Homotopies of small categories

by

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Abstract. Applying the theory of cubical sets, the author defines the homotopy groups of any object of the category Cat* of pointed small categories. The notion of Serre fibration is introduced in the category of small categories Cat, this fibration induces a long exact sequence of the homotopy groups. Moreover, it is proved that the localization of any small category with respect to the set of all its morphisms is a groupoid assigned to the respective cubical set (see [2]).

The constructions of this paper suggest that the category pro-$\mathscr{C}at$ is a model category (see [7]).

0. Preliminaries. Let $\mathscr{C}$ be any category. A *cohomotopy* (see [5]) in the category $\mathscr{C}$ is a quadruple $(P; p_0, p_1, s)$, where $P\colon \mathscr{C} \to \mathscr{C}$ is a functor, whereas $p_0, p_1\colon P \to \mathbf{1}_{\mathscr{C}}$, $s\colon \mathbf{1}_{\mathscr{C}} \to P$ are such natural transformations of functors that $p_0 s = p_1 s = \mathbf{1}_{\mathscr{C}}$.

Consider a sequence of functors $P^n\colon \mathscr{C} \to \mathscr{C}$, where $P^0 = \mathbf{1}_C$, $P^{n+1} = P(P^n)$, for $n \geqslant 0$ and natural transformations

$$d^{\delta}_{i,n} = p^{i-1} p_{\delta} P^{n-i}\colon P^n \to P^{n-1}, \quad i = 1, \ldots, n;\ \delta = 0, 1$$

and

$$s_{i,n} = P^{i-1} s P^{n+1-i}\colon P^n \to P^{n+1}, \quad i = 1, \ldots, n+1 .$$

The category $\mathscr{C}^{\square^{\mathrm{op}}}$ of contravariant functors defined on the cubical category $\square$ (see [6]), having the values in the category $\mathscr{C}$, is called a *category of cubical objects* over the category $\mathscr{C}$.

If the category $\mathscr{C}$ is a category of sets $\mathscr{S}et$ (a category of pointed sets $\mathscr{S}et^*$), then the category $\mathscr{S}et^{\square^{\mathrm{op}}}$ ($\mathscr{S}et^{*\square^{\mathrm{op}}}$) is called a *category of cubical sets* (a *category of pointed cubical sets*) (see [6]).

LEMMA (see [5]). *A sequence of functors and natural transformations* $(P^n; d^{\delta}_{i,n}, s_{i,n})n \geqslant 0$ *defines a cubical object in the category of endofunctors of the category* $\mathscr{C}$.

In particular, for $X, Y \in \operatorname{ob}\mathscr{C}$ $\big(\mathscr{C}(X, P^n(Y)), d^{\delta}_{i,n}, s_{i,n}\big)n \geqslant 0$ *is a cubical set.*

Denote by $\mathscr{C}at$ the category of small categories, and by $\mathscr{C}at^*$ — the category of pointed small categories.

Following G. Hoff (see [3], [4]), we show that the category $\mathscr{C}at$ ($\mathscr{C}at^*$) has a natural structure of a category with cohomotopy. In order to do this consider the category $\mathscr{Z}$ defined in the following way: $\mathscr{Z} = \ldots \leftarrow -2 \rightarrow -1 \leftarrow 0 \rightarrow 1 \leftarrow 2 \rightarrow \ldots$

For any small category $\mathscr{C}$ the functor $\sigma\colon \mathscr{Z} \to \mathscr{C}$ is called *finite* iff there exist $m_0, n_0 \in \mathrm{ob}\mathscr{Z}$ such that $\sigma(m) = \sigma(m_0)$, $\sigma(n) = \sigma(n_0)$ and $\sigma(m \to m') = \mathbf{1}_{\sigma(m_0)}$, $\sigma(n \to n') = \mathbf{1}_{\sigma(n_0)}$ for $m, m' \leqslant m_0$ and $n, n' \geqslant n_0$. The above conditions will be written briefly as $\sigma(-\infty) = \sigma(m_0)$ and $\sigma(+\infty) = \sigma(n_0)$. A full subcategory of the finite functors of the category $\mathscr{C}at(\mathscr{Z}, \mathscr{C})$ is denoted by $P(\mathscr{C})$.

