Remark 10. There are several known methods of assigning to a space E a polyhedral (ANR) associated system, e.g. assigning to E its Čech system [19] (also see [13]). Theorem 15 shows that the proofs of Theorems 11 and 13 offer alternative methods, which generalize the original Mardešić–Segal ANR-system approach to shape [18].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ZAGREB
Zagreb

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On the k-pseudo-symmetrical approximate differentiability *

by

Giuseppina Russo and **Santi Valenti** (Palermo)

Abstract. The purpose of this paper is to establish a connection between two ways of generalizing the notion of derivative.

1. It is well-known that a number of significant properties of differentiable functions can be expressed in terms of some symmetrical or, generally, bilateral differential quotients (see, for instance, [4] and [3]). On the other hand, a powerful way of generalizing the notion of derivative is that of picking up only these values of the differential quotient that correspond to a suitable set having positive density at a given point: so one obtains, e.g., the approximate (or asymptotical) derivative (see, for instance, [1] and [3]).

Within the present paper, our purpose is to establish a transparent connection between the first and the second way to get a notion of derivative; more precisely, we shall give a theorem who clarifies the relation between the usual approximate derivative and a new one, here called *k-pseudo-symmetrical approximate* (*or asymptotical*) *derivative*.

Such a theorem shows that this new definition, based on a method introduced elsewhere [4] by one of us (S. V.), gives place to an approximate derivative that exists, at least almost everywhere, in any measurable set where the usual one does.

As for a complete understanding of the demonstration it will be useful the knowledge of a deep and elegant theorem by A. Kintchine [2], we report here its statement: *let $f(x)$ be a measurable function, assigned on a measurable set E. Then almost all points of E do belong to one of the following sets*

$E_1 \equiv \{x \in E$: *the approximate derivative of $f(x)$ exists* (¹)$\}$;

$E_2 \equiv \{x \in E$: *its upper* (*lower*) *approximate derivates are both* $+\infty\,(-\infty)\}$.

2. Let $f(x)$ be a real function of a real variable, i.e. let $A \subset R$ and $f(x)\colon A \to R$. It is well-known that one can give the notion of *approximate* (or *asymptotical*) *derivative* of $f(x)$ at the point $x \in R$ in the following way [1]:

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DEFINITION 1. If the limit

$$\lim_{y\to x}\frac{f(y)-f(x)}{y-x} \tag{1}$$

does exist when one subtracts to A a set of density 0 at x, then this limit is called the *approximate* (or *asymptotical*) *derivative* of $f(x)$ at x (on A). (We shall denote it by $D_A f(x)$).

Furthermore, one introduces also the notion of *upper right approximate* (or *asymptotical*) *derivate* of $f(x)$, namely $D_A^+ f(x)$, at the point $x\in R$, as follows:

DEFINITION 2. The upper right approximate (or asymptotical) derivate $D_A^+ f(x)$ of $f(x)$ at x (on A) is the *least upper bound of the set of the real numbers* a such that

$$\frac{f(y)-f(x)}{y-x}\geqslant a\,, \quad y>x\,, \tag{2}$$

where y belongs to a set of density >0 at x. (One defines quite analogously the other three derivates, namely $D_{+A}f(x)$, $D_A^- f(x)$ and $D_{-A}f(x)$).

Now, following an idea contained within a previous paper [4], we introduce a new definition, which constitutes the kernel of the present paper, i.e.:

DEFINITION 3. If $k>0$ is a given number, we shall call the *k-pseudo-symmetrical approximate* (*or asymptotical*) *derivative* of $f(x)$ at x (on A) the limit (if it exists)

$$\lim_{h\to 0^+}\frac{f(x+kh)-f(x-h)}{(k+1)h}\,, \tag{3}$$

where $x+kh\in A$, $x-h\in A$ and we are allowed to subtract to the set of all possible h a set $H(x)$ of density 0 at 0. (We shall denote such a limit by $D_{kA}f(x)$).

It is our purpose to sketch the connections between this notion of approximate derivative and the ordinary one. In effect, we are going to prove that the existence of $D_{kA}f(x)$ implies the existence of the usual approximate derivative; all that, of course, being valid almost everywhere in A. (Actually, in what follows we prefer to assign $f(x)$ on an interval Δ of the real line, but the reader will easily see that our assumption introduces no restriction within the final result).

3. Now, we pass straightforward to prove the following

THEOREM. *Let Δ be an interval of the real line R, $f(x)$ a function from Δ to R and E the subset of Δ where the k-pseudo-symmetrical approximate derivative of $f(x)$ does exist. Then, if $f(x)$ is measurable, it has the approximate derivative a.e. in E.*

Proof. Suppose that our theorem does not hold. Then, there exists a subset X of E where $f(x)$ is not approximately differentiable, and such that

$$\operatorname{meas}_e(X)>0\,; \tag{4}$$

(meas_e means the Lebesgue's exterior measure).

(1) Whenever we say that a derivative exists, we mean that it is finite.

