A rest point free dynamical system on $\mathbb{R}^3$ with uniformly bounded trajectories

by

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Abstract. In this paper, we show that if $c > 0$, then there exists a $C^c$ transformation $G$ from $\mathbb{R}^3$ into $\mathbb{R}^3$ such that the unique solution $\Phi$ of the differential equation $\dot{y} = G(y)$ is a dynamical system (a continuous transformation from $\mathbb{R} \times \mathbb{R}^3$ into $\mathbb{R}^3$ such that $\Phi(t, p) = \Phi(t, \Phi(t, p)) = \Phi(t, p)$ and $\Phi(0, p) = G(p)$) with the following two properties: (1) For each point $p$ in $\mathbb{R}^3$ and each number $n$, $\Phi(t, p)$ is in the $n$-neighborhood for $p$; and (2) for each integer $n \neq 0$, $\Phi(t, p) \neq p$. Notice that Scottish Book Problem number 110 of Ulam follows as a corollary where $f(p) = \Phi(1, p)$ and the manifold is $\mathbb{R}^3$.

Introduction. In 1935 S. Ulam raised the following question [7], Problem 110: "Let $M$ be a given manifold. Does there exist a numerical constant $K$ such that every continuous mapping $f$ of the manifold $M$ into itself which satisfies the condition $|f^n(x) - x| < K$ for $n = 1, 2, \ldots$ (where $f^n(x)$ denotes the $n$th iteration of the image $f(x)$) possesses a fixed point: $f^n(x) = x$? The same under more general assumptions about $M$ (general continuum?)." In this paper, we solve this problem in the negative, where $M = \mathbb{R}^3$, $f$ is a homeomorphism onto $f$, is $C^c$, and for each $x \in \mathbb{R}^3$ and each positive integer $n$, $f^n(x) \neq x$. Moreover, $f(x) = \Phi(1, x)$, where $\Phi$ is a $C^c$ dynamical system on $\mathbb{R}^3$ with uniformly bounded trajectories.

By a dynamical system $\Phi$ on a metric space $X$ we mean a continuous mapping $\Phi: R \times X \to X$ (where $R$ is the set of real numbers) such that for each $t \in R$, $\Phi(t \times X) = X$, and such that if each of $t_1$ and $t_2$ is a number and $p \in X$ is a point, then $\Phi(t_1 + t_2, p) = \Phi(t_1, \Phi(t_2, p))$ and $\Phi(0, p) = p$. If $G$ is a transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$, then $G$ is said to generate a dynamical system $\Phi$ provided that,

$$\lim_{t \to \infty} \Phi(t, p) = G(p).$$

The set of all points $\Phi(t, p)$ for a fixed $p$ and $-\infty < t < +\infty$ is called a trajectory of the dynamical system. A point $q$ is called an $a$-limit point of a trajectory $\Phi(t, p)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots \to +\infty$ such that $lim \Phi(t_n, p) = q$. A point $q$ is called an $a$-limit point of a trajectory $\Phi(t, p)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots \to +\infty$ such that $lim \Phi(t_n, p) = q$.

A classical result which we will employ is the following: If $G$ is a transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ satisfying globally a Lipschitz condition with constant $L$, then the differential equation $\dot{y} = G(y)$ has a unique solution for each initial condition and...
the dynamical system \( \Phi \) generated by \( G \) describes this solution set. See, for instance, [6] Chapter 1.

There are two examples in the literature of dynamical systems on \( R^3 \) with all trajectories bounded and no rest points. The first is due to Jones and Yorke [5]. The main idea of this example is to describe a monotonically increasing sequence of tori in \( R^3 \), whose union is \( R^3 \), and to define a dynamical system \( \Phi(t, p) \) such that for each fixed \( t \), \( \Phi(t, p) \) restricted to the surface of any of the tori is a rotation.

