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Accepté par la Rédaction le 31. 12. 1979

A rest point free dynamical system on R^3 with uniformly bounded trajectories

by

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Abstract. In this paper, we show that if $\varepsilon > 0$, then there exists a C^∞ transformation G from R^3 into R^3 such that the unique solution Φ to the differential equation $y' = G(y)$ is a dynamical system (a continuous transformation from $R\times R^3$ into R^3 such that $\Phi(0, p) = p$, $\Phi(t_1, \Phi(t_2, p)) = \Phi(t_1+t_2, p)$ and $\partial/\partial t\Phi(0, p) = G(p)$) with the following two properties: (1) For each point p in R^3 and each number t, $\Phi(t, p)$ is in the ε-neighborhood for p; and (2) for each integer $n \neq 0$, $\Phi(n, p) \neq p$. Notice that Scottish Book problem number 110 of Ulam follows as a corollary where $f(p) = \Phi(1, p)$ and the manifold is R^3.

Introduction. In 1935 S. Ulam raised the following question [7], Problem 110: "Let M be a given manifold. Does there exist a numerical constant K such that every continuous mapping f of the manifold M into part of itself which satisfies the condition $|f^n(x)-x| < K$ for $n = 1, 2, \ldots$ (where $f^n(x)$ denotes the nth iteration of the image $f(x)$) possesses a fixed point: $f(x_0) = x_0$? The same under more general assumptions about M (general continuum?)." In this paper, we solve this problem in the negative, where $M = R^3$, f is a homeomorphism onto, f is C^∞, and for each $x \in R^3$ and each positive integer n, $f^n(x) \neq x$. Moreover, $f(x) = \Phi(1, x)$, where Φ is a C^∞ dynamical system on R^3 with uniformly bounded trajectories.

By a dynamical system Φ on a metric space X we mean a continuous mapping Φ: $R\times X \to X$ (where R is the set of real numbers) such that for each $t \in R$, $\Phi(\{t\}\times X) = X$, and such that if each of t_1 and t_2 is a number and $p \in X$ is a point, then $\Phi(t_1, \Phi(t_2, p)) = \Phi(t_1+t_2, p)$ and $\Phi(0, p) = p$. If G is a transformation from R^3 into R^3, then G is said to *generate a dynamical system* Φ provided that, for each point $p \in R^3$, $\lim_{t\to 0} \frac{\Phi(t, p)-p}{t} = G(p)$.

The set of all points $\Phi(t, p)$ for a fixed p and $-\infty < t < +\infty$ is called a *trajectory of the dynamical system*. A point q is called an *ω-limit point of a trajectory* $\Phi(t, p)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots \to +\infty$ such that $\lim \Phi(t_n, p) = q$. A point q is called an *α-limit point of a trajectory* $\Phi(t, p)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots \to -\infty$ such that $\lim \Phi(t_n, p) = q$.

A classical result which we will employ is the following: If G is a transformation from R^3 into R^3 satisfying globally a Lipschitz condition with constant L, then the differential equation $y' = G(y)$ has a unique solution for each initial condition and the dynamical system Φ generated by G describes this solution set. See, for instance, [6] Chapter I.

There are two examples in the literature of dynamical systems on R^3 with all trajectories bounded and no rest points. The first is due to Jones and Yorke [5]. The main idea of this example is to describe a monotonically increasing sequence of tori in R^3, whose union is R^3, and to define a dynamical system $\Phi(t,p)$ such that for each fixed t, $\Phi(t,p)$ restricted to the surface of any of the tori is a rotation. Therefore, it is not possible to obtain a uniform bound on the trajectories in this example. The second example was described by Brechner and Mauldin [4] and was based on the observation of Howard Cook that the acyclic Peano continuum without the fixed point property constructed by Borsuk [1], [3] can be used to define a dynamical system on R^3, with no rest points and all trajectories bounded. In this example, there is a neighborhood of the z-axis so that outside of this neighborhood, points follow circular trajectories parallel to the xy-plane with center on the z-axis. Therefore, the trajectories are not uniformly bounded.

The second part of the question of S. Ulam [7], Problem 110, has been answered in the negative by W. Kuperberg, who gave an example of a one dimensional metric continuum, which for every $\varepsilon>0$ admits a fixed point free ε-involution. Subsequently, W. Kuperberg and P. Minc, using Borsuk's example described in [3] and Cook's idea, proved that the Cartesian product of the Hilbert cube Q and the circle S^1 has the property: for every $\varepsilon>0$ there exists a dynamical system Φ on $Q\times S^1$ such that for each $p\in Q\times S^1$ the trajectory $\Phi(t,p)$ is of diameter less than ε, and $\Phi(n,p)\neq p$ for each nonzero integer n.

