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On pointed 1-movability and related notions

by

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Abstract. In this paper we discuss several problems which arose in a study of pointed 1-movability. We also prove some new theorems.

1. Introduction. The main aim of this paper is to summarize several problems in continua theory which arose in a study of pointed 1-movability and related notions. Some new results are also obtained. All spaces under discussion are at least metrizable. Terminology used is standard. The definitions of undefined terms from shape theory may be found in the book [3]. By a continuum is meant a nonvoid, compact, connected space. A one-dimensional continuum is called a curve. If N is a manifold, then $\dot{N}$ denotes its boundary and $\mathring{N}$ its interior.

Let X be a continuum lying in an ANR($\mathfrak{M}$)-space M and let x_0 be a point of X. We shall be dealing with the following properties of X:

(MOV*) (*pointed movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X which can be deformed rel. x_0 within U into any neighborhood of X [3].

(MOV) (*movability*). The same definition as above with no restriction on x_0 [3].

(1 MOV*) (*pointed* 1-*movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X such that each loop in (V, x_0) can be deformed within (U, x_0) into any neighborhood of X [comp. 3, 18 and 26].

(1 MOV) (1-*movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X such that for each neighborhood W of X and for each mapping $f\colon Y\to V$, where Y is a curve, there is a mapping $g\colon Y\to W$ homotopic to f in U [3].

(n1 MOV) (*nearly* 1-*movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X such that for each mapping $f\colon D\to V$, where D is a 2-disk, and for each neighborhood W of X there exist a sequence $D_1, \ldots, D_k$ of disjoint disks in $\mathring{D}$ and an extension $\bar{f}\colon D\setminus\bigcup_{i=1}^{k}\mathring{D}_i\to U$ of f such that $f(\bigcup_{i=1}^{k}\dot{D}_i)\subset W$ [26].

To define the next property recall that an inverse sequence $\underline{X}$ of ANR-sets is said to be *associated with* X if $X = \operatorname{invlim} \underline{X}$ [25]. An inverse sequence of groups $G_1 \xleftarrow{h_{12}} G_2 \xleftarrow{h_{23}} \ldots$ is said to be a *Mittag-Leffler sequence*, briefly: ML-sequence, if for each $n \geqslant 1$ there exists $n_0 \geqslant n$ such that $\operatorname{im} h_{nn_0} = \operatorname{im} h_{nm}$ for each $m \geqslant n_0$. By H_1 (resp. H^1) we denote in this paper the first singular homology (resp. Čech cohomology) functor with integer coefficients.

(MLH_1) there exists an ANR-sequence $\underline{X}$ associated with X such that $H_1(\underline{X})$ is a ML-sequence.

(FH^1) $H^1(X)$ is a free Abelian group.

(FDH^1) $H^1(X)$ is finitely divisible.

The finite divisibility of an Abelian group is defined in Section 4. In that section we repeat a result providing a geometric interpretation of (FDH^1).

The independence of the above mentioned notions on the ANR($\mathfrak{M}$)-spaces containing X and on the particular choice of $x_0 \in X$ is proven in the quoted papers and will also be seen from the results of the following paragraphs.

Now we shall briefly comment the following diagram of implication showing the dependences between these notions. We have numbered the implications to make the presentation more concise,

$$\begin{array}{ccccc} & \overset{(1)}{\nearrow} & (\mathrm{MOV}) & \overset{(2)}{\searrow} & \\ (\mathrm{MOV}^*) & & & & (1\mathrm{MOV}) \overset{(5)}{\Rightarrow} (n1\mathrm{MOV}) \overset{(6)}{\Rightarrow} (\mathrm{MLH}_1) \overset{(7)}{\Rightarrow} (\mathrm{FH}^1) \overset{(8)}{\Rightarrow} (\mathrm{FDH}^1). \\ & \underset{(3)}{\searrow} & (1\,\mathrm{MOV}^*) & \underset{(4)}{\nearrow} & \end{array}$$

Some of the implications are trivial. Such are (1), (2), (3), (4), (5) and (8). Implication (6) can be easily derived from the observation on nearly 1-movability made in [26] by using the well-known theorem relating $H_1(Y)$ to $\pi_1(Y)$, the fundamental group of Y, for an arcwise connected continuum Y (com. Section 4). Implication (7) is our Theorem 4.8. Not everything is known about the possibility of reversing the implications. The most significant problem, the solution of which would have serious consequences in shape theory and continua theory, concerns the possibility of reversing implication (1). Let us repeat it explicitly.

1.1. Problem [3, 17, 26]. Must a movable continuum be pointed movable?

In the last section we shall see that this problem is equivalent to other, seemingly different problems. K. Borsuk [4] constructed a locally connected non-movable continuum. Being locally connected it is pointed 1-movable (see [18, 26]). Thus this example serves as a counterexample to reversability of implications (2) and (3). In [7] J. Dydak showed that an example constructed by M. Strok is 1-movable but not pointed 1-movable. Hence (4) is not reversible. It was noticed by D. R. McMillan [26] that there exist curves serving as counterexamples to reversability of (5). The Case-Chamberlin curve [5], as it was remarked in [26], shows that (6) is not reversible. It is an open question whether or not (7) is reversible. Precisely,

1.2. Problem [1]. Suppose X is a continuum such that $H^1(X)$ is free. Does it follow that $X \in (\mathrm{MLH}_1)$?

Examples 4.4 and 4.5 show that (8) is not reversible.

It is easy to prove that all the discussed properties are pointed shape invariants. It is interesting that those notions are also unpointed shape invariant. This fact can be simply verified for all the "one-dimensional properties" except pointed 1-movability. The invariance of pointed 1-movability under unpointed shape (which implies the invariance of pointed movability (comp. the last section)) has been established by J. Dydak [9]. There are also several results about the invariance of those notions under shape domination and continuous mappings. The following are preserved by shape domination: (MOV), (1MOV), (n1MOV) [26], (MLH_1), (FH^1) and (FDH^1). The remaining case of pointed 1-movability is unsolved.

1.3. Problem (J. Dydak). Must a continuum shape dominated by a pointed 1-movable continuum be pointed 1-movable?

Since the discussed Borsuk's example is locally connected and not movable, the movability and pointed movability are not invariant under continuous transformations because the unit interval I (which can be mapped onto the example) is both movable and pointed movable. In [7] it is proved that continuous images of 1-movable continua need not be 1-movable (a result of M. Strok). The properties ($1\mathrm{MOV}^*$), (n1MOV) and (FDH^1) are preserved under continuous transformations (see resp. [18, 26], [26] and [16, 29]-comp. 4.1).

1.4. Problem [1]. Are the properties (MLH_1) and (FH^1) continuous mapping invariants?

In the next two paragraphs we discuss the problem of which continua contain subcontinua not satisfying one of the properties listed at the begining of this paragraph. This is an important problem related to the outstanding problem of topology concerning the characterization of continua embeddable in the plane or surfaces. The reason we say that is given by the result proved in [18, 26] that all continua embeddable in a surface are pointed movable. Thus each subcontinuum of a continuum sitting in a surface is pointed movable and therefore satisfies each of the properties. So, each continuum embeddable in a surface must be at least hereditarily pointed movable. We shall see that this restriction eliminates many continua far from being planar from an intuitive point of view. For instance this condition implies that the continua must be at most 2-dimensional. The classical results of Kuratowski, Mazurkiewicz, Claytor and others characterize planar continua among a class of continua with nice local properties. The force of the above notions lies in the fact that they are applicable for all continua regardless of their local structure.

It is true that the condition we have just mentioned does not provide a similar

[1] Added in proof. J. Dydak has proved that the answer is affirmative.

characterization for all continua, however it substantially restricts the continua which should be taken into account with respect to the problem of embeddability.

2. Continua with non-movable circle-like subcontinua. A continuum is said to be *snake-like* (*circle-like*) if it can be represented as the limit of an inverse sequence of intervals (circles) with surjective bonding maps. Let P be a sequence $n_1, n_2, \ldots$ of natural numbers, $P = (n_1, n_2, \ldots)$. By a P-adic snake-like continuum we mean the limit of the inverse sequence

$$(*)\qquad I \overset{q_{n_1}}{\leftarrow} I \overset{q_{n_2}}{\leftarrow} \ldots,$$

where q_k, $k \geqslant 1$, is given by the formula:

$$q_k(t) = (-1)^{\operatorname{sgn}(j-1)}kt + (-1)^{\operatorname{sgn}(j)}j + \operatorname{sgn}(j),$$

for $t \in \left[\frac{j-1}{k}, \frac{j}{k}\right]$ and $j = 1, \ldots, k$. Here $\operatorname{sgn}(m)$ is 0 for m even and 1 for m odd. Thus q_k maps i/k, $0 \leqslant i \leqslant k$, into $\operatorname{sgn}(i)$ and is linear on each interval $\left[\frac{j-1}{k}, \frac{j}{k}\right]$ (comp. [30]). The P-adic snake-like continuum will be denoted by $I(P)$. If P is a constant sequence $P = (n, n, \ldots)$, then the P-adic continuum is also called *n-adic* and is denoted by $I(n)$. Observe that $I(1)$ is homeomorphic to I, $I(2)$ is homeomorphic to the simplest Knaster indecomposable continuum with one endpoint [21, Exp. 1, p. 204], and $I(3)$ is homeomorphic to the indecomposable continuum with two endopoints [21, Ex. 3, p. 205].

By S we denote in the paper the unit circle in the complex plane.