Remark that $P\colon \mathscr{C}at \to \mathscr{C}at$ is a functor and that for any small category $\mathscr{C}$ there are functors $s(\mathscr{C})\colon \mathscr{C} \to P(\mathscr{C})$, $p_0(\mathscr{C})$, $p_1(\mathscr{C})\colon P(\mathscr{C}) \to (\mathscr{C})$ defined in the following way: $s(\mathscr{C})(\mathscr{C})(k) = C$, for $C \in \mathrm{ob}\mathscr{C}$, $k \in \mathrm{ob}\mathscr{Z}$ and $p_0(\mathscr{C})(\sigma) = \sigma(-\infty)$, $p_1(\mathscr{C})(\sigma = \sigma(+\infty)$, for $\sigma \in \mathrm{ob}P(\mathscr{C})$.

It will now be shown that the functor $P\colon \mathscr{C}at \to \mathscr{C}at$ is prorepresentable. In order to do that consider, for all $m \leqslant n$, a full subcategory ${}_m\mathscr{Z}_n$ of the category $\mathscr{Z}$ such that $k \in \mathrm{ob}_m\mathscr{Z}_n$ iff $m \leqslant k \leqslant n$ and a full embedding $\alpha_n\colon {}_{-n+1}\mathscr{Z}_{n+1} \to {}_{-n}\mathscr{Z}_n$ such that $\alpha_n(k) = k$, for $-n \leqslant k \leqslant n$ and $\alpha_n(\pm(n+1)) = \pm n$. Then $P(\mathscr{C}) = \operatorname{colim}_n \mathscr{C}at({}_{-n}\mathscr{Z}_n, \mathscr{C})$.

The functors $s(\mathscr{C})\colon \mathscr{C} \to P(\mathscr{C})$, $p_0(\mathscr{C})$, $p_1(\mathscr{C})\colon P(\mathscr{C}) \to \mathscr{C}$ induce natural transformations $s\colon \mathbf{1} \to P$, $p_0, p_1\colon P \to \mathbf{1}$ such that $p_0 s = p_1 s = \mathbf{1}$. Thus the quadruple $(P; p_0, p_1, s)$ is a cohomotopy in the category $\mathscr{C}at$. By the lemma the cohomotopy $(P; p_0, p_1, s)$ in the category $\mathscr{C}at$ induces the functor $Q\colon \mathscr{C}at \times \mathscr{C}at \to \mathscr{S}et^{\square^{\mathrm{op}}}$ where $Q(\mathscr{C}, \mathscr{D}) = (\mathscr{C}at(\mathscr{C}, P^n(\mathscr{D})), d_{i,n}, s_{i,n})n \geqslant 0$, for $\mathscr{C}, \mathscr{D} \in \mathrm{ob}\mathscr{C}at$.

For $\mathscr{C} = *$ ($*$ is a point category), we get the functor $Q\colon \mathscr{C}at \to \mathscr{S}et^{\square^{\mathrm{op}}}$, where $Q(\mathscr{C}) = (\zeta(P^n(\mathscr{C})); d^{\delta}_{i,n}, s_{i,n})n \geqslant 0$, while $\zeta\colon \mathscr{C}at \to \mathscr{S}et$ is a forgetful functor, i.e. $\zeta(\mathscr{C}) = \mathrm{ob}\mathscr{C}$, for $\mathscr{C} \in \mathrm{ob}\mathscr{C}at$; of course, any functor $F\colon \mathscr{C} \to \mathscr{D}$ induces a map of sets $\zeta(\mathscr{F})\colon \zeta(\mathscr{C}) \to \zeta(\mathscr{D})$.