The example. Suppose that $\varepsilon>0$. We will construct a C^∞ transformation G from R^3 into R^3 satisfying globally a Lipschitz condition with constant L such that the dynamical system Φ generated by G satisfies the following two properties:

(1) If t is a number and p is a point, then $\Phi(t,p)$ is in the ε-neighborhood of p;

(2) If n is an integer distinct from zero, then $\Phi(n,p)\neq p$.

Set $\delta=\varepsilon/400$. G will first be defined on the closed solid cylinder C consisting of those points (x,y,z) satisfying $\sqrt{x^2+y^2}\leqslant 4\delta$ and $0\leqslant z\leqslant 6\delta$. Now, set $T=\{(x,y,z)\colon \delta\leqslant\sqrt{x^2+y^2}\leqslant 2\delta\}$ and for each number b set $T_b=\{(x,y,z)\in T\colon z=b\}$. G will satisfy the following eight conditions: (1) for each point $p\in C$ in the δ-neighborhood of the boundary of C, $G(p)=(0,0,1)$; (2) if $p\in C$ and $\Phi(t,p)\in C$, then p and $\Phi(t,p)$ are equidistant from the z-axis; (3) each of the annuli $T_{2\delta}$ and $T_{4\delta}$ is invariant under Φ, and Φ is a rotation on $T_{2\delta}$ and on $T_{4\delta}$ such that for each integer n distinct from zero $\Phi(n,p)\neq p$; (4) if $0<b<2\delta$ and $p\in T_b$, then there is a negative number t such that $\Phi(t,p)\in T_0$ and each ω-limit point of the trajectory $\Phi(t,p)$ is on $T_{2\delta}$; (5) if $2\delta<b<4\delta$ and $p\in T_b$, then each α-limit point of the trajectory $\Phi(t,p)$ is on $T_{2\delta}$ and each ω-limit point of the trajectory $\Phi(t,p)$ is on $T_{4\delta}$; (6) if $4\delta<b<6\delta$ and $p\in T_b$, then each α-limit point of the trajectory $\Phi(t,p)$ is on $T_{4\delta}$ and there is a positive number t such that $\Phi(t,p)\in T_{6\delta}$; (7) if $(x,y,z)\in C\setminus(T_{2\delta}\cup T_{4\delta})$ and $\Phi(t,(x,y,z))=(u,v,w)$ for some $t>0$, then $w>z$; (8) if $(x,y,z)\in C\setminus T$

and $z=0$, then there is a $t>0$ such that if $\Phi(t,(x,y,z))=(u,v,w)$ then $u=x$, $v=y$, and $w=6\delta$.

The construction of a dynamical system with the above eight properties in C, will be made possible by rotating $T_{2\delta}$ and $T_{4\delta}$ in opposite directions. Property (8) is accomplished by making sure that points in $C\setminus T$ are trajectories that "unwind" in the top half of C by the same amount that they "wound up" in the bottom half of C. See Figure 1. The fact that this can be accomplished in a C^∞ fashion will now be demonstrated.

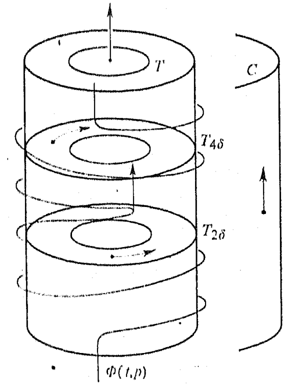
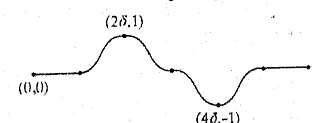


Fig. 1

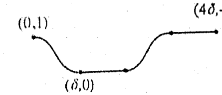
Let h denote a strictly increasing C^∞ function on $[0,\delta]$ such that $h(0)=0$, $h(\delta)=1$, and all of the derivatives of h at zero and one are zero. Now define the five real valued functions f, g, α, β, and γ as follows.

$$f(r)=\begin{cases}1 & \text{if } 0\leqslant r\leqslant 2\delta,\\ h(3\delta-r) & \text{if } 2\delta\leqslant r\leqslant 3\delta,\\ 0 & \text{if } 3\delta\leqslant r\leqslant 4\delta,\end{cases}$$

