Criteria of openness for relations

by

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Abstract. Under some hypotheses, nearly open graph-closed relations are open. The domain spaces considered are: Čech-complete spaces and groups, uniformly Čech-complete spaces, B-groups, B-complete vector spaces.

1. Introduction. The general problem studied in the paper is this: When is a nearly open and/or graph-closed relation open? It contains simultaneously the problems of openness and of continuity of functions, and was considered e.g. in [9], [14], [12] and [11]. Our main results are Theorems 1, 2 and 15, which state some criteria in topological spaces, topological groups and uniform spaces, respectively. They imply, among other things, some results of [1], [9], [7], [2], [3], [11] and [17]. The domain spaces of the relations considered here are mostly assumed to be Čech-complete. In the case of uniform spaces this topological assumption is not satisfactory; in Section 6 the notion of uniform Čech-completeness is introduced and investigated. B-groups [7] and B-complete vector spaces [13] are also considered, as domains.

2. Separating relations. Let X and Y be topological spaces, and let $R\subset X\times Y$. R is said to be *separating* [11] if for each pair of distinct points x_1, x_2 in X there are neighbourhoods U_i of x_i such that $\overline{R[U_1]}\cap R[U_2] = 0$. Such a relation is injective, i.e., $x_1 \neq x_2$ implies $R[x_1]\cap R[x_2] = 0$ (equivalently, R^{-1} is a function), $R^{-1}[Y]$ is a T_2-space, and R is a closed subset of $X\times R[X]$ (because $R\ni (x_\sigma, y_\sigma)\to (x_1, y)$, $(x_2, y)\in R$ and $x_1 \neq x_2$ lead to a contradiction). The last property implies that all images (pre-images of compact subsets of X (of $R[X]$) are closed in $R[X]$ (in X, respectively) [9; 6.A]. R is called *open* (*closed*) if all images of open (closed) subsets of X are open (closed) in Y; R is called *lower* (*upper*) *semicontinuous* if R^{-1} is open (closed). If R is a closed subset of $X\times Y$, we sometimes say that R is graph-closed. The following is a relation version of Proposition 7 from [17] (it can be proved similarly).

PROPOSITION 1. *Consider the following conditions*:

(1) *If* $x_\sigma\to x$ *and* $(x_\sigma, y_\sigma)\in R$, *then the net* $\{y_\sigma\}$ *has a cluster point in* $R[x]$ $(x_\sigma, x\in X,\ y_\sigma\in Y)$.

(2) *R is upper semicontinuous and all images of points are compact.*

(3) *R is graph-closed and all images of compact sets are compact.*

Then (1) $\Leftrightarrow$ (2). If $Y \in T_2$, then (1) $\Rightarrow$ (3). If X is a k-space, then (3) $\Rightarrow$ (2).

If X and Y are uniform spaces (i.e., $T_{3\frac{1}{2}}$-spaces with fixed uniformities $\mathscr{U}$ and $\mathscr{V}$, resp.) then R is said to be *uniformly open* if for every $U \in \mathscr{U}$ there is a $V \in \mathscr{V}$ such that $R[U[x]] \supset V[R[x]]$ for all $x \in X$; R is said to be *uniformly lower semicontinuous* if R^{-1} is uniformly open.

PROPOSITION 2. *Suppose that R is injective and $X \in T_2$. Each of the following conditions implies that R is separating:*

(4) *R is open.*

(5) *$Y \in T_2$ and R is a continuous* 1-1 *function.*

(6) *$Y \in T_2$ and* 1 (or (2)) *holds.*

(7) *X is a k-space, $Y \in T_2$ and* (3) *holds.*

(8) *$Y \in T_4$, R is upper semicontinuous and all images of points are closed.*

(9) *X is locally compact and R is graph-closed.*

(10) *Y is compact and R is graph-closed.*

(11) *X and Y are uniform spaces, R is uniformly lower semicontinuous and all images of points are compact.*

(12) *X and Y are* (T_0) *topological groups and R is a closed subgroup of $X \times Y$.*

Proof. Assertions (4), (6), and (8) are mentioned in [11]; (9) is equally easy. By Proposition 1, condition (7) implies (6); (5) and (10) also imply (6).