By a P-adic solenoid we mean the limit of the inverse sequence

$$(**)\qquad S \overset{p_{n_1}}{\leftarrow} S \overset{p_{n_2}}{\leftarrow} \ldots,$$

where $p_k(z) = z^k$. We denote it by $S(P)$. If P is constant, $P = (n, n, \ldots)$, then $S(P)$ will be also denoted by $S(n)$ and called *n-adic solenoid.* Note that $S(1)$ is homeomorphic to S. A P-adic solenoid not homeomorphic to S will be called a *solenoid.* This is the case if P contains infinite number of elements $\neq 1$. Thus the circle S is not regarded as a solenoid.

In the sequel we identify two homeomorphic spaces.

Let us make a remark about the functions q_n. Consider the square $D = I \times I$. If $x, y \in D$, then by $[xy]$ we denote the segment in D with endpoints at these points. A broken line $S_1 = [ab] \cup [bc] \cup [cd] \cup [da]$ in D forming a simple closed curve will be called a *regular circle* if $a, b, c, d \in D$, the segments $[ab]$ and $[cd]$ are parallel to the diagonal $[(0,0)(1,1)]$, the segments $[bc]$ and $[ad]$ are paralel to $[(1,0)(0,1)]$ and $a = (t, 0)$ for some $0 < t < 1$. It is clear that a regular circle in D is uniquely determined by the point a. In fact, if $a = (t, 0)$, then $b = (1, 1-t)$, $c = (1-t, 1)$ and $d = (0, t)$. Therefore S_1 will be called a regular circle generated by a, and a its generator.

An important observation about regular circles and the maps q_n is included in the following lemma whose proof is easy and is left to the reader (see Fig. 1).

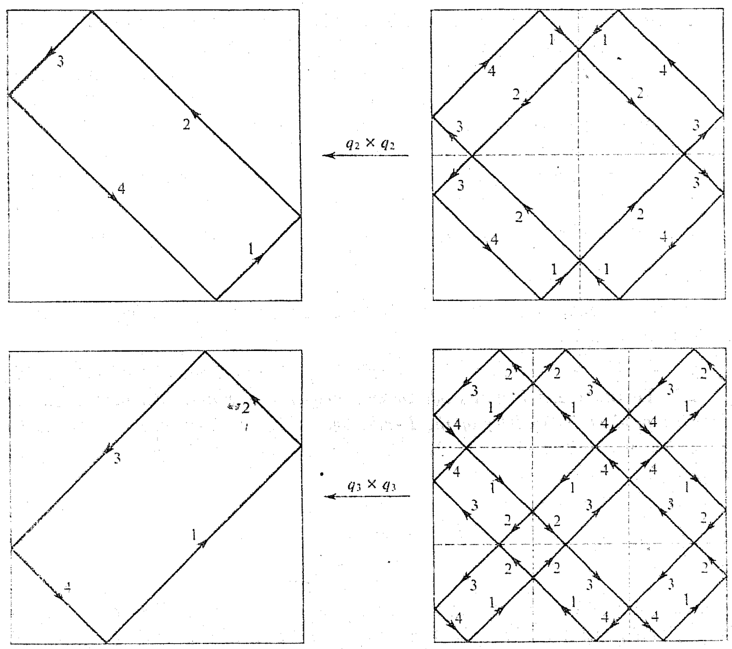


Fig. 1

2.1. LEMMA. *Let $n \geqslant 1$. If S_1 is a regular circle in D, then $(q_n \times q_n)^{-1}(S_1)$ is the union of n regular circles. If S_2 is one of them, then the map $\bar{p}_n\colon S_2 \to S_1$ determined by $q_n \times q_n$ is a covering projection of degree n. If $(t, 0)$, $0 < t < 1$, is the generator of S_1, then the n regular circles are generated by the points $(s, 0)$, where $s \in q_n^{-1}(t)$. If n is odd, then all vertices of D and the centre of D are fixed points of $q_n \times q_n$; if n is even then all vertices of D are mapped under $q_n \times q_n$ into $(0, 0)$ and the centre of D is mapped into $(0, 0)$ or $(1, 1)$.*

Let us note that for $m, n \geqslant 1$ we have

$$(\overset{**}{*})\qquad p_m \circ p_n = p_{mn} \quad \text{and} \quad q_m \circ q_n = q_{mn}\,.$$

From this remark it follows that every P-adic solenoid (snake-like continuum) X is homeomorphic to a P'-adic solenoid (snake-like continuum), where $P' = (k_1, k_2, \ldots)$ satisfies one of the following:

(i) $k_i = 1$ for each $i \geqslant 1$,
(ii) $k_i > 1$ is odd for each $i \geqslant 1$,
(iii) k_i is even for each $i \geqslant 1$.

Sequences satisfying (i) or (ii) will be called *odd*; the ones satisfying (iii) — *even*. The unique one with property (i) will be also called *trivial*.

It is an open problem which cartesian products of two continua contain non-movable subcontinua. See the detailed discussion of this question in Section 3. D. Bellamy has pointed out to me in a conversation that solenoids (which are not movable [3]) can be embedded into products of snake-like indecomposable continua. However his proof is purely algebraic and gives no geometric picture of the nature of the embeddings. Here we present a stronger result with a geometric proof.

Notation. If P is an odd or even sequence, then by $E(P)$ we denote a subset of $\prod_{i=1}^{\infty} D_i$, $D_i = I\times I$, given by the conditions:

(α) if P is odd, then $E(P)$ is the five-point set $\{(x, x, \ldots)$: x is either a vertex of D or its centre$\}$,

(β) if P is even, then $E(P)$ is the one-point set $\{(x, x, \ldots)\}$, where $x = (0, 0)$.

2.2. THEOREM. *Let P be an odd or even sequence of natural numbers. Then the cartesian product $I(P)\times I(P)$ of the P-adic snake-like continua minus the set $E(P)$ is a union of a collection of P-adic solenoids each two of which intersect and have at most four points in common.*

Proof. This result is a simple consequence of Lemma 2.1. In fact, let $P = (n_1, n_2, \ldots)$. Then $I(P)\times I(P)$ may be identified with the limit of the sequence:

$$\text{(S)} \qquad D_1 \xleftarrow{q_{n_1}\times q_{n_1}} D_2 \xleftarrow{q_{n_2}\times q_{n_2}} \ldots,$$

where $D_n = I\times I$ for each $n \geqslant 1$. Now we construct a collection $\mathcal{C}$ of P-adic solenoids in $I(P)\times I(P)$ such that $I(P)\times I(P)\setminus E(P) = \bigcup \mathcal{C}$. An element C of $\mathcal{C}$ is constructed as follows. Choose an index $i \geqslant 1$ and let S_i be a regular circle in D_i. By repeated application of 2.1 we may construct a sequence $S_i, S_{i+1}, \ldots$ such that S_j, $j \geqslant i$, is a regular circle in D_j and the map $S_{j+1} \to S_j$ determined by $q_{n_j}\times q_{n_j}$ is a covering projection of degree n_j. By a standard procedure one can extend the resulting sequence of S_j's to a subsequence of (S). The element C is defined as the limit of the subsequence. Clearly, C is a P-adic solenoid in $I(P)\times I(P)$.

To complete the proof it remains to show that for each $z \in I(P)\times I(P)\setminus E(P)$ there is $C \in \mathcal{C}$ containing z. So, let $z = (z_1, z_2, \ldots)$ be such a point.

Consider two cases:

(α) P is odd. Then by 2.1 and the definition of $E(P)$ there is an index $i \geqslant 1$ such that z_i is neither a vertex of D_i nor the centre of D_i. Also $z_{j+1} \in (q_{n_j}\times q_{n_j})^{-1}(z_j)$ for each $j \geqslant i$. Hence begining the above construction of C from a regular circle S_i in D_i which contains z_i one can extend it in such a way that $z_j \in S_j$ for $j \geqslant i$, since by 2.1 $(q_{n_j}\times q_{n_j})^{-1}(S_j)$ is a union of regular circles. The continuum C will then contain z, what was to be proved.

(β) P is even. Then there is an index $i_0 \geqslant 1$ such that $z_{i_0} \neq (0, 0)$. Let $i = i_0+2$. It follows from 2.1 that z_i is neither a vertex of D_i nor the centre of D_i. Then, for the same reason as in (α), there is an element of $\mathcal{C}$ containing z.

This completes the proof.

Remark. As a particular case of 2.2 we have: if e is the endpoint of the simplest indecomposable Knaster continuum X, then $X\times X\setminus\{(e, e)\}$ is a union of dyadic solenoids.

This results lead to the following:

2.3. PROBLEM. Suppose $I(P) \neq I$. Does there exist two disjoint solenoids in $I(P)\times I(P)$?

Observe that if $I(P) \neq I$, then $I(P)$ is indecomposable and each proper non-degenerate subcontinuum of $I(P)$ is an arc. Thus the above problem is related to the following

2.4. PROBLEM. Suppose $I(P) \neq I$. Does there exist a solenoid in $I(P)\times I$? Does there exist any snake-like continuum X such that $X\times I$ contains a solenoid?

As the next theorem shows the product $I(3)\times I$ contains a circle-like subcontinuum having the shape of a solenoid. Consequences of this fact are discussed in Section 3.