Therefore, it is not possible to obtain a uniform bound on the trajectories in this example. The second example was described by Brechner and Mauldin [4] and was based on the observation of Howard Cook that the acyclic Peano continuum without the fixed point property constructed by Borsuk [1], [3] can be used to define a dynamical system on \( R^3 \), with no rest points and all trajectories bounded. In this example, there is a neighborhood of the \( z \)-axis so that outside of this neighborhood, points follow circular trajectories parallel to the \( xy \)-plane with center on the \( z \)-axis. Therefore, the trajectories are not uniformly bounded.

The second part of the question of S. Ulam [7], Problem 110, has been answered in the negative by W. Kuperberg, who gave an example of a one dimensional metric continuum, which for every \( \varepsilon > 0 \) admits a fixed point free \( \varepsilon \)-involution. Subsequently, W. Kuperberg and P. Minic, using Borsuk's example described in [3] and Cook's idea, proved that the Cartesian product of the Hilbert cube \( Q \) and the circle \( S^1 \) has the property: for every \( \varepsilon > 0 \) there exists a dynamical system \( \Phi \) on \( Q \times S^1 \) such that for each \( p \in Q \times S^1 \) the trajectory \( \Phi(t, p) \) is of diameter less than \( \varepsilon \), and \( \Phi(n, p) \neq p \) for each nonzero integer \( n \).

**The example.** Suppose that \( \varepsilon > 0 \). We will construct a \( C^\infty \) transformation \( G \) from \( R^2 \) into \( R^3 \) satisfying globally a Lipschitz condition with constant \( L \) such that the dynamical system \( \Phi \) generated by \( G \) satisfies the following two properties:

1. If \( t \) is a number and \( p \) is a point, then \( \Phi(t, p) \) is in the \( \varepsilon \)-neighborhood of \( p \);
2. If \( n \) is an integer distinct from zero, then \( \Phi(n, p) \neq p \).

Set \( \delta = \varepsilon/400 \). \( G \) will first be defined on the closed solid cylinder \( C \) consisting of all points \((x, y, z)\) satisfying \( \sqrt{x^2 + y^2} \leq \delta \) and \( 0 \leq z \leq 6 \). Now, set \( T = \{(x, y, z) : \delta < \sqrt{x^2 + y^2} \leq 2\delta \} \) and for each number \( b \) set \( T_b = \{(x, y, z) \in T : z = b\} \).

\( G \) will satisfy the following eight conditions:

1. For each point \( p \) in \( C \) the \( \varepsilon \)-neighborhood of \( p \) is connected;
2. If \( p \in C \) and \( \Phi(t, p) \in C \), then \( p \) and \( \Phi(t, p) \) are equidistant from the \( z \)-axis;
3. Each of the annuli \( T_b \) and \( T_{b + \varepsilon} \) is invariant under \( \Phi \), and \( \Phi \) is a rotation on \( T_{b + \varepsilon} \) and on \( T_b \) such that for each integer \( n \) distinct from zero \( \Phi(n, p) \neq p \); (4) if \( 0 < \varepsilon < 2\delta \) and \( p \in T_b \), then there is a negative number \( t \) such that \( \Phi(t, p) \in T_0 \) and each \( \omega \)-limit point of the trajectory \( \Phi(t, p) \) is on \( T_0 \); (5) if \( 2\delta - \varepsilon < 4\delta \) and \( p \in T_b \), then each \( \omega \)-limit point of the trajectory \( \Phi(t, p) \) is on \( T_{b + \varepsilon} \) and each \( \omega \)-limit point of the trajectory \( \Phi(t, p) \) is on \( T_{b + \varepsilon} \); (6) if \( \delta < \varepsilon < 2\delta \), and \( p \in T_b \), then there is a positive number \( t \) such that \( \Phi(t, p) \in T_{b + \varepsilon} \); (7) if \( (x, y, z) \in T \), then \( \Phi(t, p) \in T \), and \( \Phi(t, (x, y, z)) = (u, v, w) \) for some \( t > 0 \), then \( w > z \); (8) if \( (x, y, z) \in C \), then \( \Phi(t, p) \) is connected.