(11) We will prove that (1) holds. Given $V \in \mathscr{V}$, there is a $U \in \mathscr{U}$ such that $R^{-1}[V[y_\sigma]] \subset U[x_\sigma]$ for all σ; there are indices $\sigma(V)$, $V \in \mathscr{V}$, such that for $\sigma \geqslant \sigma(V)$ we have $z_\sigma^V \in R[x] \cap V[y_\sigma]$. The net $\{z_\sigma^V,\ V \in \mathscr{V}$ and $\sigma \geqslant \sigma(V)\}$ has a cluster point in $R[x]$. It follows that y is a cluster point of $\{y_\sigma\}$.

(12) Assume R is not separating. There are points x, v, x_U^V, v_U^V in X and y_U^V, z_U^V in Y such that $x \neq v$, $x_U^V \in xU \cap R^{-1}[y_U^V]$, $v_U^V \in vU \cap R^{-1}[z_U^V]$ and $y_U^V(z_U^V)^{-1} \in V$ for all neighbourhoods U, V of 1_X, 1_Y, resp. The net $\{(x_U^V(v_U^V)^{-1}, y_U^V(z_U^V)^{-1}).\ (U, V) \ni (1_X, 1_y)\} \subset R$ converges to $(xv^{-1}, 1_Y)$. Hence the last point is in R, so that $xv^{-1} = 1_X$; a contradiction.

Remark 1. Let $X \in T_2$ and $R = g^{-1}$, where g is a function on Y to X. If g has a Δ-closed graph in the sense of [17], then R is separating (proof a contrario as in (12)).

3. Criteria in topological spaces. Let us start with a characterization of openness in terms of nets (proof omitted).

PROPOSITION 3. *Suppose $R[X] \subset Y$. Then R is open iff $y_\sigma \to y \in R[x]$ implies there is a subnet $\{y_{\sigma'}\}$ and points $x_{\sigma'} \in R^{-1}[y_{\sigma'}]$ with $x_{\sigma'} \to x$.*

A relation R is called *nearly open* [13, 14] (or *almost open*) if for each open subset U of X, $R[U]$ is nearly open in Y, i.e., $R[U] \subset \operatorname{Int}\overline{R[U]}$. In case R is injective with $R^{-1} = f$, R is (nearly) open iff f is (nearly) continuous.

THEOREM 1. *Let X and Y be topological spaces, X being Čech-complete. Let R be a separating relation in $X \times Y$. If R is nearly open, then R is open as a relation from X to $R[X]$.*

Proof. $R \subset X \times R[X]$ is also separating and nearly open; we may suppose that $R[X] = Y$. Assume, to get a contradiction, that R is not open. There exists an open set G in X such that $\overline{R[G]} \cap R[X \setminus \bar{G}] \neq 0$ (otherwise $\overline{R[G]} \subset R[\bar{G}]$ for all G, and near-openness of R and regularity of X imply openness of R). Put $V_0 = G$ and $W_0 = X \setminus \bar{G}$. Let $\{\mathscr{U}_i\}$ be a complete sequence of open covers of X (see the beginning of Section 6). Let $y \in \overline{R[V_0]} \cap R[w]$ for a certain $w \in W_0$, and let W_1 be an open neighbourhood of w with diameter less than $\mathscr{U}_1$ and $\overline{W}_1 \subset W_0$. Since $\overline{R[W_1]}$ is a neighbourhood of y, there is a $z \in \overline{R[W_1]} \cap R[v]$ for a certain $v \in V_0$. Let V_1 be an open neighbourhood of v with diameter less than $\mathscr{U}_1$ and $\overline{V}_1 \subset V_0$. Then $\overline{R[V_1]}$ is a neighbourhood of z, so that $\overline{R[V_1]} \cap R[W_1] \neq 0$. Inductively, there are open sets V_i, $W_i \subset X$ such that $\overline{V}_{i+1} \subset V_i$, $\overline{W}_{i+1} \subset W_i$, V_i, W_i are of diameter less than $\mathscr{U}_i$ and $\overline{R[V_i]} \cap R[W_i] \neq 0$ for $i = 1, 2, \ldots$ Put $C = \bigcap_i V_i$ and $K = \bigcap_i W_i$; they are non-empty compact sets with open bases $\{V_i\}$, $\{W_i\}$, resp. (Up to this moment we have followed the proof of Theorem in [3].) Let us say that V, W separate A, B if V, W are open subsets of X containing A, B, resp., $\overline{R[V]} \cap R[W] = 0$. Fix $v \in C$. There are V_w, W_w that separate $\{v\}$, $\{w\}$ ($w \in K$). Let $w_1, \ldots, w_n \in K$ be such that $K \subset W_V = \bigcup_{i=1}^{n} W_{w_i}$. Put $V_V = \bigcap_{i=1}^{n} V_{w_i}$. Then, as can easily be seen, V_V, W_V separate $\{v\}$, K ($v \in C$). Let $v_1, \ldots, v_m \in C$ be such that $C \subset V = \bigcup_{i=1}^{m} V_{V_i}$. Put $W = \bigcap_{i=1}^{m} W_{V_i}$. Then, as can easily be checked, V, W separate C, K. There is an index i_0 such that $V_{i_0} \subset V$ and $W_{i_0} \subset W$. V_{i_0}, W_{i_0} separate C, K; a contradiction.