2.5. THEOREM. *The cartesian product $I(3)\times I$ contains a circle-like subcontinuum which can be mapped onto the dyadic solenoid $S(2)$ under a mapping which is a shape equivalence.*

Proof. We only give an idea of the argument leaving the details to the reader. The continuum $I(3)\times I$ is homeomorphic to the limit of the inverse sequence

$$D_1 \xleftarrow{q_3\times \mathrm{id}\, I} D_2 \xleftarrow{q_3\times \mathrm{id}\, I} \ldots,$$

where $D_n = I\times I$ for each $n \geqslant 1$. We claim that for $n \geqslant 1$ there exist a subset B_n of D_n homeomorphic to $S\times I$ such that $q_3\times \mathrm{id} I$ maps B_{n+1} into B_n and the map $g_n\colon B_{n+1} \to B_n$ determined by $q_3\times \mathrm{id} I$ is of degree 2 with respect to fundamental

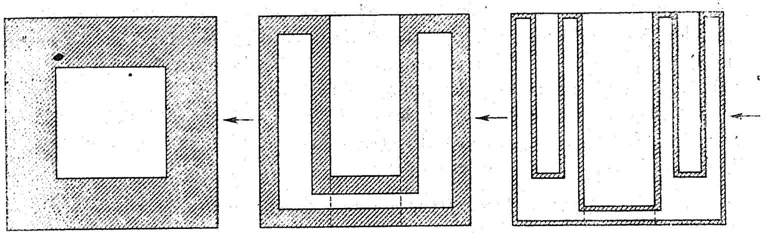


Fig. 2

groups. We are not going to describe precisely the sets B_n. The author believs that the geometric picture of the sets B_1, B_2, B_3 given above completely explains how to construct all the other B's (see Fig. 2). Now it is easy to construct a sequence of mappings $f_n\colon B_n \to S$, $n\geqslant 1$ (using the homotopy lifting theorem [31, p. 76]) such that: f_n is a homotopy equivalence and $p_2\circ f_{n+1} = f_n\circ g_n$ for each $n\geqslant 1$. Then f's determine a map from $X = \operatorname{invlim}\{B_n, g_n\}$ onto $S(2)$ which is a shape equivalence. Choosing the bands B_n thin enough we may assure X to be a circle-like continuum. Since $X\subset I(3)\times I$, the proof is completed.

2.6. Problem. Let X be a snake-like continuum. Is it true that each two nonmovable subcontinua of $X\times I$ intersect?

2.7. Problem. Let X be a snake-like continuum such that $X\times I$ contains nonmovable subcontinua. Determine their shapes. Is it true that each such subcontinuum shape dominates the dyadic solenoid? Can we map such a subcontinuum onto the dyadic solenoid?

It is easy to see that the shape of every subcontinuum of $X\times I$ and $X\times Y$ is represented by a curve if X and Y are snake-like. Thus the first four conditions from the Introduction are equivalent for this continua [33]. The reader should compare the last question in the above problem with the fact that the Case–Chamberlin curve [5] cannot be mapped onto any solenoid [16].

The next theorem is well-known. The priority of its discovery belongs to R. H. Bing. Recall that by a simple triod we mean a continuum which is the union of three arcs having exactly one point in common being an endpoint of each of them. The definition of a simple n-od, $n\geqslant 3$, is analogues.

2.8. Theorem (R. H. Bing). *Let X_0 be a simple triod. Then the cartesian product $X_0\times I$ ("the three-page book") contains all solenoids.*

2.9. Problem (1). Is it true that each nonmovable subcontinuum of $X_0\times I$ shape dominates a solenoid? Classify the shapes of nonmovable subcontinua of $Y\times I$, where Y is an n-odd.

Now we shall show that there exists relatively simple type of continua, completely different from those considered in Theorems 2.2, 2.5 and 2.8 which still contain nonmovable circle-like continua. In fact, we shall construct a continuum which we call "a glued chain" — the name proposed by D. Bellamy — which contains all solenoids. This continuum is an "almost 2-manifold with boundary" in the sense that there is a point in it whose complement is a non-compact 2-manifold with boundary. Now we pass to a description of the glued chain.

Let Q be a pyramid in the 3-space R^3 with the vertex $v = (0,0,0)$ and the base $\{1\}\times I\times I$. For each $n\geqslant 1$ let $Q_n = \{(x_1, x_2, x_3)\in Q\colon 1/n+1\leqslant x_1\leqslant 1/n\}$. By L_n we denote the nth link given by

$$L_n = \begin{cases} Q_n\cap[\Pi(x_1 = 1/n)\cup\Pi(x_1 = 1/n+1)\cup\Pi(x_3 = 0)\cup\Pi(x_1 = x_3)] & \text{for } n \text{ odd},\\ Q_n\cap[\Pi(x_1 = 1/n)\cup\Pi(x_1 = 1/n+1)\cup\Pi(x_2 = 0)\cup\Pi(x_1 = x_2)] & \text{for } n \text{ even},\end{cases}$$

(1) Added in proof. Recently J. Olędzki and S. Spież proved the following unexpected result: any curve embeds up to shape into $X_0\times I$.

where $\Pi(\varphi)$ is the plane in R^3 given by the equation φ. The glued chain is defined to be the continuum $L = \{v\}\cup\bigcup_{n\geqslant 1} L_n$ (see Fig. 3). The set $L\setminus\{v\}$ is a 2-manifold with boundary.

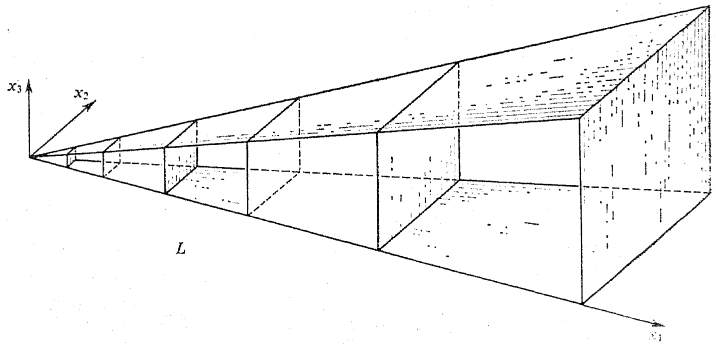


Fig. 3

2.10. Theorem. *The glued chain L contains every solenoid.*

Proof. We present the main idea of the proof, leaving the details to the reader.

Let $P = (n_1, n_2, \ldots)$. Let $T_1, T_2, \ldots$ be a decreasing sequence of tori in E^3 such that there exists a sequence of homeomorphisms $\{h_i\colon T_i\to S\times B^2\}$, where B^2 is the unit ball in the complex plane, satisfying the conditions (where $r\colon S\times B^2\to S$ is the projection on the first factor):

(i) $\operatorname{diam}(r\circ h_i)^{-1}(s) < 1/i$ for $s\in S$,

(ii) $p_{n_i}\circ r\circ h_{i+1} = r\circ h_i|T_{i+1}$ for $i\geqslant 1$.

Then the P-adic solenoid $S(P)$ is homeomorphic to $Y = \bigcap_{i\geqslant 1} T_i$.

Now we show how to embed $S(P)$ into L. It is easy to see that there exist three sequences: a decreasing sequence $X_1, X_2, \ldots$ of subcontinua of L; a decreasing sequence $T'_1, T'_2, \ldots$ of tori in E^3 such that $X_i\subset T'_i$, and a sequence of homeomorphisms $\{g_i\colon T_i\to T'_i\}$, with the following properties:

(1) $\bigcap_{i\geqslant 1} X_i = \bigcap_{i\geqslant 1} T'_i$,

(2) the maps $\{g_i|Y\colon Y\to E^3\}$ form a Cauchy sequence in the function space $(E^3)^Y$ and the set $g_i(Y)\subset T'_i$ is η_i-dense in T'_i, where $\eta_i \xrightarrow[i\to\infty]{} 0$,

(3) for each pair of points $x, y\in Y$ there exists an $\varepsilon>0$ such that $\varrho(g_i(x), g_i(y))\geqslant\varepsilon$ for almost all i.

Since the function space $(E^3)^Y$ is complete, the maps $\{g_i|Y\}$ converge to a map $g\colon Y\to E^3$. Since $\eta_i \xrightarrow[i\to\infty]{} 0$, g maps Y onto $\bigcap_{i\geqslant 1} T'_i$. This jointly with (3) implies that g

maps Y homeomorphically onto $\bigcap_{i\geq 1} T_i'$. The latter set is by (1) a subset of L. Hence the proof is completed because $Y = S(P)$. (In Figure 4 we pictured the sets X_1 and X_2 for the case $P = (2, 2, \ldots)$).

In [4] K. Borsuk constructed a non-movable locally connected continuum B in E^3 which is locally R^2 at all but a one point. He raised the question whether every proper subset of B is movable. It is easy to see that the glued chain embeds into B.

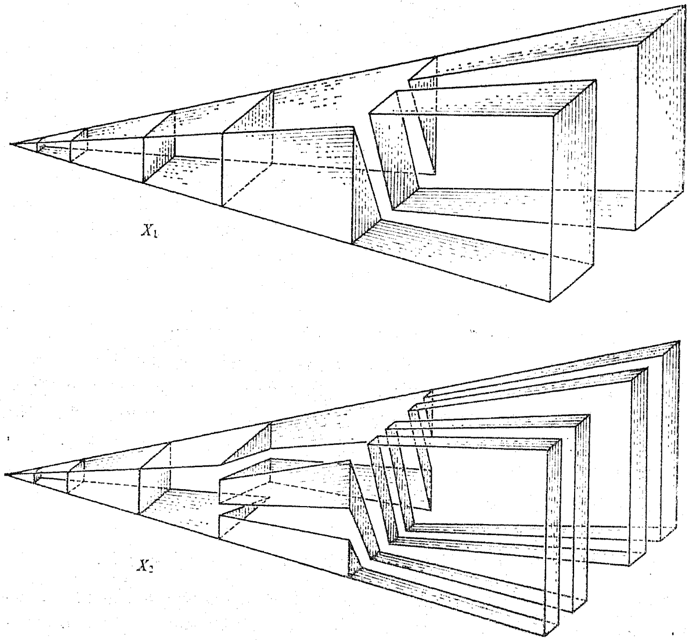


Fig. 4

Thus the above theorem provides a negative answer to Borsuk's question. In connection with these facts we have the following

2.11. Problem. Must every non-movable continuum contain a finite dimensional non-movable subcontinuum?

2.12. Problem. Must every n-dimensional non-movable continuum contain an $(n-1)$-dimensional non-movable subcontinuum?

2.13. Problem. Must every non-movable continuum contain a non-movable curve?

The list of this type problems may be extended replacing the word "movable" by any one of the related notions listed in the Introduction.