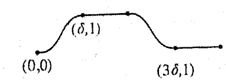
$$g(z)=\begin{cases}0 & \text{if } 0\leqslant z\leqslant \delta,\\ h(z-\delta) & \text{if } \delta\leqslant z\leqslant 2\delta,\\ h(3\delta-z) & \text{if } 2\delta\leqslant z\leqslant 3\delta,\\ -g(6\delta-z) & \text{if } 3\delta\leqslant z\leqslant 6\delta,\end{cases}$$



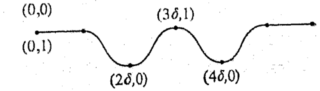
$$\alpha(r)=\begin{cases}h(\delta-r) & \text{if } 0\leqslant r\leqslant \delta,\\ 0 & \text{if } \delta\leqslant r\leqslant 2\delta,\\ h(r-2\delta) & \text{if } 2\delta\leqslant r\leqslant 3\delta,\\ 1 & \text{if } 3\delta\leqslant r\leqslant 4\delta,\end{cases}$$



$$\beta(r)=1-\alpha(r),$$



$$\gamma(r)=\begin{cases}-g(z) & \text{if } 0\leqslant z\leqslant 3\delta,\\ g(z) & \text{if } 3\delta\leqslant z\leqslant 6\delta.\end{cases}$$



Now for each point $p = (r\cos(\theta), r\sin(\theta), z)$ of C, set

$$G(p) = \left(-f(r)g(z)r\sin(\theta), f(r)g(z)r\cos(\theta), \alpha(r)+\beta(r)\gamma(z)\right).$$

Notice that there is a uniform bound on all of the partial derivatives of G over C and therefore, G will satisfy a global Lipschitz condition over C. G will be extended to all of R^3 in such a way that G will satisfy a Lipschitz condition with the same constant everywhere and for each point $p \in R^3$ of the δ-neighborhood of the boundary of C, $G(p) = (0, 0, 1)$. Under these conditions, we will now observe that G and its generated dynamical system Φ have the eight desired properties outlined above. G is defined so that properties (1), (3), and (7) are satisfied. Property (2) is satisfied because $(r\cos(\theta), r\sin(\theta), 0)$ and $(-f(r)g(z)r\sin\theta, f(r)g(z)r\cos(\theta), 0)$ are orthogonal. Properties (4), (5), and (6) follow from properties (2), (3), and (7), and from the fact that $y' = G(y)$ has a unique solution. Property (8) follows from the reversed symmetry of G in the upper and lower halves of C, and from the uniqueness of the solution of $y' = G(y)$.

Now extend G to the set of all points (u, v, w) such that $0 \leqslant w \leqslant 6\delta$ as follows. If there exists an integer pair (i, j) and a point (x, y, z) of C such that $(u, v, w) = (x+8i\delta, y+8j\delta, z)$ then set $G(u, v, w) = G(x, y, z)$; otherwise set $G(u, v, w) = (0, 0, 1)$. Now extend G to the set of all points (u, v, w) such that $0 \leqslant w \leqslant (6\times 64)\delta$ as follows. Let $[a_0, a_1, \ldots, a_{63}]$ denote the point sequence

$$[(0, 0), (0, \delta), \ldots, (0, 7\delta), (\delta, 0), \ldots, (\delta, 7\delta), \ldots, (7\delta, 7\delta)].$$

Let i denote the integer such that $6i\delta < w \leqslant 6(i+1)\delta$ and set $G(u, v, w) = G(x, y, z)$ where $(x, y)+a_i = (u, v)$ and $z+6i\delta = w$. Extend G to all of R^3 as follows. If (u, v, w) is a point of R^3 such that w is not in $[0, (6\times 64)\delta]$, let i denote the integer such that

$$(6\times 64)i\delta < w \leqslant (6\times 64)(i+1)\delta.$$

Now set

$$G(u, v, w) = G(u, v, w-(6\times 64)i\delta).$$

This completes the description of the example.