Remark 2. R need not be open to Y (take X dense and co-dense in Y, $R = \Delta_X$). Theorem 1 contains the theorem of [3] (by Proposition 2 (7)) and Theorem 5 of [11] (X complete metric); in fact, it was inspired by [11] and its proof modifies that of [3]. Theorem 1 contains also the theorem of [17] (by Remark 1), and is closely related to the Theorem (X the Rudin-complete Moore space) and the Conjecture ($X \in T_3$ strongly countably complete), both from [11; Added in revision]. Proposition 2 yields, directly or indirectly, many more consequences of Theorem 1; most of them can be found in [3], [17] and references therefrom. What is most important, Theorem 1 induces strong results in the case of groups (Section 4).

Now, let us restate two negative results (they answer some questions from [12]):

1° [3] There exists a continuous nearly open function f on X onto Y which in not open, where X is a certain separable complete metric space and Y is the unit interval [0, 1].

2° [18] There exists a nearly continuous nearly open graph-closed one-to-one function f on X onto Y which is neither continuous nor open, where X and Y are certain separable metric spaces, X — complete.

4. Criteria in topological groups. All groups (and vector spaces) considered in the paper are T_0-spaces (hence $T_{3\frac{1}{2}}$-spaces).

THEOREM 2. *Let G and H be topological groups, G being Čech-complete. Let R be a closed subgroup of $G\times H$. If R is nearly open, then R is open.*

Proof. It is sufficient to prove the theorem under the additional assumption that $R^{-1}[H]$ is dense in G (then consider R in $\overline{R^{-1}[H]}\times H$). The assumption guarantees that the closed subgroup $K = R^{-1}[1_H]$ of G is invariant. The quotient group $G_1 = G/K$ is Čech-complete and complete in its two-sided uniformity [2]. Consider the induced injective relation R_1 in $G_1\times H((\dot{x}, y)\in R_1$ iff $(x, y)\in R)$; R_1 is a subgroup of $G_1\times H$. To prove that R_1 is closed in $G_1\times H$, we will prove that $R_1\ni(x^1_\sigma, y)\to(\dot{x}, y)$ implies $(\dot{x}, y)\in R_1$. Since the quotient homomorphism of G onto G_1 is open, there is a subnet $\{x^1_{\sigma'}\}$ and points $x_{\sigma'}\in G$ with $\dot{x}_{\sigma'} = x^1_{\sigma'}$ and $x_{\sigma'}\to x$ (Proposition 3). Now $R\ni(x_{\sigma'}, y_{\sigma'})\to(x, y)$, so that $(x, y)\in R$. By Proposition 2 (12), R_1 is separating. Evidently R_1 is nearly open. By Theorem 1, R_1 is open as a relation from G_1 to $H_1 = R_1[G_1] = R[G]$. Hence R is open as a relation from G to H_1. It now remains to prove that H_1 is open in H. We will prove that H_1 is closed in H; then it is open, being nearly open. Let $H_1\ni y^1_\sigma\to y\in H$. Since $g = R_1^{-1}$ is a continuous homomorphism of H_1 to G_1, the net $\{g(y^1_\sigma)\}$ is two-sided Cauchy in G_1, and so converges to a point x_1 in G_1. Since R_1 is a closed subset of $G_1\times H$, $(x_1, y)\in R_1$. Hence $y\in H_1$, and the proof is complete.

Theorem 2 contains Theorem 6.R of [9] (additional hypotheses: G-locally compact or metrizable left-complete, R — a function on G or R^{-1} — a function on H), Theorem 31.3 of [7] ($\tilde{G}$-metrizable, R — a continuous function on G onto H), Theorem 4 of [2] (R — a continuous function on G). Corollaries 4 and 7 of [17] (R — a function on G or R^{-1} — a function on H).

G is called a *B-group* [7] if each continuous nearly open homomorphism of G onto another group is open (such are all Čech-complete groups).