The last problem seems to be very important.

3. Non-movable subsets of continua. In the preceding section we gave some examples of continua with non-movable circle-like subcontinua. Using some of these examples we prove in this section that many other continua contain non-movable subcontinua. The notion of a weakly confluent mapping is very usefull in these considerations. Recall that a mapping $f\colon X \to Y$ is said to be *weakly confluent* if for each continuum $C \subset Y$ there is a continuum $D \subset X$ such that $f(D) = C$. An important result on weakly confluent mappings has been proved by S. Mazurkiewicz [27]. To state it we need the following definition. A mapping $f\colon X \to I^n$, where I^n is the n-cube, is said to be AH-*essential* if each extension g of the map $f|f^{-1}(\dot{I}^n)\colon f^{-1}(\dot{I}^n) \to I^n$ over X transforms X onto I^n.

3.1. Theorem [1]. *If X is compact and* $\dim X \geq n$, *then there exists an* AH-*essential map from X onto I^n.*

The theorem of Mazurkiewicz reads as follows.

3.2. Theorem [27]. *An* AH-*essential mapping from a compact space X onto I^n is weakly confluent.*

The reader is referred to [11] for a discussion and some interesting generalization of this result.

3.3. Corollary. *If X is compact and* $\dim X \geq n$, *then for every subcontinuum Y of the euclidean space E^n there exists a subcontinuum of X which can be mapped onto Y.*

In the sequel the following lemma plays an important role.

3.4. Lemma. *If X and Y are continua and $f\colon X \to I$, $g\colon Y \to I$ are surjective mappings, then the product of mappings $f\times g\colon X\times Y \to I^2$ is* AH-*essential.*

Proof. Suppose the lemma fails. Then there is a mapping $h\colon X\times Y \to \dot{I}^2$ such that

(1) $$h|(f\times g)^{-1}(\dot{I}^2) = f\times g|(f\times g)^{-1}(\dot{I}^2)\,.$$

Consider X and Y as subsets of the Hilbert cube Q and let $\bar{f}\colon Q \to I$, $\bar{g}\colon Q \to I$ be extensions of f and g respectively. Since $\dot{I}^2 \in$ ANR, there exist a neighborhood G of $X\times Y$ in $Q\times Q$ and an extension $\bar{h}\colon G \to \dot{I}^2$ of the map h. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two nested sequences of locally connected continua in Q such that $\bigcap_n X_n = X$ and $\bigcap_n Y_n = Y$. Choose four points $a_0 \in f^{-1}(0)$, $a_1 \in f^{-1}(1)$, $b_0 \in g^{-1}(0)$ and $b_1 \in g^{-1}(1)$. Define $M = \{a_0, a_1\}\times Y \cup X\times\{b_0, b_1\}$, and $M_n = \{a_0, a_1\}\times Y_n \cup X_n\times\{b_0, b_1\}$ for $n \geq 1$. Since $M \subset (f\times g)^{-1}(\dot{I}^2)$ and M_n's form a decreasing sequence of compacta converging to M, then by the fact that $\bar{f}\times\bar{g}(M_n) \subset \dot{I}^2 \in$ ANR and by condition (1) we infer that there exists an index $m \geq 1$ such that

(2) $$X_m\times Y_m \subset G\,,$$

(3) $$\bar{f}\times\bar{g}|M_m \simeq \bar{h}|M_m \quad \text{in } \dot{I}^2\,.$$

Let A be an arc in X_m with endpoints a_0 and a_1; let B be an arc in Y_m with endpoints b_0 and b_1, and let D be the disk $A\times B$. Clearly, $\dot{D}\subset M_m$. It is easy to see that $\bar{f}\times\bar{g}|\dot{D}\colon \dot{D}\to\dot{I}^2$ is an essential mapping (comp. [21, p. 234]), i.e. not homotopic to a constant mapping. Form (3) it follows that $\bar{h}|\dot{D}\colon \dot{D}\to\dot{I}^2$ is essential. But this contradicts the fact that $\bar{h}|\dot{D}$ has an extension over D, because by (2) we have $D\subset G$. This completes the argument.

The above lemma combined with 3.2 gives the following

3.5. COROLLARY. *If $f\colon X\to I$ and $g\colon Y\to I$ are surjective mappings of continua then $f\times g\colon X\times Y\to I^2$ is weakly confluent.*

This in turn implies the following

3.6. THEOREM. *Let X and Y be continua and let M and N be snake-like continua. If $f\colon X\to M$ and $g\colon Y\to N$ are surjective mappings, then $f\times g\colon X\times Y\to M\times N$ is weakly confluent.*

Proof. Let $\underline{M}=\{I_n,\alpha_{nm}\}$ and $\underline{N}=\{I_n,\beta_{nm}\}$ be two inverse sequences with surjective bonding maps, where $I_n=I$ for each $n\geqslant 1$, such that $M=\operatorname{invlim}\underline{M}$ and $N=\operatorname{invlim}\underline{N}$. Then $M\times N=\operatorname{invlim}\{I_n\times I_n,\alpha_{nm}\times\beta_{nm}\}$.

Now let C be a subcontinuum of $M\times N$ and for each $n\geqslant 1$ let $C_n=\alpha_n\times\beta_n(C)$, where α_n, β_n are the projections. Since the maps $\alpha_n\circ f$ and $\beta_n\circ g$ are surjective, by 3.5 there exists a continuum $D_n\subset X\times Y$ such that

$$(\alpha_n\circ f)\times(\beta_n\circ g)(D_n)=\alpha_n\times\beta_n\circ f\times g(D_n)=C_n\,.$$

Without loss a generality we may assume that the sequence $\{D_n\}$ converges to a continuum D. It is easy to verify that $f\times g(D)=C$, which completes the proof.

Since each nondegenerate indecomposable continuum can be mapped onto $I(P)$, where P is an arbitrary sequence (see [30]), from the above theorem we obtain

3.7. COROLLARY. *If X and Y are nondegenerate indecomposable continua and P, P' are two arbitrary sequences of natural numbers, then there exists a weakly confluent mapping from $X\times Y$ onto $I(P)\times I(P')$.*

This jointly with the results of Section 2 gives the following

3.8. THEOREM. *Let X and Y be nondegenerate continua. Then we have:*

(a) *if X and Y are indecomposable, then for every solenoid C there exists a subcontinuum of $X\times Y$ which can be mapped onto C.*

(b) *if X is indecomposable, then there exists a subcontinuum of $X\times Y$ which can be mapped onto the dyadic solenoid.*

A continuum X is said to be a *triod of the form* $(X_0;X_1,X_2,X_3)$ if X_i are subcontinua of X and the following holds:

(i) $X=X_1\cup X_2\cup X_3$,

(ii) $X_i\cap X_j=X_0$ for $i,j=1,2,3$, $i\neq j$,

(iii) $X_0\neq X_i$ for $i=1,2,3$.

3.9. LEMMA. *Let X be a triod of the form $(X_0;X_1,X_2,X_3)$ and let $T=a_0a_1\cup a_0a_2\cup a_0a_3$ be a simple triod. Let $f\colon X\to T$ be a mapping such that*

$f^{-1}(a_0)=X_0$ and $f(X_i)=a_0a_i$, $i=1,2,3$. Let Y be a continuum and let $g\colon Y\to I$ be a monotone surjection. Then $f\times g\colon X\times Y\to T\times I$ is weakly confluent.

Proof. Let $D_1=a_0a_1\times I$, $D_2=(a_0a_2\cup a_0a_3)\times I$, $Z_1=(f\times g)^{-1}(D_1)$ and $Z_2=(f\times g)^{-1}(D_2)$. Then D_1 and D_2 are two disks such that

(1) $$p\in D_1\cap D_2\Rightarrow (f\times g)^{-1}(p)\ \text{is a continuum.}$$

Consider a continuum C in $T\times I$ and let V be a neighbourhood of C in $T\times I$. Let W_i, $i=1,2$, be a polyhedron in $D_i\cap V$ containing $C\cap D_i$ such that each component of W_i meets C. Let $K_{i1},\ldots,K_{in_i}$, $i=1,2$, be the components of W_i such that $K_{ir}\cap K_{is}=\emptyset$ for $r\neq s$. By Corollary 3.5 the map from Z_i into D_i induced by $f\times g$ is weakly confluent. Hence there exist continua $K'_{ij}\subset Z_i$, $i=1,2$, $j=1,\ldots,n_i$, such that $f\times g(K'_{ij})=K_{ij}$. Since C is connected, the union $K=\bigcup_{\substack{i=1,2\\ j=1,\ldots,n_i}}K_{ij}$ is a continuum. Hence we may arrange all the continua $\{K_{ij}\}$ into a sequence $K_1,K_2,\ldots,K_n$ such that $K_l\cap K_{l+1}\neq 0$ for $l=1,\ldots,n-1$. It follows that $K_l\cap K_{l+1}\subset D_1\cap D_2$. Let $p_l\in K_l\cap K_{l+1}$. Then by (1) it follows that $K'_l\cup K'_{l+1}\cup(f\times g)^{-1}(p_l)$ is a continuum. Hence $K_V=\bigcup_{l=1}^{n-1}K'_l\cup K'_{l+1}\cup(f\times g)^{-1}(p_l)$ is a continuum in $X\times Y$ such that $f\times g(K_V)=K$. Thus for each neighbourhood V of C in $T\times I$ there exists a continuum K_V in $X\times Y$ such that $C\subset(f\times g)(K_V)\subset V$. This easily yields the existence of a continuum in $X\times I$ which is mapped by $f\times g$ onto C. This completes the proof.