Let \( h \) denote a strictly increasing \( C^\infty \) function on \([0, \delta]\) such that \( h(0) = 0 \), \( h(\delta) = 1 \), and all of the derivatives of \( h \) at zero and one are zero. Now define the five real valued functions \( f, g, \alpha, \beta, \gamma \) as follows.

\[
\begin{align*}
f(t) &= \begin{cases} 1 & \text{if } 0 < r < 2\delta, \\
\frac{r}{2\delta - r} & \text{if } 2\delta - r < 2\delta, \\
0 & \text{if } 3\delta < r < 4\delta, \\
\end{cases} \\
g(t) &= \begin{cases} 0 & \text{if } 0 < z < \delta, \\
h(z - \delta) & \text{if } \delta < z < 2\delta, \\
h(2\delta - z) & \text{if } 2\delta < z < 3\delta, \\
0 & \text{if } 3\delta < z < 6\delta, \\
\end{cases} \\
\alpha(t) &= \begin{cases} 0 & \text{if } 0 < \alpha < \delta, \\
h(-\alpha) & \text{if } \delta < \alpha < 2\delta, \\
0 & \text{if } 2\delta < \alpha < 3\delta, \\
\end{cases} \\
\beta(t) &= 1 - \alpha(t), \\
\gamma(t) &= \begin{cases} -h(z) & \text{if } 0 < z < \delta, \\
h(z - \delta) & \text{if } \delta < z < 2\delta, \\
0 & \text{if } 3\delta < z < 6\delta. \\
\end{cases}
\end{align*}
\]
Now for each point \( p = (r \cos(\theta), r \sin(\theta), z) \) of \( C \), set
\[
G(p) = (-f(r)g(z)\sin(\theta), f(r)g(z)\cos(\theta), \alpha(r) + \beta(r)\gamma(z)).
\]

Notice that there is a uniform bound on all of the partial derivatives of \( G \) over \( C \) and therefore, \( G \) will satisfy a global Lipschitz condition over \( C \). \( G \) will be extended to all of \( \mathbb{R}^3 \) in such a way that \( G \) will satisfy a Lipschitz condition with the same constant everywhere and for each point \( p \in \mathbb{R}^3 \) of the \( \delta \)-neighborhood of the boundary of \( C \), \( G(p) = (0, 0, 1) \). Under these conditions, we will now observe that \( G \) and its generated dynamical system \( \Phi \) have the eight desired properties outlined above. \( G \) is defined so that properties (1), (3), and (7) are satisfied. Property (2) is satisfied because \( (r \cos(\theta), r \sin(\theta), 0) \) and \( (-f(r)g(z)\sin(\theta), f(r)g(z)\cos(\theta), 0) \) are orthogonal. Properties (4), (5), and (6) follow from properties (2), (3), and (7), and from the fact that \( y' = G(y) \) has a unique solution. Property (8) follows from the reversed symmetry of \( G \) in the upper and lower halves of \( C \) and from the uniqueness of the solution of \( y' = G(y) \).

Now extend \( G \) to the set of all points \( (u, v, w) \) such that \( 0 \leq w \leq 6\delta \) as follows. If there exists an integer pair \((i, j)\) and a point \((x, y, z) \in C \) such that \( (u, v, w) = (x + 8i\delta, y + 8j\delta, z) \) then set \( G(u, v, w) = G(x, y, z) \); otherwise set \( G(u, v, w) = (0, 0, 1) \). Extend \( G \) to all of \( \mathbb{R}^3 \) as follows. If \((u, v, w) \) is a point of \( \mathbb{R}^3 \) such that \( w \) is not in \([0, (6 \times 64)\delta]\), let \( i \) denote the integer such that \((6 \times 64)\delta < w \leq (6 \times 64)(i + 1)\delta \).

Now set
\[
G(u, v, w) = G(u, v, w - (6 \times 64)i\delta). \]

This completes the description of the example.