After, consider a mapping Γ, from the set N^+ of the positive integers onto a subset $\{X_n\}$ of the power set $\mathscr{P}(X)$ of X, so defined:

$$\forall n\in N^+,\ \Gamma(n)=X_n\equiv\{x\in X\colon 0<h<1/n,\ h\notin H(x) \Rightarrow f(x+kh)-f(x-h)\leqslant(k+1)hn\}\,. \tag{5}$$

The following equivalence is obvious:

$$f(x+kh)-f(x-h)\leqslant(k+1)hn \Leftrightarrow f(x+kh)-n(x+kh)-f(x-h)+n(x-h)\leqslant 0\,, \tag{6}$$

so that, by means of the measurable functions

$$F_n(x)=f(x)-nx\,, \tag{7}$$

one obtains the other equivalence

$$x\in X_n \Leftrightarrow \{0<h<1/n,\ h\notin H(x) \Rightarrow F_n(x+kh)-F_n(x-h)\leqslant 0\}\,. \tag{8}$$

Besides, by virtue of (4) and since

$$X=\bigcup_{n=1}^{\infty}X_n\,, \tag{9}$$

we can find an index, say $r\in N^+$, such that

$$\operatorname{meas}_e(X_r)>0\,; \tag{10}$$

on the other hand, there is a perfect set of continuity for $F_r(x)$, say $P\subset\Delta$, such that

$$\operatorname{meas}(P)>\operatorname{meas}(\Delta)-\operatorname{meas}_e(X_r) \tag{11}$$

so that, for its density set, say D, one draws from (11)

$$\operatorname{meas}_e(D\cap X_r)>0\,. \tag{12}$$

Consider, then, a density point of $D\cap X_r$ and call it z: at present, we shall prove that the measurable set

$$A\equiv\{x\colon x>z,\ F_r(x)>F_r(z)\} \tag{13}$$

is of density 0 at z.

In effect, if the density of A at z was >0, we could find a positive number τ, with $\tau<1/r$, such that, for all $y\in\,]z,z+\tau[$, the following inequality can be written:

$$\operatorname{meas}_e(X_r\cap\,]z,y[)>(y-z)\left[1-\frac{\operatorname{dens}(A,z)}{2k+2}\right]; \tag{14}$$

(dens means the density of a given set at a given point). Further, it would be

$$\begin{aligned}&\operatorname{meas}(D\cap\,]z,y[)>\tfrac{3}{4}(y-z)\operatorname{dens}(A,z)\,,\\ &\operatorname{meas}(A\cap\,]z,y[)>\tfrac{3}{4}(y-z)\operatorname{dens}(A,z)\,;\end{aligned} \tag{15}$$

which implies

(16) $$\operatorname{meas}(A \cap D \cap]z, y[) > \tfrac{1}{2}(y-z)\operatorname{dens}(A, z).$$

Consider, now, a mapping T, from the set $B \equiv A \cap D \cap]z, y[$ into R, defined as follows:

(17) $$\forall x \in B,\ x \to T(x) \equiv \hat{x} \equiv \frac{kz+x}{k+1};$$

formula (16) ensures us that the set $T(B)$ has measure greater than the quantity $[(y-z)\operatorname{dens}(A, z)]/(2k+2)$, so that, according to (14), we would have

(18) $$\operatorname{meas}_e[T(B) \cap X_r] > 0.$$

Then, set $\hat{x}_r \in [T(B) \cap X_r]$; by virtue of this choice one has:

(19) $$x_r \equiv T^{-1}(\hat{x}_r) = (k+1)\hat{x}_r - kz \in A \cap D \cap]z, y[, \quad F_r(x_r) > F_r(z);$$

but the restriction of $F_r(x)$ to P is continuous, so that we can find a neighborhood of z, say $\beta(z) =]z-\beta, z+\beta[$, contained in $]y-1/r, y[$, and a neighborhood of x_r, say $\alpha(x_r)$, contained in $]x_r-k\beta, x_r+k\beta[\cap]\hat{x}_r, y[$, such that:

(20) $$x_\alpha \in \alpha(x_r) \cap P, \quad z_\beta \in \beta(z) \cap P \Rightarrow F_r(x_\alpha) > F_r(z_\beta).$$

Then, consider another mapping $T^{\cdot\cdot}$, from the set $P \cap \alpha(x_r)$ into R, defined as follows:

(21) $$\forall x_\alpha \in P \cap \alpha(x_r), \quad x_\alpha \to T^{\cdot\cdot}(x_\alpha) \equiv \hat{x}^{\cdot\cdot} = \frac{(k+1)\hat{x}_r - x_\alpha}{k};$$

as the measurable set $P \cap \alpha(x_r)$ has density 1 at x_r, also its image $T^{\cdot\cdot}[P \cap \alpha(x_r)]$ will have the same density at z, whence the set $\hat{X}^{\cdot\cdot} \equiv P \cap T^{\cdot\cdot}[P \cap \alpha(x_r)]$ has density >0 at z. On the other hand, from (20) and (21) we draw

(22) $$\hat{x}^{\cdot\cdot} \in \hat{X}^{\cdot\cdot} \Rightarrow F_r(x_\alpha) \equiv F_r[\hat{x}_r + k(\hat{x}_r - \hat{x}^{\cdot\cdot})] > F_r(\hat{x}^{\cdot\cdot}) \equiv F_r[\hat{x}_r - (\hat{x}_r - \hat{x}^{\cdot\cdot})]$$
$$\Rightarrow F_r[\hat{x}_r - k(\hat{x}_r - \hat{x}^{\cdot\cdot})] - F_r[\hat{x}_r - (\hat{x}_r - \hat{x}^{\cdot\cdot})] > 0;$$

but this is in contradiction with (8), because

(23) $$\operatorname{dens}(\hat{X}^{\cdot\cdot}, z) > 0, \quad 0 < \hat{x}_r - \hat{x}^{\cdot\cdot} < 1/r.$$

So, in effect, the density of A at z is 0.

All that means, obviously:

(24) $$[D_\Delta^+ F_r(t)]_{t=z} \leqslant 0$$

and this holds for all density points of $D \cap X_r$.

But such a circumstance contradicts (12), because the fundamental theorem quoted at the beginning [2] ensures us that if the approximate derivative of any measurable function does not exist a.e. in a measurable set, then every subset where (24) holds must be negligeable.

Concluding, $F_r(x)$ [and consequently $f(x)$] has the approximate derivative $D_\Delta F_r(x)[D_\Delta f(x)]$ a.e. in Δ and our theorem is proved.

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