From the preceding lemma we have the following

3.10. COROLLARY. *If X and Y are continua such that X contains a triod and Y contains a subcontinuum which admits a monotone surjection onto I, then there exists a weakly confluent mapping from $X\times Y$ onto $T\times I$, where T is a simple triod. Hence by 2.8 for each solenoid C there exists a subcontinuum of $X\times Y$ which can be mapped onto C.*

Every irreducible hereditarily decomposable continuum admits a monotone surjection onto I [21, p. 216]. Then by 3.8 and 3.10 we have the following

3.11. THEOREM. *If X and Y are nondegenerate continua such that $X\times Y$ contains no subcontinuum which can be mapped onto the dyadic solenoid, then X and Y are hereditarily decomposable and atriodic.*

3.12. PROBLEM. Suppose X and Y are hereditarily decomposable and atriodic continua. Must $X\times Y$ be hereditarily movable?

3.13. PROBLEM. Are the conditions "to be hereditarily movable" and "to be hereditarily pointed 1-movable" equivalent for continua?

3.14. PROBLEM [1]. Does there exist a (1-dimensional) continuum which is hereditarily non-movable?

We close this section with some remarks and problems on hereditarily indecomposable continua. From results in [19] it follows that the class of shapes which are

(1) Added in proof. L. Oversteegen and the author have constructed such a curve.

represented by hereditarily indecomposable continua is limited. In that paper it is also proved that we have plenty of 1-dimensional hereditarily indecomposable continua. Thus the following sort of problems is of an interest:

3.15. PROBLEM ([1]). Does there exist a hereditarily indecomposable continuum of dimension >1 with one of the following properties: a) movable, b) acyclic, c) of trivial shape, d) pointed 1-movable, etc.?

3.16. PROBLEM. Suppose X is a 1-dimensional hereditarily indecomposable continuum with infinite dimensional hyperspace of subcontinua. Must X be non-movable?

This problem is related to a recent result of W. Lewis [23] who proved that such hereditarily indecomposable 1-dimensional continua exist.

It is easy to prove that hereditarily indecomposable continua with infinite dimensional hyperspaces must contain many non-movable subcontinua (which can be mapped onto solenoids). To prove this it suffices to combine 3.3 with the results in [10]. Hence hereditarily movable hereditarily indecomposable continua containing more than one point must have 2-dimensional hyperspaces of subcontinua, which follows from some results in [10]. Since all subcontinua of 2-manifolds are hereditarily movable [18, 26], the result in [15] may be generalizated as follows.

3.17. THEOREM. *If X is a nondegenerate hereditarily indecomposable continuum embeddable into a 2-manifold, then its hyperspace of subcontinua $C(X)$ is 2-dimensional.*

4. On the properties (MLH_1), (FH^1) **and** (FDH^1). Let g be an element of an Abelian group G. Then we say that g is finitely divisible if there exists a natural number n_0 such that for no $n \geqslant n_0$ the equation $g = n\cdot x$ has a solution in G. If each element $g \neq 0$ of G is finitely divisible, then G is said to be *finitely divisible*. By H^1 we denote in this paper the first Čech cohomology functor with integer coefficients. If $f\colon X \to Y$ is a mapping between compacta, then the induced homomorphism from $H^1(Y)$ into $H^1(X)$ will be denoted by f^*. Finite divisibility of the group $H^1(X)$, that is: the condition (FDH^1), has a geometric interpretation given by the following

4.1. THEOREM ([16, 29]). *If X is a continuum, then $H^1(X)$ is finitely divisible if and only if there is no continuous surjection from X onto any solenoid.*

It is obvious that each free Abelian group is finitely divisible. The converse is not true even for the group $H^1(X)$, where X is a curve. An appropriate example is given below. Before we present it we recall some results and terminology.

Consider the unit circle S as an Abelian group with multiplication of complex numbers as a group operation. If X is a compactum and $f, g\colon X \to S$ are mappings, then we define the product $f\cdot g\colon X \to S$ by $f\cdot g(x) = f(x)\cdot g(x)$. It is obvious that if $f \simeq g$ and $f' \simeq g'$, then $f\cdot f' \simeq g\cdot g'$. Thus the product of mappings induces a group structure in the set of homotopy classes of mappings from X into S. This set with the group operation is denoted by $\pi^1(X)$ and called the *Bruschlinsky group of X*.

([1]) Added in proof. M. Smith has proved that there is no such a continuum with trivial shape. The author has proved that no such a continuum can be acyclic.

An element of $\pi^1(X)$ with a representative f will be denoted by $[f]$. The symbol π^1 will be also regarded as a functor from the category of compacta to category of Abelian groups. If $h\colon X \to Y$ then the induced homomorphism $\pi^1(h)\colon \pi^1(Y) \to \pi^1(X)$ is also denoted by h^* and is given by $h^*([f]) = [f\circ h]$. Let γ denote a generator of the infinite cyclic group $H^1(S)$. Let $a = [f] \in \pi^1(X)$. To the map $f\colon X \to S$ there corresponds the element $f^*(\gamma) \in H^1(X)$, which does not depend on the choice of a particular map f representing a. In this way we obtain a function $\chi\colon \pi^1(X) \to H^1(X)$ defined by $\chi([f]) = f^*(\gamma)$.

4.2. BRUSCHLINSKY THEOREM [6, p. 226]. *The function χ is a natural equivalence of functors π^1 and H^1.*

If G is an Abelian topological group, then by char G we denote the character group of G.

4.3. STEENROD THEOREM [32]. *Let Y be a compact connected Abelian topological group. Then char Y is isomorphic to $H^1(Y)$.*

4.4. EXAMPLE. There exists a compact connected (metrizable) Abelian topological group Y of dimension 2 such that $H^1(Y)$ is finitely divisible but not free.

Proof. In [28, p. 43] L. S. Pontrjagin gave an example of an Abelian torsion free group G of rank 2 which is finitely divisible but not free. Consider G as a topological group with discrete topology. Let $Y = \operatorname{char} G$. Then Y is compact. Since G is countable, Y is metrizable because it has a countable basis. Since G is torsion free, Y is connected [28, p. 239]. Since rank $G = 2$, $\dim Y = 2$ [28, p. 240]. By the Steenrod theorem we have $H^1(Y) \approx \operatorname{char} Y = \operatorname{char}(\operatorname{char} G)$. From the Pontrjagin duality theorem [28, p. 233] we conclude that $H^1(Y) \approx G$, which completes the proof.

Now we are prepared to present the promised example.

4.5. EXAMPLE. There exists a curve X such that $H^1(X)$ is finitely divisible but not free.

Proof. Let Y be as in the above example. It follows from a result of Wilson [34] that there exist a curve X and a surjective mapping $f\colon X \to Y$ such that $f^{-1}(y)$ is a locally connected continuum for each $y \in Y$. We shall show that X possesses the desired properties. Since f is a monotone surjection, by a result of A. Lelek [22] the induced homomorphism $f^*\colon H^1(Y) \to H^1(X)$ is a monomorphism. Thus $H^1(X)$ contains a subgroup which is not free. Therefore $H^1(X)$ is not free. It remains to prove that $H^1(X)$ is finitely divisible. Suppose not. Then by 4.1 there exists a surjective mapping $g\colon X \to \Sigma$, where Σ is a solenoid. Let M be a subset of $Y \times \Sigma$ defined by

$$M = \bigcup \{\{y\} \times g(f^{-1}(y))\colon y \in Y\}.$$

Observe that M is a continuum. Let $p\colon M \to Y$ and $q\colon M \to \Sigma$ be the mappings determined by projections. Since $g(f^{-1}(y))$ is a locally connected subcontinuum of Σ, it is an arc or a point. Hence p is an acyclic map and therefore $H^1(M) \simeq H^1(Y)$ because $p^*\colon H^1(Y) \to H^1(M)$ is an isomorphism [2]. Since q is a surjection, then by 4.1 we infer that $H^1(M)$ is not finitely divisible. This contradicts the fact that $H^1(M)$ is isomorphic to G. Hence $H^1(X)$ is finitely divisible, which completes the proof.

The next considerations are preparatory for the proof of Theorem 4.8 in which we show that certain class of continua has the first cohomology group free.

Recall that a subgroup H of an Abelian group G admits division in G if for each $g \in G$ and each natural number n the condition $n \cdot g \in H$ implies $g \in H$. The following proposition follows from [28, p. 243].

4.6. Proposition. *Let G be an Abelian countable torsion free group. Suppose there exists an increasing sequence $G_1, G_2, \ldots$ of finitely generated subgroups of G such that $G = \bigcup_{n \geqslant 1} G_n$ and G_n admits division in G for each $n \geqslant 1$. Then G is free.*

By $H_1(X)$ we denote the first singular homology group of X with integer coefficients. It is known that for ANR-sets the singular homology is naturally equivalent to the Čech homology (based on arbitrary open coverings) [24, Th. 1, p. 30].

Let x_0 be a point of X and let $\omega\colon (I, \dot{I}) \to (X, x_0)$ be a loop in X. Then ω is a representative of the element $[\omega] \in \pi_1(X, x_0)$. But $\omega\colon I \to X$ can be also treated as a singular symplex in X which is a cycle in X. Hence ω is a representative of an element $\langle\omega\rangle \in H_1(X)$. It is easy to see that this procedure defines a homomorphism $h\colon \pi_1(X, x_0) \to H_1(X)$. Observe that if $f\colon X \to Y$ is a mapping, then for the induced homomorphism $f_*\colon H_1(X) \to H_1(Y)$ we have $f_*(\langle\omega\rangle) = \langle f \circ \omega\rangle$.