It is easy to observe that any element $\sigma \in Q_k(\mathscr{C}) = \mathrm{ob}P^k(\mathscr{C})$ induces a functor $\bar{\sigma}\colon \mathscr{Z}^k \to \mathscr{C}$ such that

$$\bar{\sigma}(m_1, \ldots, m_{i-1}, \pm\infty, m_{i+1}, \ldots, m_k) = \bar{\sigma}(m_1, \ldots, m_{i-1}, \pm n_0, m_{i+1}, \ldots, m_k)$$

for a certain $n_0 \in \mathrm{ob}\mathscr{Z}$ and $i = 1, \ldots, k$, ($\mathscr{Z}^k$ denotes the kth power of the category $\mathscr{Z}$).

The elements $\sigma \in Q_k(\mathscr{C})$ will be identified with the appropriate functors $\bar{\sigma}\colon \mathscr{Z}^k \to \mathscr{C}$.

Thus we have $(d^{\delta}_i\sigma)(m_1, \ldots, m_{k-1}) = \sigma(m_1, \ldots, m_{i-1}, (-1)^{\delta+1}\infty, m_i, \ldots, m_{k-1})$ and $(s_i\sigma)(m_1, \ldots, m_{k+1}) = \sigma(m_1, \ldots, \hat{m}_i, \ldots, m_{k+1})$.

For such functors $\sigma, \sigma'\colon \mathscr{Z}^k \to \mathscr{C}$ and for the natural transformation $\varphi\colon \sigma \to \sigma'$ there are defined natural transformations $d^{\delta}_i\varphi\colon d^{\delta}_i\sigma \to d^{\delta}_i\sigma'$ and $s_i\varphi\colon s_i\sigma \to s_i\sigma'$, where

$$(d^{\delta}_i\varphi)(m_1, \ldots, m_{k-1}) = \varphi(m_1, \ldots, m_{i-1}, (-1)^{\delta+1}\infty, m_i, \ldots, m_{k-1}),$$

while

$$(s_i\varphi)(m_1, \ldots, m_{k+1}) = \varphi(m_1, \ldots, \hat{m}_i, \ldots, m_{k+1}).$$

1. Homotopy groups of the small category. Put $h([1]^n) = \square([1]^n)$, where $h\colon \square \to \mathscr{S}et^{\square^{\mathrm{op}}}$ is a Yoneda functor, and $\sqcap^{(\varepsilon,k)}([1]^n)$ the smallest subset of the cubical set $\square([1]^n)$ containing the elements $d^{\delta}_{i,n}(\mathbf{1}_{[1]^n})$, $i = 1, \ldots, n$, $\delta = 0, 1$ and $(\delta, i) \neq (\varepsilon, k)$. A map $f\colon X \to Y$ of the cubical sets is called a *Kan fibration* if, for every $n \geqslant 0$, $1 \leqslant k \leqslant n$, $\varepsilon = 0, 1$, any commutative square

$$\begin{array}{ccc} \sqcap^{(\varepsilon,k)}([1]^n) & \longrightarrow & X \\ \cap\downarrow & \nearrow & \downarrow f \\ \square([1]^n) & \longrightarrow & Y \end{array}$$

is diagonalizable. If $*$ is a final object of the category $\mathscr{S}et^{\square^{\mathrm{op}}}$ (i.e. all the components consist of a single point), it is said that the cubical set X satisfies a *Kan extension condition* when the map $f\colon X \to *$ is a Kan fibration.

