subsets of $(0, \infty]$ to S. We note that the particular case of the theorem in which S is the power set of a one-element set is the main result of [2]; the use of the observation made in § 1 here enables us to avoid taking the common refinements of [2] and leads to an argument neater than that given there.

Although it concerns $[0, \infty]$-valued functions, the theorem proved above has an implication for real-valued functions. Let $\mathscr{M}$ be the set of real-valued functions on X which are measurable with respect to the σ-ring S on X. Then each f in $\mathscr{M}$ can be written as a difference of non-negative functions in $\mathscr{M}$. It follows that f is the sum of a pointwise absolutely convergent series $\sum \alpha_n \chi_{a_n}$ where $\alpha_n \in R$ and $a_n \in S$; conversely, each such series defines a function in $\mathscr{M}$. Suppose that two such series define the same function in $\mathscr{M}$. Then this fact may be proved using only the axioms for magnitude modules together with the relations (R) and the rule (T): $f_1+f_4 = f_2+f_3$ implies $f_1-f_2 = f_3-f_4$. To see this, take the negative terms in each of the two series to the other side (of the equation formed by equating the two series) — the result is an equation of the type we have shown to be thus provable and (T) then allows us to transpose back the terms moved. Similar remarks apply to functions taking their values in the extended reals. Of course, with the reals or the extended reals, infinite sums are not always defined (nor can they be, if they are to satisfy axioms (i) and (ii) and agree with ordinary addition for finite sums, witness $1-1+1-1+\ldots$). This means that one only has *partial* algebras and therefore cannot expect so simple a formulation of the appropriate freeness/universal property as one obtains for algebras proper. Using magnitude modules (as opposed to vector spaces together with some topological and/or order structure) we lose something: everywhere-defined subtraction (the extended reals do not even provide that) and we win something: everywhere-defined infinite sums.

References

[1] S. K. Berberian, *Measure and Integration*, Macmillan, New York 1965.

[2] D. Higgs, *A universal characterization of* $[0, \infty]$, Nederl. Akad. Wetensch. Proc. Ser. A, 81 (= Indag. Math. 40) (1978), pp. 448–455.

[3] F. E. J. Linton, *Functorial measure theory, in Functional Analysis*, Proceedings of the Conference in Irvine, California, 1966, edited by B. R. Gelbaum, pp. 36–49 (Thompson Book Company, Washington 1967).

PURE MATHEMATICS DEPARTMENT
UNIVERSITY OF WATERLOO
Waterloo, Ontario

Accepté par la Rédaction le 2. 7. 1979