The following is a classical result about h.

4.7. Theorem. [12, p. 45]. *If X is pathwise connected, then the homomorphism $h\colon \pi_1(X, x_0) \to H_1(X)$ is an epimorphism with the commutator subgroup of $\pi_1(X, x_0)$ as its kernel. Moreover, h establishes a natural transformation of the functor π_1 into H_1.*

Since $\pi_1(S, 1)$ is an infinite cyclic group, it follows from the above theorem that each $a \in H_1(S)$ is of the form $a = \langle\omega\rangle$ for some loop ω in $(S, 1)$. Moreover if $\langle\omega_1\rangle = \langle\omega_2\rangle$, then $\omega_1 \simeq \omega_2$ rel. $\dot{I}$, i.e. $[\omega_1] = [\omega_2]$.

From now on to the end of this section the symbol p_n, $n \geqslant 1$, stands for the map $p_n\colon S \to S$ given by $p_n(z) = z^n$. We shall use the fact that p_n is a covering projection and the simple observation that if $[f] \in \pi^1(X)$, then $n \cdot [f] = [p_n \circ f]$.

Now we are ready to prove

4.8. Theorem. *Let $\underline{X}$ be an inverse sequence of connected ANR-sets such that $H_1(\underline{X})$ is an ML-sequence. Then $H^1(\mathrm{invlim}\,\underline{X})$ is a free group.*

Proof. Let $\underline{X} = \{X_n, f_n^m\}$ and let $X = \mathrm{invlim}\,\underline{X}$. Let $(f_n^m)_*\colon H_1(X_m) \to H_1(X_n)$, $1 \leqslant n \leqslant m$, denote the induced homomorphisms. Without loss of generality we may assume that

(1) $$\mathrm{im}(f_n^{n+1})_* = \mathrm{im}(f_n^m)_* \quad \text{for each } 1 \leqslant n < m.$$

By the continuity of Čech cohomology and the Bruschlinsky theorem we have $H^1(X) \approx \mathrm{dirlim}\, H^1(\underline{X}) \approx \mathrm{dirlim}\, \pi^1(\underline{X})$, where the induced homomorphisms

$$(f_n^m)^*\colon \pi^1(X_n) \to \pi^1(X_m)$$

are given by $(f_n^m)^*([\alpha]) = [\alpha \circ f_n^m]$. Thus the proof will be completed when we show that $G^\infty = \mathrm{dirlim}\, \pi^1(\underline{X})$ is free.

Let $\eta_n\colon \pi^1(X_n) \to G^\infty$ be the natural homomorphisms. For each $n \geqslant 1$ define

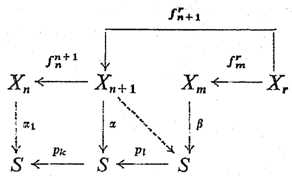
$$G_n = \{g \in G^\infty\colon \text{there exist } [\alpha] \in \pi^1(X_{n+1}) \text{ and } k \geqslant 1$$
$$\text{such that } g = \eta_{n+1}([\alpha]) \text{ and } k \cdot [\alpha] \in \mathrm{im}(f_n^{n+1})^*\}.$$

Observe that $\mathrm{im}\,\eta_n \subset G_n \subset \mathrm{im}\,\eta_{n+1}$. Since the groups $\pi^1(X_n)$ are free of finite rank it follows that G_n's form an increasing sequence of finitely generated subgroups of G^∞ such that $G^\infty = \bigcup_{n \geqslant 1} G_n^\infty$. Since G^∞ is an Abelian countable torsion free group, then by 4.6 our task is reduced to showing that

(2) G_n admits division in G^∞.

Suppose, to the contrary, that there exist an element $g \in G^\infty \setminus G_n$ and a natural number l such that $l \cdot g \in G_n$. From the construction of G_n it follows that there exist $[\alpha] \in \pi^1(X_{n+1})$ and $k \geqslant 1$ such that $l \cdot g = \eta_{n+1}([\alpha])$ and $k \cdot [\alpha] \in \mathrm{im}(f_n^{n+1})^*$. Let $k \cdot [\alpha] = (f_n^{n+1})^*([\alpha_1])$. There exist an index m and an element $[\beta] \in \pi^1(X_m)$ such that $g = \eta_m([\beta])$. Since $\mathrm{im}\,\eta_j \subset G_n$ for $j \leqslant n$ and $g \notin G_n$ it follows that $m \geqslant n+1$. Since $\eta_{n+1}([\alpha]) = \eta_m(l \cdot [\beta])$, there exists an index $r > m$ such that $(f_{n+1}^r)^*([\alpha]) = (f_m^r)^*(l \cdot [\beta])$.

Thus we have the following diagram commuting up to homotopy:



Now we shall show that there is a mapping $\tilde{\alpha}\colon X_{n+1} \to S$ (corresponding to the dotted arrow in the diagram) such that $\alpha \simeq p_l \circ \tilde{\alpha}$. Let x_0 be a point of X_{n+1}. Without loss of generality we may assume that $\alpha(x_0) = 1$. The existence of $\tilde{\alpha}$ will follow from the lifting theorem [31, p. 76] when we prove that for each loop ω_0 in (X_{n+1}, x_0) there exists a loop ω_1 in $(S, 1)$ such that $\alpha \circ \omega_0 \simeq p_l \circ \omega_1$ rel. $\dot{I}$. So, let ω_0 be a loop in (X_{n+1}, x_0). Let $\langle\omega_0\rangle$ be an element of $H_1(X_{n+1})$ determined by ω_0. Then by (1) there exists an element $a \in H_1(X_r)$ such that $(f_n^{n+1})_*(\langle\omega_0\rangle) = (f_n^r)_*(a)$. Let ω_1 be a loop in $(S, 1)$ such that $\langle\omega_1\rangle = (\beta \circ f_m^r)_*(a)$. Now we prove that $\alpha \circ \omega_0 \simeq p_l \circ \omega_1$ rel. $\dot{I}$. It suffices to show that $\langle\alpha \circ \omega_0\rangle = \langle p_l \circ \omega_1\rangle$. Since the functor H_1 makes the diagram commutative and the homomorphism $(p_k)_*$ is a monomorphism, the last equality will follow when we prove that $(p_k)_*(\langle\alpha \circ \omega_0\rangle) = (p_k)_*(\langle p_l \circ \omega_1\rangle)$. But this is true because we have:

$$\begin{aligned}(p_k)_*(\langle\alpha \circ \omega_0\rangle) &= \langle p_k \circ \alpha \circ \omega_0\rangle = (\alpha_1)_*(f_n^{n+1})_*(\langle\omega_0\rangle) = (\alpha_1)_*(f_n^r)_*(a) \\ &= (\alpha_1)_*(f_n^{n+1})_*(f_{n+1}^r)_*(a) = (p_k)_*(\alpha)_*(f_{n+1}^r)_*(a) \\ &= (p_k)_*(p_l)_*(\beta \circ f_m^r)_*(a) = (p_k)_*(\langle p_l \circ \omega_1\rangle).\end{aligned}$$

Thus we have proved that there is an element $[\tilde{\alpha}]\in\pi^1(X_{n+1})$ such that

(3) $$[\alpha] = l\cdot[\tilde{\alpha}].$$

Return now to the proof of condition (2). We have $l\cdot g = \eta_{n+1}([\alpha])$. Using (3) we infer that $l\cdot g = l\cdot\eta_{n+1}([\tilde{\alpha}])$. Since G^∞ is torsion free we conclude that $g = \eta_{n+1}([\tilde{\alpha}])$. Also $k\cdot l[\tilde{\alpha}] = k\cdot[\alpha]\in\operatorname{im}(f_n^{n+1})^*$. These properties show that $g\in G_n$, a contradiction completing the proof of (2). As we observed before this proves the whole theorem.

It follows from 2.8 that every continuum X satisfying one of the following conditions has the group $H^1(X)$ free: (MOV*), (MOV), (1MOV*), (1MOV), (n1MOV), (MLH_1).

In [13] J. Keesling proved several interesting results on the group $H^1(X)$, where X is an Abelian topological group. The following is a corollary to his results.

4.9. Theorem (J. Keesling). *Let X be a continuum and let $A_X = \operatorname{char}H^1(X)$, considering $H^1(X)$ as a discrete group. Then there is a mapping $f\colon X\to A_X$ such that $f^*\colon H^1(A_X)\to H^1(X)$ is an isomorphism.*

Let T^n be the n-torus, where $n = 1, 2, \ldots, \infty$. By definition T^n is the cartesian product of n-copies of the circle S, hence it is the character group of a discrete free Abelian group of rank n.

4.10. Corollary. *Let X be a continuum such that $H^1(X)$ is free of rank n. Then there is a mapping $f\colon X\to T^n$ such that $f^*\colon H^1(T^n)\to H^1(X)$ is an isomorphism.*

Now we shall see that all the properties listed in the Introduction except (FDH^1) are equivalent for continua which admit a group structure under which they are Abelian topological groups.

The following is a corollary to Theorems 2.5 and 2.7 in [14].

4.11. Theorem (J. Keesling). *Let A be a compact connected (metrizable) Abelian topological group. Then the following are equivalent:*

(i) *A is movable,*
(ii) *A is locally connected,*
(iii) *A is arcwise connected,*
(iv) *$H^1(A)$ is free.*

From 4.3 and the Pontrjagin duality theorem it follows that the list of equivalent conditions can be extended to the following:

(v) *A is pointed movable,*
(vi) *A is pointed 1-movable,*
(vii) *A is 1-movable,*
(viii) *A is nearly 1-movable,*
(ix) *there exists an ANR-sequence $\underline{A}$ associated with A such that $H_1(\underline{A})\in$ ML,*
(x) *A is isomorphic to T^n, where $n = \operatorname{rank}H^1(A)$.*

Remark. For non metrizable Abelian topological groups these conditions are not equivalent (see [14, Ex. 2.8]).