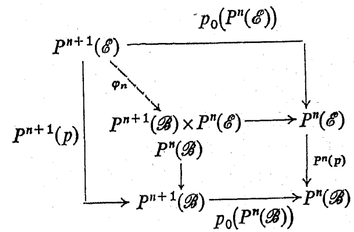
By applying the definition of cohomotopy $(P; p_0, p_1, s)$ in the category $\mathscr{C}at$, it is not difficult to prove the following lemma:

LEMMA 1.1. *If for the functor $p\colon \mathscr{E} \to \mathscr{B}$ the commutative diagrams*

$$\begin{array}{ccc} \sqcap^{(0,1)}([1]^n) & \longrightarrow & Q(\mathscr{E}) \\ \cap\downarrow & & \downarrow Q(p) \\ \square([1]^n) & \longrightarrow & Q(\mathscr{B}) \end{array}$$

are diagonalizable for every $n > 0$, then a map of cubical sets $Q(p)\colon Q(\mathscr{E}) \to Q(\mathscr{B})$ is a Kan fibration.

The functor $p\colon \mathscr{E} \to \mathscr{B}$ is called a *Serre fibration* iff for every $n \geqslant 0$ in the diagram



the functor φ_n is a surjection on the objects.

It is easy to observe that the functor $p\colon \mathscr{E} \to \mathscr{B}$ is a Serre fibration iff, for $\sigma \in \mathrm{ob}P^n(\mathscr{E})$, $\omega \in \mathrm{ob}P^{n+1}(\mathscr{B})$ such that $p\sigma(-, \ldots, -) = \omega(-\infty, -, \ldots, -)$, there exists $\gamma \in \mathrm{ob}P^{n+1}(\mathscr{E})$ such that $p\gamma = \omega$ and $\gamma(-\infty, -, \ldots, -) = \sigma(-, \ldots, -)$.

THEOREM 1.2. *For the functor $p\colon \mathscr{E} \to \mathscr{B}$ the following conditions are equivalent*:

a) *the functor $p\colon \mathscr{E} \to \mathscr{B}$ is a Serre fibration,*

b) *the map of cubical sets $Q(p)\colon Q(\mathscr{E}) \to Q(\mathscr{B})$ is a Kan fibration.*

Proof. In the proof of the implication b) $\Rightarrow$ a), by induction with respect to k, the following three sequences, $(1, k)$, $(2, k)$, $(3, k)$, of conditions will be proved:

$(1, k)$ — for the functors $\sigma_k \in \mathrm{ob}P^{k-1}(\mathscr{E})$, $\omega_k \in \mathrm{ob}P^k(\mathscr{B})$, $k \geqslant 1$ such that $p\sigma_k(-, \ldots, -) = \omega_k(-\infty, -, \ldots, -)$ there exists such a functor $\tau_k(\sigma_k, \omega_k) \in \mathrm{ob}P^k(\mathscr{E})$ that $(*)$ $p\tau_k(\sigma_k, \omega_k) = \omega_k$ and $\tau_k(\sigma_k, \omega_k)(-\infty, -, \ldots, -) = \sigma_k(-, \ldots, -)$.

Remark. The functor $\tau_k(\sigma_k, \omega_k)$ is not uniquely determined, and in spite of that, it is convenient for technical reasons to denote by $\tau_k(\sigma_k, \omega_k)$ any functor satisfying conditions (*).

$(2, k)$ — for natural transformations of the functors $\varphi_k\colon \sigma_k \to \sigma'_k$, $\psi_k\colon \omega_k \to \omega'_k$ satisfying the condition $\psi_k(-\infty, -, \ldots, -) = p\varphi_k(-, \ldots, -)$ there exist functors $\tau_k(\sigma_k, \omega_k)$, $\tau_k(\sigma'_k, \omega'_k)$ satisfying conditions $(1, k)$ and there exists a natural transformation of these functors $\tau_k(\varphi_k, \psi_k)\colon \tau_k(\sigma_k, \omega_k) \to \tau_k(\sigma'_k, \omega_k)$ such that $p\tau_k(\varphi_k, \psi_k) = \psi_k$, $\tau_k(\varphi_k, \psi_k)(-\infty, \ldots, -) = \varphi_k(-, \ldots, -)$.