THEOREM 3. *Let G and H be topological groups, G being an Abelian B-group. Let R be a closed subgroup of $G\times H$. If R is nearly open, then R is open as a relation from G to $R[G]$.*

Proof (notation as in Theorem 2). G_1 is a B-group as well; g is a nearly continuous graph-closed homomorphism of H_1 to G_1. By Theorem 2 of [8], g is continuous. This yields the assertion.

Remark 3. In [16] there is an example of an Abelian B-group G which is not complete (in its canonical uniformity). The embedding of G in its completion $\tilde{G}$ is continuous and nearly open but not open.

THEOREM 4. *Let G be a separable or Lindelöf group, and let H be a second category group. Let R be a closed subgroup of $G\times H$ with $R[G] = H$. If G is a Čech-complete group or an Abelian B-group, then R is open.*

Proof. In view of Theorems 2 and 3, it suffices to verify near-openness of R. In the Lindelöf case the proof proceeds just as in [9; 6.R]. The separable case needs slightly more care. Let D be a countable dense subset of G. Let U be a symmetric neighbourhood of 1_G. Put $D_1 = \{d\in D\colon dU\cap R^{-1}[H]\neq 0\}$, and choose $(x_d, y_d)\in R$ with $x_d\in dU$ for $d\in D_1$. Then

$$R^{-1}[H]\subset\bigcup_{d\in D_1} dU\subset\bigcup_{d\in D_1} x_d U^2 .$$

Hence

$$H = \bigcup_{d\in D_1} R[x_d U^2] = \bigcup_{d\in D_1} y_d R[U^2] .$$

Since H is second category, $\operatorname{Int}\overline{R[U^2]}\neq 0$. It follows that $1_H\in\operatorname{Int}\overline{R[U^4]}$, which proves the near-openness of R.

5. Criteria in topological vector spaces. The next theorem is an obvious consequence of Theorem 2 and well-known arguments e.g. from [9; 6.R] and [15; IV.8]. It unifies some classical open mapping and closed graph theorems of functional analysis.

THEOREM 5. *Let E and F be topological vector spaces, E being complete metric. Let R be a closed vector subspace of $E\times F$. If R is nearly open, then R is open. In particular, R is open provided $R[E] = F$ and either* (i) *F is of second category, or* (ii) *E and F are locally convex spaces and F is barrelled.*

Recall that E is complete metric iff E is Čech-complete [2]. A locally convex space E is called *B-complete* [13] (or a Pták space [15]) if each continuous nearly open linear mapping of E onto another locally convex space is open. Banach [1] essentially proved that each Fréchet space (i.e., complete metric l.c.s.) is B-complete. This is also provided by Theorem 5. Let us recall Theorem 1, based on duality theory, from [14], slightly improved.

THEOREM 6. *The assertion of Theorem 5 remains true if E and F are locally convex spaces, E being a Pták space.*

Proof. By Theorem 1 of [14]. R is open as a relation from E to $R[E]$. $E/R^{-1}[0]$ is a Pták space, hence complete (cf. [13] or [15]). Proceeding as in Theorem 2, one can prove that $R[E]$ is closed, and so open in F.

6. Uniformly Čech-complete spaces. Let C be a collection of covers of X; C is said to be *complete* [5] if any centred family $\mathcal{F}$ of closed subsets of X has a non-empty intersection, provided for each $\mathcal{C}\in C$ there is an $F\in\mathcal{F}$ of diameter less than $\mathcal{C}$ (i.e., $F\subset C$ for a certain $C\in\mathcal{C}$). A $T_{3\frac{1}{2}}$-space X is Čech-complete iff there exists a countable complete family of open covers of X (cf. [5] or [4]). Let X be a uniform space, i.e., a $T_{3\frac{1}{2}}$-space with a uniformity $\mathcal{U}$ on X (inducing the given topology). X is complete iff the family of all uniform covers of X is complete. X is uniformly locally compact iff there exists a uniform cover $\mathcal{C}$ of X such that the one-element family $\{\mathcal{C}\}$ is complete [9; 6.T]. In view of these facts, we think that the following definition is quite natural.

DEFINITION. A uniform space X is *uniformly Čech-complete* (UČC) if there exists a countable complete family of uniform covers of X.

If X is UČC, then X is complete and (topologically) Čech-complete. If X is uniformly locally compact, then X is UČC. In case $\mathcal{U}$ is metrizable, X is UČC iff X is complete [4; 4.3.10]. If X is UČC and $\mathcal{U}'$ on X is finer than $\mathcal{U}$, then X is UČC with respect to $\mathcal{U}'$. A closed subspace of a UČC space is UČC.