Since each continuum embeddable into a 2-dimensional manifold is movable, conditions (i) and (x) from the preceding theorem imply the following fact, probably known to people working in topological groups.

4.12. Corollary. *Let X be a nondegenerate continuum embeddable into 2-dimensional manifold M. Then if X admits a group structure under which it is an Abelian topological group, then X is homeomorphic either to the circle S or to the ordinary torus T^2. Hence if M is the sphere S^2, then X is homeomorphic to S.*

5. Some remarks on pointed 1-movable continua. It was observed in [18] and [26] that pointed movable continua are exactly those which are simultaneously movable and pointed 1-movable. In symbols we can state this as follows:

$$(\text{MOV*}) = (\text{MOV})+(\text{1MOV*}).$$

It follows that Problem 1.1 from the Introduction is equivalent to the following

(1.1)′. Problem. Must movable continua be pointed 1-movable?

Thus we see the importance of the notion of pointed 1-movability in shape theory and continua theory. We refer the reader to [17], [18], [20] and [26] for detailed discussion of this property.

5.1. Theorem. *A continuum X is pointed 1-movable if and only if it is joinable* [20].

For the convenience of the reader we repeat here the definition of joinability introduced in [20]. In the definition we consider a continuum X to be a subset of a space $M\in\text{ANR}(\mathfrak{M})$, but it is easy to check that the notion does not depend on the choice of a particular M.

Consider two points $x, y\in X$. Then we say that x and y *are joinable in X* if there exists a mapping $\varphi\colon I\times[0,\infty)\to M$ satisfying the conditions:

(a) $\varphi(0,t) = x$, $\varphi(1,t) = y$ for each $t\in[0,\infty)$,

(b) for each neighborhood U of X in M there is a real $t_0\geqslant 0$ such that $\varphi(I\times[t_0,\infty))\subset U$.

5.2. Problem [1]. Does there exist a nondegenerate continuum X such that no two different points of X are joinable?

If each pair of points of X is joinable then X is said to be *joinable*.

If U_0 is a fixed neighborhood of X in M then it is reasonable to consider a weaker form of joinability, so called U_0-joinability. We obtain this notion by replacing condition (b) with the following:

(c) $\varphi(I\times[0,\infty))\subset U_0$.

The next theorem, proved in [20] in an equivalent form, shows how complicated global structure a continuum must have when it is not pointed 1-movable.

5.3. Theorem [20]. *Let X be a continuum lying in a space $M\in\text{ANR}(\mathfrak{M})$. If X is not pointed 1-movable, then there exist a neighbourhood U_0 of X in M and a subset A of X such that $\operatorname{Card}A = \mathfrak{c}$ and no two different points of A are U_0-joinable.*

(1) Added in proof. The example of L. Oversteegen and the author is such a continuum (see 3.14).

5.4. Problem. Can we choose A to be a closed subset of X?

The relations of joinability and U_0-joinability partition the continuum X into classes of equivalent elements. We call them joinability components and U_0-joinability components of X in an analogy to arc-components of X. In [20] they were called approximative path-components. For some simple continua, such as solenoids, these components coincide with the usual composants. It should be noted however that their structure is different from the structure of composants — for instance they need not be dense in the continuum.

The structure of joinability components will be studied in another paper by the author. Here we present only a construction and simple observations on an interesting fibration associated with a given joinability component which makes the study possible.

Let B be a joinability component of a continuum X and let $x_0 \in B$. Let $(\underline{X}, x_0) = \{(X_n, x_n), f_{nm}\}$ be an inverse sequence of connected ANR-sets associated with X. For each $n \geqslant 1$ let $p_n\colon (\tilde{X}_n, \tilde{x}_n) \to (X_n, x_n)$ be the universal covering projection. Then there exist mappings $\tilde{f}_{nm}\colon (\tilde{X}_m, \tilde{x}_m) \to (\tilde{X}_n, \tilde{x}_n)$, $n \leqslant m$, such that $p_n \circ \tilde{f}_{nm} = f_{nm} \circ p_m$. Let $(E, \tilde{x}_0)$ be the limit of the inverse sequence $\{(\tilde{X}_n, \tilde{x}_n), \tilde{f}_{nm}\}$ and let $p'\colon (E, \tilde{x}_0) \to (X, x_0)$ be the mapping induced by p_n's. Then E is in a sense simply connected and one can prove that

5.5. Theorem. *$p'(E) = B$ and the map $p\colon E \to B$ determined by p' is a Hurewicz fibration with 0-dimensional fibres.*

Thus with every joinability component of X we have associated a Hurewicz fibration. An important observation is this. If X is joinable, then $B = X$ and the above procedure gives us a Hurewicz fibration $p\colon E \to X$ with E being "simply connected". This indicates that for pointed 1-movable continua we may construct a reasonable fibration theory. This will be done in another paper.

Now we shall prove that the 1-dimensional "holes" in a pointed 1-movable continuum can be filled in by a 0-dimensional boundle of open disks. The detailed construction and appropriate results are given below.

Let X be a pointed 1-movable continuum and let $x_0 \in X$. There exist an ANR-sequence $(\underline{X}, x_0) = \{(X_n, x_n), f_{nm}\}$ associated with (X, x_0) and a sequence $F_1, F_2, \ldots$ of finite sets such that each X_n is connected, each F_n generates $\pi_1(X_n, x_n)$, in symbols: $\langle F_n \rangle = \pi_1(X_n, x_n)$, and $(f_{n,n+1})_\#(F_{n+1}) = F_n$ for each $n \geqslant 1$ (see the proof of 3.1 in [18]). Now we shall construct a new ANR-sequence whose inverse limit contains X and has some nice properties. Using polar coordinates, for each $n \geqslant 1$, define

$$D_n = \{(r, \theta)\colon 0 \leqslant r \leqslant n,\ 0 \leqslant \theta < 2\pi\} \subset R^2 .$$

Let $F_n = \{e_{n1}, \ldots, e_{nk_n}\}$. Let the mapping

$$\varphi_{nj}\colon (\dot{D}_n, d_n) \to (X_n, x_n), \quad d_n = (n, 0),$$

represent the element e_{nj}, $1 \leqslant j \leqslant k_n$. Let X_n' be a space obtained from X_n attaching to X_n the disks D_n by means of the maps φ_{nj}. Let σ_{nj}, $1 \leqslant j \leqslant k_n$, be the 2-cells and let $\overline{\varphi}_{nj}\colon D_n \to \sigma_{nj}$ be the characteristic maps. Thus $\overline{\varphi}_{nj}$ extends φ_{nj}. (It is assumed as usual that the interiors of the 2-cells are mutually disjoint). By the Borsuk–Whitehead theorem the space X_n' is an ANR and it is easy to observe that it is simply connected (comp. [31, p. 146]). Now we define the mappings

$$f'_{n,n+1}\colon (X'_{n+1}, x_n) \to (X'_n, x_n) .$$

For $x \in X_{n+1}$ let $f'_{n,n+1}(x) = f_{n,n+1}(x)$. On the attached 2-cell $\sigma_{n+1,j}$, $1 \leqslant j \leqslant k_{n+1}$, $f'_{n,n+1}$ acts as follows. Let $(f_{n,n+1})_\#(e_{n+1,j}) = e_{ni}$. Let $P = \{(r, \theta)\colon n \leqslant r \leqslant n+1\}$. Since φ_{ni} represents e_{ni} and $\varphi_{n+1,j}$ represents $e_{n+1,j}$ there is a mapping (homotopy)

$$\alpha\colon P \to X_n$$

such that $\alpha(n, \theta) = \varphi_{ni}(n, \theta)$, $\alpha(n+1, \theta) = f_{n,n+1} \circ \varphi_{n+1,j}(n+1, \theta)$ and $\alpha(r, 0) = x_n$ for each $n \leqslant r \leqslant n+1$ and $0 \leqslant \theta < 2\pi$. Then if $x = \overline{\varphi}_{n+1,j}(y) \in \sigma_{n+1,j}$, then we set

$$f'_{n,n+1}(x) = \begin{cases} \overline{\varphi}_{ni}(y) & \text{for } y \in D_n , \\ \alpha(y) & \text{for } y \in P . \end{cases}$$

This completes the description of the maps f'_{nm}.

Let us observe that the following implication holds:

$$(*) \qquad 1 \leqslant k < l \wedge e_{ki} = (f_{kl})_\#(e_{lj}) \Rightarrow f'_{kl} \cdot \overline{\varphi}_{lj} | D_k = \overline{\varphi}_{ki} .$$

Let

$$X' = \operatorname{invlim}\{X'_n, f'_{nm}\} .$$

It follows that X' is approximatively simply connected and obviously X is a subset of X'.

Consider F_n's as spaces with discrete topologies. Then define

$$F = \operatorname{invlim}\{F_n, \tilde{f}_{nm}\} ,$$

where $\tilde{f}_{nm}$ is determined by $(f_{nm})_\#$. The set F is a compact 0-dimensional space. Using this set we can determine the structure of the complement $X' \setminus X$. In fact, we are able to prove that the set is homeomorphic to $R^2 \times F$. To see this we define an embedding

$$h\colon R^2 \times F \to X'$$

such that $h(R^2 \times F) = X' \setminus X$. Let $e = (e_{1j_1}, e_{2j_2}, \ldots)$ be a point of F. Denote $\overline{\varphi}_n = \overline{\varphi}_{nj_n}$, $n \geqslant 1$. Let $y \in R^2$. To define $h(y, e)$ it suffices to determine its ith co-ordinate $h(y, e)_i$ and this is defined as follows. If $y = (r, \theta)$, where $n-1 \leqslant r < n$, then

$$h(y, e)_i = \begin{cases} f'_{in} \circ \overline{\varphi}_n(y) & \text{for } i \leqslant n , \\ \overline{\varphi}_i(y) & \text{for } i > n . \end{cases}$$

Using $(*)$ it is an easy exercise to check that h has the desired properties.