For integers \( i, j, k, n, \) and \( k, 0 \leq k \leq 63, \) put
\[
A_{i,j,k} = \{(x, y) \in \mathbb{R}^2 : \delta \leq \| (x, y) - (a_i + (8i, 8j)) \| \leq 2\delta \}.
\]

Notice that the union of all annuli \( A_{i,j,k} \) is \( \mathbb{R}^2 \). Denote by \( B_{i,j,k,n} \) (where \( \lambda = (i, j, k, n, m) \)), \( i, j, k, n, \) and \( m \) are integers, \( 0 \leq k \leq 63, \) and \( m = 2 \) or \( 4 \) the annulus in \( \mathbb{R}^2 \),
\[
[(x, y) \in \mathbb{R}^2 : (x, y) \in A_{i,j,k} \text{ and } z = [(6 \times 64)n + 64k + m] \delta].
\]

Each of \( B_{i,j,k} \) is invariant under \( \Phi \), and if \( p \in B_{i,j,k} \)
\[
|\Phi(t, p) - p| \leq 4\delta + \varepsilon. \]

If \( p \) is not on one of these annuli then either: (1) There are two annuli \( B_{i,j,k} \) and \( B_{i,j,k,n} \), where \( A_1 = (i, j, k, n, 2) \) and \( A_2 = (i, j, k, n, 4) \) such that the trajectory \( \Phi(t, p) \) is "between" \( B_{i,j,k} \) and \( B_{i,j,k,n} \) with the \( a \)-limit points of the trajectory \( \Phi(t, p) \) on \( B_{i,j,k} \), and the \( a \)-limit points of the trajectory \( \Phi(t, p) \) on \( B_{i,j,k,n} \). Then
\[
|\Phi(t, p) - p| \leq (4 + 2\delta + \varepsilon);
\]
or (2) There is a number \( t_0 \) and an integer \( d \) such that if \( \Phi(t_0, p) \) is denoted by \( (u, v, w) \) then \( w = 66\delta \). The line perpendicular to the \( xy \)-plane, and passing through \( \Phi(t_0, p) \) intersects two annuli \( B_{i,j,k} \) (below \( \Phi(t_0, p) \)), and \( B_{i,j,k,n} \) (above \( \Phi(t_0, p) \)) so that the \( a \)-limit points of the trajectory \( \Phi(t, p) \) are on \( B_{i,j,k} \) and the \( a \)-limit points of the trajectory \( \Phi(t, p) \) are on \( B_{i,j,k,n} \) (see Figure 2). The distance between \( B_{i,j,k} \) and \( B_{i,j,k,n} \) is less than \((64 \times 64)\delta \). The projection of the trajectory \( \Phi(t, p) \) on the \( xy \)-plane has diameter less than \( 16\delta \), since the projection of \( \Phi(t, p) \), for any \( t \), on the \( xy \)-plane is in a distance less than \( 6\delta \) from the projection of \( B_{i,j,k} \) on the \( xy \)-plane. Therefore,
\[
|\Phi(t, p) - p| \leq [(6 \times 64) + 16] \delta = 400\delta = \varepsilon.
\]

The only periodic trajectories are on the annuli \( B_{i,j,k} \). Hence, for no non-zero integer \( n, \Phi(n, p) = p \).

**Remark.** A similar dynamical system can be constructed on a closed manifold \( S^3 \times S^1 \times S^1 \), where \( S^1 \) denotes the one dimensional sphere.
Cardinal functions on compact $F$-spaces and on weakly countably complete Boolean algebras

by

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Abstract. We investigate limitations on the cardinals $\kappa$ which occur as the value of cardinal functions on infinite compact $F$-spaces (or on weakly countably complete Boolean algebras). We find limitations of the form $\kappa^\aleph_0 = \kappa$, or else $\text{cf}(\kappa) = \omega$, or at least "$\kappa$ is not a strong limit with $\text{cf}(\kappa) = \omega$", and show that all infinite cardinals $\kappa$ with $\kappa^\aleph_0 = \kappa$ do occur (for cardinality one needs the additional restriction $\kappa \leq 2^\omega$, as is well known).

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1. Introduction. This is a paper on the behavior of cardinal functions on compact $F$-spaces. The Boolean algebras which occur as the algebras of clopen (= closed and open) sets of a zero-dimensional compact $F$-space are the weakly countably complete Boolean algebras, or WCC algebras for short, see § 6 for the definition. This class includes the class of countably complete Boolean algebras and has the pleasant property of being closed under homomorphisms. (However, it is consistent

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