Let $\mathcal{U}_0 \subset \mathcal{U}$; a net $\{x_\sigma\}$ is said to be *$\mathcal{U}_0$-Cauchy* if for every U in $\mathcal{U}_0$ there is an index σ_U such that $(x_\sigma, x_{\sigma_U}) \in U$ whenever $\sigma \geqslant \sigma_U$.

PROPOSITION 4. *X is UČC iff there exists a countable subfamily $\mathcal{U}_0$ of $\mathcal{U}$ with the property that each $\mathcal{U}_0$-Cauchy net has a cluster point.*

Let d be a pseudometric on X, we say that d is *perfect* if the quotient mapping of X onto the metric space X/d is perfect. This is so iff $X \in T_2$, d is continuous and each d-convergent net has a cluster point in X (cf. Proposition 1).

THEOREM 7. *Let X be a uniform space. The following statements are equivalent:*

(i) *X is UČC;*

(ii) *X has a perfect complete uniform pseudometric d;*

(iii) *The family of all perfect complete pseudometrics on X generates the uniformity.*

Proof. (i) ⇒ (iii) Given $U_0 \in \mathcal{U}$, construct $\mathcal{U}_0 = \{U_i\}_{i=1}^{\infty} \subset \mathcal{U}$ so that $3U_i \subset U_{i-1}$ for $i = 1, 2, \ldots$ and $\mathcal{U}_0$ has the property from Proposition 4. By [4; 8.1.10], there exists a pseudometric d on X satisfying $U_i \subset U^d_{2^{-i}} \subset U_{i-1}$ for $i = 1, 2, \ldots$ The inclusions imply that d is uniform and each d-Cauchy net has a cluster point in X. Hence d is perfect and complete. Since U_0 was arbitrary, (iii) follows.

(ii) ⇒ (i) $\mathcal{U}_0 = \{U_i\}_{i=1}^{\infty}$, where $U_i = U^d_{i^{-1}}$, has the property from Proposition 4.

COROLLARY 1. *If there exists a uniformly continuous perfect mapping of X onto a UČC space Y, then X is UČC.*

COROLLARY 2. *The Cartesian product of countably many UČC spaces is UČC.*

Proof. The key argument is that a countable (in fact, arbitrary) product of perfect mappings is perfect (cf. [6] or [4]).

THEOREM 8. *Let X be a $T_{3\frac{1}{2}}$-space. The following conditions are equivalent:*

(i) *X is UČC with respect to some uniformity $\mathcal{U}$ on X;*

(ii) *X is UČC with respect to the finest uniformity $\mathcal{U}_f$ on X;*

(iii) *X is Čech-complete and paracompact.*

Proof. The first two conditions are clearly equivalent. By Theorem 7 (ii) and [4; 3.9.10 and 5.1.35], (i) implies (iii). The implication (iii) ⇒ (ii) follows from the fact that each open cover of X is uniform with respect to $\mathcal{U}_f$ provided X is paracompact [4; 8.5.13(d)].

If Y is a UČC space containing the uniform space X as a dense subspace, then $Y = \tilde{X}$, because the completion $\tilde{X}$ is (essentially) unique [4; 8.3.12]. Therefore X possesses "uniform Čech-completion" iff $\tilde{X}$ is UČC.

THEOREM 9. *Let X be a uniform space. The following statements are equivalent:*

(i) *$\tilde{X}$ is UČC;*

(ii) *There exists a countable subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that each $\mathcal{U}_0$-Cauchy net has a $\mathcal{U}$-Cauchy subnet;*

(iii) *There exists a uniform pseudometric d on X such that each d-Cauchy net has a $\mathcal{U}$-Cauchy subnet;*

(iv) *The family of all such pseudometrics d generates the uniformity.*

Proof. In view of Theorem 8, it suffices to prove that (ii) implies (i). Let $\mathcal{U}_0 = \{U_n\}_{n=1}^{\infty}$ be as in (ii). There are $\hat{U}_n \in \tilde{\mathcal{U}}$ with $U_n \supset \hat{U}_n \cap X \times X$ and $\hat{U}_n \supset 3\hat{U}_{n+1}$ for $n = 1, 2, \ldots$ Put $\hat{\mathcal{U}}_0 = \{\hat{U}_n\}_{n=1}^{\infty}$, and let $S = \{x_\sigma, \sigma \in \Sigma\}$ be a $\hat{\mathcal{U}}_0$-Cauchy net in $\tilde{X}$. Choose $x_\sigma^V \in X$ with $(x_\sigma, x_\sigma^V) \in V$ for $\sigma \in \Sigma$ and $V \in \hat{\mathcal{U}}$. The net