Remark. The transformations $\tau_k(\varphi_k, \psi_k)$ are not determined uniquely either.

$(3, k)$ — for commutative squares of natural transformations

$$\begin{array}{ccc} \sigma_k & \xrightarrow{\varphi_k} & \sigma'_k \\ {\scriptstyle \overline{\varphi}_k}\uparrow & & \uparrow{\scriptstyle \overline{\varphi}'_k} \\ \hat{\sigma}_k & \xrightarrow{\hat{\varphi}_k} & \hat{\sigma}'_k \end{array} \qquad \begin{array}{ccc} \omega_k & \xrightarrow{\psi_k} & \omega'_k \\ {\scriptstyle \overline{\psi}_k}\uparrow & & \uparrow{\scriptstyle \overline{\psi}'_k} \\ \hat{\omega}_k & \xrightarrow{\hat{\psi}_k} & \hat{\omega}'_k \end{array}$$

satisfying conditions $(2, k)$ there exists a commutative square of pairs (φ_k, ψ_k), (φ'_k, ψ'_k), $(\overline{\varphi}_k, \overline{\psi}_k)$ and $(\overline{\varphi}'_k, \overline{\psi}'_k)$:

$$\begin{array}{ccc} \tau_k(\sigma_k, \omega_k) & \xrightarrow{\tau_k(\varphi_k, \psi_k)} & \tau_k(\sigma'_k, \omega'_k) \\ {\scriptstyle \tau_k(\overline{\varphi}_k, \overline{\psi}_k)}\uparrow & & \uparrow{\scriptstyle \tau_k(\overline{\varphi}'_k, \overline{\psi}'_k)} \\ \tau_k(\hat{\sigma}_k, \hat{\omega}_k) & \xrightarrow{\tau_k(\hat{\varphi}_k, \hat{\psi}_k)} & \tau_k(\hat{\sigma}'_k, \hat{\omega}'_k) \end{array}$$

Remark. The transformations $\tau_k(\varphi_k, \psi_k)$, $\tau_k(\varphi'_k, \psi'_k)$, $\tau_k(\overline{\varphi}_k, \overline{\psi}_k)$ and $\tau_k(\overline{\varphi}'_k, \overline{\psi}'_k)$ are not determined uniquely.

In the proof of the implication a) $\Rightarrow$ b), also by induction with respect to k, auxiliary constructions are carried out. Applying Lemma 1.1, in order to show that the map of cubical sets $Q(p)\colon Q(\mathscr{E}) \to Q(\mathscr{B})$ is a Kan fibration, it is enough to prove the diagonalizability of the following commutative square:

$$(*) \qquad \begin{array}{ccc} \sqcap^{(0,1)}([1]^k) & \xrightarrow{\Sigma_k} & Q(\mathscr{E}) \\ \cap\downarrow & & \downarrow{\scriptstyle Q(p)} \\ \square([1]^k) & \xrightarrow{\Omega_k} & Q(\mathscr{B}) \end{array} \qquad \text{for } k \geqslant 1\,.$$

The following three sequences, $(1', k)$, $(2', k)$, $(3', k)$, of conditions will be proved:

$(1', k)$ — the functors $\sigma_i^\delta\colon \mathscr{Z}^{k-1} \to \mathscr{E}$ determined by the map

$$\Sigma_k = \{\sigma_i^\delta\}\colon \sqcap^{(0,1)}([1]^k) \to Q(\mathscr{E})$$

induce a certain functor $\tau_k(\Sigma_k)\colon \mathscr{Z}^{k-1} \to \mathscr{E}$, the natural transformation

$$\Phi_k = \{\varphi_i^\delta\}\colon \Sigma_k = \{\sigma_i^\delta\} \to \Sigma'_k = \{\sigma_i'^\delta\}$$