For integers i, j, and k, $0 \leqslant k \leqslant 63$, put

$$A_{i,j,k} = \{(x, y) \in R^2\colon\ \delta \leqslant \|(x, y)-[a_k+(8i\delta, 8j\delta)]\| \leqslant 2\delta\}.$$

Notice that the union of all annuli $A_{i,j,k}$ is R^2. Denote by B_λ (where $\lambda = (i, j, k, n, m)$, i, j, k, and n are integers, $0 \leqslant k \leqslant 63$, and $m = 2$ or 4) the annulus in R^3,

$$\{(x, y, z) \in R^3\colon\ (x, y) \in A_{i,j,k} \text{ and } z = [(64\times 6)n+6k+m]\delta\}.$$

Each of B_λ is invariant under Φ, and if $p \in B_\lambda$ then

$$\|\Phi(t, p)-p\| \leqslant 4\delta < \varepsilon.$$

If p is not on one of these annuli then either: (1) There are two annuli B_{λ_1} and B_{λ_2} where $\lambda_1 = (i, j, k, n, 2)$ and $\lambda_2 = (i, j, k, n, 4)$ such that the trajectory $\Phi(t, p)$ is "between" B_{λ_1} and B_{λ_2} with the α-limit points of the trajectory $\Phi(t, p)$ on B_{λ_1}, and the ω-limit points of the trajectory $\Phi(t, p)$ on B_{λ_2}. Then

$$\|\Phi(t, p)-p\| < (4+2)\delta < \varepsilon;$$

or (2) There is a number t_0 and an integer d such that if $\Phi(t_0, p)$ is denoted by (u, v, w) then $w = 6d\delta$. The line perpendicular to the xy-plane, and passing through $\Phi(t_0, p)$ intersects two annuli B_{λ_1} (below $\Phi(t_0, p)$), and B_{λ_2} (above $\Phi(t_0, p)$) so that the α-limit points of the trajectory $\Phi(t, p)$ are on B_{λ_1} and the ω-limit points of the trajectory $\Phi(t, p)$ are on B_{λ_2} (see Figure 2). The distance between B_{λ_1} and B_{λ_2} is less than $(64\times 6)\delta$. The projection of the trajectory $\Phi(t, p)$ on the xy-plane has diameter less than 16δ, since the projection of $\Phi(t, p)$, for any t, on the xy-plane is in a distance less than 6δ from the projection of B_{λ_1} on the xy-plane. Therefore,

$$\|\Phi(t, p)-p\| \leqslant [(64\times 6)+16]\delta = 400\delta = \varepsilon.$$

The only periodic trajectories are on the annuli B_λ. Hence, for no non-zero integer n, $\Phi(n, p) = p$.

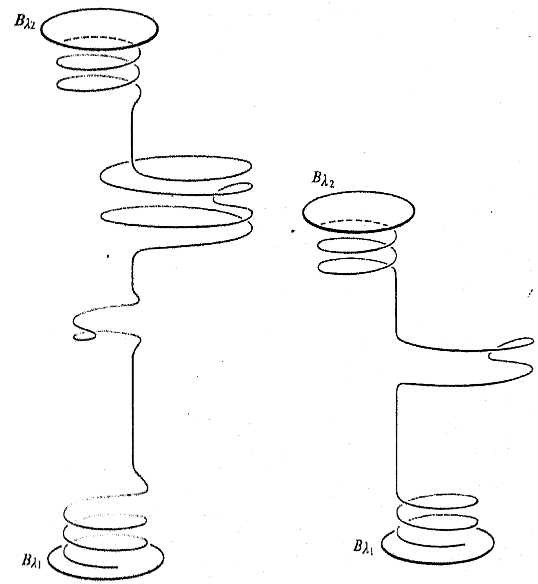


Fig. 2

Remark. A similar dynamical system can be constructed on a closed manifold $S^1\times S^1\times S^1$, where S^1 denotes the one dimensional sphere.

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Accepté par la Rédaction le 21. 2. 1980

Cardinal functions on compact F-spaces and on weakly countably complete Boolean algebras *

by

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Abstract. We investigate limitations on the cardinals $\varkappa$ which occur as the value of cardinal functions on infinite compact F-spaces (or on weakly countably complete Boolean algebras). We find limitations of the form $\varkappa^\omega = \varkappa$, or else $\mathrm{cf}(\varkappa) = \omega$, or at least "$\varkappa$ is not a strong limit with $\mathrm{cf}(\varkappa) = \omega$", and show that all infinite cardinals $\varkappa$ with $\varkappa^\omega = \varkappa$ do occur (for cardinality one needs the additional restriction $\varkappa \geqslant 2^{2^\omega}$, as is well known).

Contents

1. Introduction. This is a paper on the behavior of cardinal functions on compact F-spaces. The Boolean algebras which occur as the algebra of clopen (= closed and open) sets of a zero-dimensional compact F-space are the weakly countably complete Boolean algebras, or WCC algebras for short, see § 6 for the definition. This class includes the class of countably complete Boolean algebras and has the pleasant property of being closed under homomorphisms. (However, it is consistent

* Completed while supported by NSF-Grant MCS78-09484.