$$S' = \{x_\sigma^V, (\sigma, V) \in \Sigma \times \hat{\mathcal{U}}\}$$

is $\mathcal{U}_0$-Cauchy, and so has a $\mathcal{U}$-Cauchy subnet. The subnet is also $\hat{\mathcal{U}}$-Cauchy, and hence converges to some point x in $\tilde{X}$. It follows that x is a cluster point of S.

Now let G be a topological group. G_t (G_l) will denote the two-sided (left) uniform space of G, the corresponding uniformity denoted by $\mathcal{U}_t$ ($\mathcal{U}_l$). Notice that if G is locally compact, then both G_t and G_l are UČC spaces. Given a pseudometric d on G, define $d^*(x, y) = d(x, y) + d(x^{-1}, y^{-1})$ for $x, y \in G$ [9; 6.Q]. The next theorems of this section are based on the results of [2].

THEOREM 10. *Let G be a topological group. The following statements are equivalent:*

(i) *G_t is UČC;*

(ii) *G is Čech-complete;*

(iii) *There exists a compact subgroup K of G such that G/K is completely metrizable;*

(iv) *There exists a left-invariant pseudometric d on G such that d^* is perfect and complete;*

(v) *The family of all such pseudometrics d^* generates $\mathcal{U}_t$.*

Proof. The equivalence of (ii), (iii) and (iv) is proved in [2]. By Theorem 7, (iv) implies (i). Let $\{d_\alpha\}_{\alpha \in A}$ be the family of all left-invariant continuous pseudometrics on G. If (iv) holds, then the family $\{(d+d_\alpha)^*\}_{\alpha \in A}$ generates $\mathcal{U}_t$ and consists of perfect complete pseudometrics; therefore (v) holds. The implications (i) ⇒ (ii) and (v) ⇒ (iv) are obvious. The implication (i) ⇒ (iii) yields the following

COROLLARY. *Let E be a topological vector space over a number field, with its translation-invariant uniformity. E is UČC iff E is completely metrizable.*

THEOREM 11. *Let G be a topological group. The following statements are equivalent:*

(i) *G_l is UČC;*

(ii) *G is Čech-complete and G_l is complete;*

(iii) *There exists a perfect complete left-invariant pseudometric d on G;*

(iv) *$\mathcal{U}_l$ is generated by the family of all such pseudometrics d.*

Proof. (ii) $\Leftrightarrow$ (iii) is proved in [2]. By Theorem 7, (iii) implies (i). Clearly (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) (see the proof of Theorem 10).

THEOREM 12. *Let H be a closed invariant subgroup of a topological group G. (i) If G_t is UČC, then so is $(G/H)_t$. (ii) If G_l is UČC, then so is $(G/H)_l$.*

Proof. If G is Čech-complete, then so is G/H [2]. Taking into account Theorems 10 and 11, we get (i) and (ii), the latter — provided we know that $(G/H)_l$ is complete. This fact is a consequence of the forthcoming Theorem 16.

THEOREM 13. *Let G be a topological group. The following statements are equivalent:*

(i) *$\tilde{G}_t$ is UČC;*

(ii) *There exists a closed $\mathscr{U}_t$-totally bounded subgroup K of G such that G/K is metrizable;*

(iii) *There exists a continuous left-invariant pseudometric d on G with the property that each d^*-Cauchy net has a $\mathscr{U}_t$-Cauchy subnet;*

(iv) *$\mathscr{U}_t$ is generated by the family of all such pseudometrics d^*.*

Note. A subgroup is $\mathscr{U}_t$-totally bounded iff is $\mathscr{U}_r$-totally bounded.

Proof. Theorem 10 shows that (i) implies (iv)($\Rightarrow$ (iii)). A set K is totally bounded iff each net in K has a Cauchy subnet. It follows that (iii) implies (ii). Finally, we must prove that (ii) implies the Čech-completeness of the group $\tilde{G}_t$. In case K is compact, this is exactly Corollary 2 to Theorem 1 of [2]; its proof can be slightly modified so as to cover the general case.

It would be interesting to investigate when the uniform space $\tilde{G}_l$ is UČC.