Denote

$$a = (1, 0) \quad \text{and} \quad R_1 = \{(r, \theta) \in R^2\colon r \geqslant 1 \wedge \theta = 0\} .$$

Note that $\overline{h(R_1\times F)} = h(R_1\times F)\cup\{x_0\}$ is homeomorphic to the cone over F, where $\{a\}\times F$ corresponds to the base and x_0 to the vertex of the cone. We simply identify this set with the cone. Thus we have proved the following.

5.6. THEOREM. *Let X be a pointed 1-movable continuum and let x_0 be a point of X. Then there exist an approximatively simply connected continuum X' containing X and a compact 0-dimensional set F such that $X'\setminus X$ is homeomorphic to $R^2\times F$. Moreover, there exists an embedding h: $R^2\times F\to X'$ such that $h(R^2)\times F = X'\setminus X$ and $h(R_1\times F)\cup\{x_0\}$ is a cone with the base $\{a\}\times F$ and the vertex x_0.*

Roughly speaking this theorem expresses the fact that in pointed 1-movable continua we can fill in the 1-dimensional holes adding to the space a 0-dimensional boundle of open disks which nicely attain a given point in X.

The property stated in the theorem does not characterize the pointed 1-movability. Namely, J. Dydak [8] constructed an example of a non-movable curve X containing a simple closed curve S such that after attaching to X a 2-cell by means of a map which is a homeomorphism on the boundary of a disk one gets a continuum with trivial shape.

5.7. THEOREM. *Let X be a subcontinuum of a nearly 1-movable continuum X' such that $X'\setminus X$ is homeomorphic to $R^2\times F$, where F is a closed 0-dimensional set. Then X is nearly 1-movable.*

Proof. Let D be a closed disk in R^2. Consider the continuum $X'' = X'\setminus\mathring{D}\times F$. Let Y be the quotient space obtained from X'' by identifying each circle $\dot{D}\times\{a\}$, $a\in F$, to a point. Let f: $X''\to Y$ be the quotient map. Then Y is homeomorphic to X' and for each $y\in Y$ the set $f^{-1}(y)$ is either a point or a circle. Thus $f^{-1}(y)$ is a nearly 1-movable continuum. Since Y is nearly 1-movable, it follows from [8, Th. 2] that X'' is nearly 1-movable. Since $\mathrm{Sh}X'' = \mathrm{Sh}X$, then X is nearly 1-movable.

5.8. PROBLEM. Let X be a nearly 1-movable continuum. Do there exist an approximatively 1-connected continuum $X'\supset X$ and a closed 0-dimensional set F such that $X'\setminus X$ is homeomorphic to $R^2\times F$?

We finish this section with a result showing the importance of joinability in the study of pointed shape. The class of solenoids shows that there exist continua which have the same shape with respect to points which are not joinable. However we shall prove that joinability of two points in a continuum is sufficient for the space to have the same shape with respect to the points. First we prove a lemma.

If ω is a path in a space Y and f, g: $X\to Y$ are two mappings, and x_0 is a base point in X, then f and g are said to be *ω-homotopic* if there is a homotopy h from f to g such that $h(x_0, t) = \omega(t)$ for each $t\in I$.

5.9. LEMMA. *Let ω be a path in an ANR-set X. Then there exist mappings g, g': $X\to X$ such that $\mathrm{id}X$ is ω-homotopic to g and $\mathrm{id}X$ is ω^{-1}-homotopic to g'. Any such mappings satisfy relations: $g'\circ g\simeq \mathrm{id}X$ rel. $\omega(0)$ and $g\circ g'\simeq \mathrm{id}X$ rel. $\omega(1)$ (it is understood that $\mathrm{id}X$ is ω-homotopic to g rel. $\omega(0)$ and $\omega(1)$ is the base point for the other homotopy).*

Proof. The existence of g and g' follows from [31, p. 380]. Then it follows that $\mathrm{id}X$ is $\omega*\omega^{-1}$-homotopic to $g'\circ g$ and $\mathrm{id}X$ is $\omega^{-1}*\omega$-homotopic to $g\circ g'$. Since the loops are contractible, the conclusion of the lemma follows from [31, p. 380].

5.10. THEOREM. *Let x and y be two points of a continuum X. If x and y are joinable in X, then $\mathrm{Sh}(X, x) = \mathrm{Sh}(X, y)$.*

Proof. Let $\underline{X} = \{X_n, f_{nm}\}$ be an inverse sequence of ANR-sets such that $X = \mathrm{invlim}\underline{X}$. Let x_n, y_n denote the nth coordinates of the points x and y respectively. Since x and y are joinable, there exists a sequence of paths

$$\{\omega_n\colon (I, 0, 1)\to (X_n, x_n, y_n)\}$$

such that

$$\omega_n\simeq f_{n,n+1}\circ\omega_{n+1}\ \text{rel. } \dot{I}\,. \tag{1}$$

The theorem will be proved when we show that there exist sequences of mappings $\{g_n, g_n'\colon X_n\to X_n\}$ such that: $g_n(x_n) = y_n$, $g_n'(y_n) = x_n$ $g_n'\circ g_n\simeq \mathrm{id}X_n$ rel. x_n, $g_n\circ g_n'\simeq \mathrm{id}X_n$ rel. y_n and $g_n\circ f_{n,n+1}\simeq f_{n,n+1}\circ g_{n+1}$ rel. x_{n+1}.

Let g_n: $X_n\to X_n$ be such that $\mathrm{id}X_n$ is ω_n-homotopic to g_n and let g_n': $X_n\to X_n$ be such that $\mathrm{id}X_n$ is ω_n^{-1}-homotopic to g_n'. The existence of such mappings follows from the above lemma. Using this lemma the proof will be done when we check the relation $g_n\circ f_{n,n+1}\simeq f_{n,n+1}\circ g_{n+1}$ rel. x_{n+1}.

But $f_{n,n+1} = \mathrm{id}X_n\circ f_{n,n+1}$ is ω_n-homotopic to $g_n\circ f_{n,n+1}$, where f_{n+1} is regarded as a map from (X_{n+1}, x_{n+1}) to (X_n, x_n). Moreover, $f_{n,n+1} = f_{n,n+1}\circ \mathrm{id}X_{n+1}$ is $f_{n,n+1}\circ\omega_{n+1}$-homotopic to $f_{n,n+1}\circ g_{n+1}$ (x_{n+1} is the base point of X_{n+1}).

By (1) and [31, p. 380] we infer that $f_{n,n+1}$ is ω_n-homotopic to $f_{n,n+1}\circ g_{n+1}$. Thus $f_{n,n+1}$: $(X_{n+1}, x_{n+1})\to (X_n, x_n)$ is ω_n-homotopic to both $g_n\circ f_{n,n+1}$ and $f_{n,n+1}\circ g_{n+1}$. By [31, p. 380] it follows that $g_n\circ f_{n,n+1}\simeq f_{n,n+1}\circ g_{n+1}$ rel. x_{n+1}, which completes the proof.

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Approximate polyhedra, resolutions of maps and shape fibrations

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Abstract. Shape fibrations between compact metric spaces were introduced by T. B. Rushing and the author in [15]. In this paper one extends the definition so as to apply to maps $p: E \to B$ between arbitrary topological spaces. This is done by considering certain morphisms in pro-Top $\underline{p}: \underline{E} \to \underline{B}$, called resolutions of p. In the compact case resolutions reduce to inverse limit expansions. One requires also that the systems $\underline{E}$ and $\underline{B}$ consist of ANR's, polyhedra or more generally of spaces called approximate polyhedra (AP). A map p is a shape fibration provided it admits an AP-resolution $\underline{p}$, which has a certain approximate homotopy lifting property. Resolutions of spaces are characterized and compared with the inverse limit expansions. Moreover, existence of ANR-resolutions and polyhedral resolutions is demonstrated.

1. Introduction. Shape fibrations $p: E \to B$ between compact metric spaces (more generally, proper shape fibrations between locally compact metric spaces) were introduced and studied by T. B. Rushing and the author in [15], [16], [17]. Further contributions to this theory were made by Z. Čerin, L. S. Husch, M. Jani, J. Keesling, S. Mardešić, A. Matsumoto and T. C. McMillan. For a survey of results on approximate fibrations and shape fibrations see [14] and [22].

This paper originated from an attempt to extend the notion of shape fibration from the rather special case of maps between metric compacta to the general case of maps between arbitrary topological spaces. Results concerning this question are contained in Sections 4 and 8 of this paper.

The main idea consists in considering certain expansions $\underline{p}: \underline{E} \to \underline{B}$ of the map $p: E \to B$, called resolutions of p. They are related to inverse limit expansions of p and appear to be of interest on their own. For resolutions of p one defines the approximate homotopy lifting property (AHLP) as in ([15], § 9). If one allows as members of $\underline{E}$ and $\underline{B}$ only "nice" spaces, then the property AHLP does not depend on the choice of the resolution, but depends only on the map p. Maps which have this property are, by definition, shape fibrations. In § 8 we give a "categorical" definition of shape fibrations.

In § 2 we define and study "nice" spaces under the name of approximate polyhedra. We show that they include ANR's (for metric spaces), CW-complexes and n-dimensional LC^{n-1} paracompacta. In the compact metric case approximate