7. Criteria in uniform spaces. Let X and Y be uniform spaces (with uniformities $\mathscr{U}$ and $\mathscr{V}$, resp.), and let R be a relation in $X\times Y$. R is called *uniformly open* if for every $U\in\mathscr{U}$ there is a $V\in\mathscr{V}$ such that for all $x\in X$

$$(*)\qquad R[U[x]]\supset V[R[x]]\,.$$

If the set on the left is replaced with its closure, the notion of uniform near-openness appears.

Notice that R is *closed* iff for each $y\in Y$ and each open set U in X with $U\supset R^{-1}[y]$ there is a neighbourhood V of y such that $U\supset R^{-1}[V]$. Thus R may be called *uniformly closed* if for every $U\in\mathscr{U}$ there is a $V\in\mathscr{V}$ such that for all $y\in Y$

$$(**)\qquad U[R^{-1}[y]]\supset R^{-1}[V[y]]\,.$$

PROPOSITION 5. *R is uniformly open iff R is uniformly closed.*

Proof. Notice that $y\in R[U[x]]$ iff $x\in U[R^{-1}[y]]$, $y\in V[R[x]]$ iff $x\in R^{-1}[V[y]]$. Therefore $(*)$ holds for all $x\in X$ iff $(**)$ holds for all $y\in Y$.

R is called *open* at y_0 if for each point $x\in R^{-1}[y_0]$ and each neighbourhood U of x, $R[U]$ is a neighbourhood of y_0. R is called *graph-closed* at y_0 if

$$R\ni(x_\sigma,y_\sigma)\to(x,y_0)\quad\text{implies}\quad(x,y_0)\in R\,.$$

LEMMA. *Let X be UČC, and let R be uniformly nearly open. Put $C=\{y\in Y\colon R$ is graph-closed at $y\}$. For every $U\in\mathscr{U}$ there is a $V\in\mathscr{V}$ such that $(**)$ holds for all $y\in C$, i.e., R is uniformly closed at the points of C. Hence, if R is injective and $R[X]=Y$, then for every $U\in\mathscr{U}$ there is a $V\in\mathscr{V}$ such that*

$$R[U[x]]\supset V[y]\,,\qquad x=R^{-1}[y]\,,\qquad y\in C\,,$$

i.e., R is uniformly open at the points of C.

Proof. Let d be a perfect complete uniform pseudometric on X, $\varepsilon>0$, $y\in C$. There are $V_n\in\mathscr{V}$ such that

$$V_n[R[x]]\subset\overline{R[U^d_{\varepsilon 2^{-n}}[x]]}\,,\qquad x\in X\,,\ n\in N\,,$$

where $U^d_\delta=\{(x_1,x_2)\colon d(x_1,x_2)<\delta\}$. Let $(x',y')\in R$ and $(y,y')\in V_2$; in view of Theorem 7 (iii) and the free choice of d and ε, it is sufficient to prove that $x'\in U^d_\varepsilon[R^{-1}[y]]$. Let $\mathscr{B}(y)$ be the set of all neighbourhoods of y directed by inclusion; put $G_n=V_n[y]\in\mathscr{B}(y)$. Since $y\in V_2[y']\subset\overline{R[U^d_{\varepsilon 2^{-2}}[x']]}$, there are $(x_2^G,y_2^G)\in R$ with $d(x',x_2^G)<\varepsilon 2^{-2}$ and $y_2^G\in G$ for $G\in\mathscr{B}(y)$. Since $y\in V_3[y_2^{G_3}]\subset\overline{R[U^d_{\varepsilon 2^{-3}}[x_2^{G_3}]]}$, there are $(x_3^G,y_3^G)\in R$ with $d(x_2^{G_3},x_3^G)<\varepsilon 2^{-3}$ and $y_3^G\in G$ for $G\in\mathscr{B}(y)$. Inductively, there are $(x_n^G,y_n^G)\in R$ with $d(x_{n-1}^{G_n},x_n^G)<\varepsilon 2^{-n}$ and $y_n^G\in G$ for $G\in\mathscr{B}(y)$, $n=2,3,\ldots$ The net $\{x_n^G,(n,G)\in N\times\mathscr{B}(y)\}$ is, as can easily be seen, d-Cauchy, and so has a cluster point $x\in X$, which satisfies also $d(x',x)\leqslant\varepsilon 2^{-1}$. The corresponding net $\{y_n^G\}$ converges to y. Since $y\in C$, $(x,y)\in R$. This proves what was required.

Remark 4. The lemma remains true if Y is any topological space and $\mathscr{V}$ is any family of symmetric neighbourhoods of the diagonal Δ_Y (i.e., sets V with $V=-V$ and $\operatorname{Int}V\supset\Delta_Y$, or if $(Y,\mathscr{V})$ is a Morita uniform space (cf. [11]).

The lemma and Proposition 5 yield the following two results:

THEOREM 14. *Let X and Y be uniform spaces, X — UČC. Let R be an injective uniformly nearly open relation in $X\times Y$ with $R[X]=Y$. If R is graph-closed at y_0, then R is open at y_0.*

THEOREM 15. *Let X and Y be uniform spaces, X being UČC. Let R be a closed subset of $X\times Y$. If R is uniformly nearly open, then R is uniformly open.*

For a complete metric space X, Theorem 15 gives Kelley's uniformly open relation theorem [9; 6.36]. Pettis [12] asked whether Kelley's theorem holds for an X complete in a more general sense, while in [11] it is proved for uniformly locally compact space X. Theorem 15 answers the question. It proves also that each UČC space is B-complete in the sense of [10]. The next theorem, for the metrizable case, in [9; 6.37].

THEOREM 16. *Let X and Y be uniform spaces, X being UČC. If there exists a continuous uniformly nearly open mapping of X onto Y, then Y is complete.*

Proof. The mapping f, considered as a map into $\tilde{Y}$, is also continuous and uniformly nearly open. By Theorem 15 and Proposition 5, f is closed. Hence $Y=\tilde{Y}$.

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A rest point free dynamical system on R^3 with uniformly bounded trajectories

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Abstract. In this paper, we show that if $\varepsilon > 0$, then there exists a C^∞ transformation G from R^3 into R^3 such that the unique solution Φ to the differential equation $y' = G(y)$ is a dynamical system (a continuous transformation from $R\times R^3$ into R^3 such that $\Phi(0,p) = p$, $\Phi(t_1, \Phi(t_2, p)) = \Phi(t_1+t_2, p)$ and $\partial/\partial t\Phi(0,p) = G(p)$) with the following two properties: (1) For each point p in R^3 and each number t, $\Phi(t,p)$ is in the ε-neighborhood for p; and (2) for each integer $n \neq 0$, $\Phi(n,p) \neq p$. Notice that Scottish Book problem number 110 of Ulam follows as a corollary where $f(p) = \Phi(1,p)$ and the manifold is R^3.

Introduction. In 1935 S. Ulam raised the following question [7], Problem 110: "Let M be a given manifold. Does there exist a numerical constant K such that every continuous mapping f of the manifold M into part of itself which satisfies the condition $|f^n(x)-x|<K$ for $n = 1, 2, \ldots$ (where $f^n(x)$ denotes the nth iteration of the image $f(x)$) possesses a fixed point: $f(x_0) = x_0$? The same under more general assumptions about M (general continuum?)." In this paper, we solve this problem in the negative, where $M = R^3$, f is a homeomorphism onto, f is C^∞, and for each $x \in R^3$ and each positive integer n, $f^n(x) \neq x$. Moreover, $f(x) = \Phi(1,x)$, where Φ is a C^∞ dynamical system on R^3 with uniformly bounded trajectories.

By a dynamical system Φ on a metric space X we mean a continuous mapping $\Phi\colon R\times X \to X$ (where R is the set of real numbers) such that for each $t \in R$, $\Phi(\{t\}\times X) = X$, and such that if each of t_1 and t_2 is a number and $p \in X$ is a point, then $\Phi(t_1, \Phi(t_2, p)) = \Phi(t_1+t_2, p)$ and $\Phi(0,p) = p$. If G is a transformation from R^3 into R^3, then G is said to *generate a dynamical system* Φ provided that, for each point $p \in R^3$, $\lim\limits_{t\to 0}\dfrac{\Phi(t,p)-p}{t} = G(p)$.

The set of all points $\Phi(t,p)$ for a fixed p and $-\infty < t < +\infty$ is called a *trajectory of the dynamical system*. A point q is called an *ω-limit point of a trajectory* $\Phi(t,p)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots \to +\infty$ such that $\lim \Phi(t_n, p) = q$. A point q is called an *α-limit point of a trajectory* $\Phi(t,p)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots \to -\infty$ such that $\lim\Phi(t_n,p) = q$.

A classical result which we will employ is the following: If G is a transformation from R^3 into R^3 satisfying globally a Lipschitz condition with constant L, then the differential equation $y' = G(y)$ has a unique solution for each initial condition and