(i.e., a natural transformation $\varphi_i^\delta\colon \sigma_i^\delta \to \sigma_i'^\delta$ such that $d_i^\delta \varphi_j^\eta = d_{j-1}^\eta \varphi_j^\delta$, for $i<j$, (δ, i), $(\eta, j) \neq (0, 1)$) induces the natural transformation $\tau_k(\Phi_k)\colon \tau_k(\Sigma_k) \to \tau_k(\Sigma'_k)$ and the functor $\sigma\colon \mathscr{Z}^k \to \mathscr{E}$ induces certain functors $\varrho_k(\sigma)\colon \mathscr{Z}^k \to \mathscr{E}$, $\nu_k(\sigma)\colon \mathscr{Z}^{k+1} \to \mathscr{E}$, whereas the natural transformation $\varphi\colon \sigma \to \sigma'$ induces natural transformations $\varrho_k(\varphi)\colon \varrho_k(\sigma) \to \varrho_k(\sigma')$, $\nu_k(\varphi)\colon \nu_k(\sigma) \to \nu_k(\sigma')$ in such a way that the following conditions are satisfied:

(i) if the diagram

$$\begin{array}{ccc} \Sigma_k & \xrightarrow{\Phi_k} & \Sigma'_k \\ {\scriptstyle \overline{\Phi}_k}\uparrow & & \uparrow{\scriptstyle \overline{\Phi}'_k} \\ \hat{\Sigma}_k & \xrightarrow{\hat{\Phi}_k} & \hat{\Sigma}'_k \end{array}$$

is commutative, (i.e., the diagrams

$$\begin{array}{ccc} \sigma_i^\delta & \xrightarrow{\varphi_i^\delta} & \sigma_i'^\delta \\ {\scriptstyle \overline{\varphi}_i^\delta}\uparrow & & \uparrow{\scriptstyle \overline{\varphi}_i'^\delta} \\ \hat{\sigma}_i^\delta & \xrightarrow{\hat{\varphi}_i^\delta} & \hat{\sigma}_i'^\delta \end{array}$$

are commutative, for $(\delta, i) \neq (0, 1)$), then the diagram

$$\begin{array}{ccc} \tau_k(\Sigma_k) & \xrightarrow{\tau_k(\Phi_k)} & \tau_k(\Sigma'_k) \\ {\scriptstyle \tau_k(\overline{\Phi}_k)}\uparrow & & \uparrow{\scriptstyle \tau_k(\overline{\Phi}'_k)} \\ \tau_k(\hat{\Sigma}_k) & \xrightarrow{\tau_k(\hat{\Phi}_k)} & \tau_k(\hat{\Sigma}'_k) \end{array}$$

is also commutative;

(ii) if the map $\Sigma_k(\sigma)\colon \sqcap^{(0,1)}([1]^k) \to Q(\mathscr{E})$ is determined by the elements $\sigma_i^\delta = d_i^\delta \sigma$, while the natural transformation $\Sigma_k(\varphi)\colon \Sigma_k(\sigma) \to \Sigma_k(\sigma')$ by the natural transformations $d_i^\delta \varphi\colon d_i^\delta \sigma \to d_i^\delta \sigma'$, for $(\delta, i) \neq (0, 1)$, then

$$d_1^1 \varrho_k(\sigma) = \tau_k(\Sigma_k(\sigma)), \quad d_1^0 \varrho_k(\sigma) = d_1^0 \sigma$$

and

$$d_1^1 \varrho_k(\varphi) = \tau_k(\Sigma_k(\varphi)), \quad d_1^0 \varrho_k(\varphi) = d_1^0 \varphi\,;$$

(iii) for any $\sigma\colon \mathscr{Z}^k \to \mathscr{E}$ we have $\nu_k(\sigma)(+\infty, m_1, \ldots, m_k) = \sigma(-m_k, m_1, \ldots, m_{k-1})$ and $\nu_k(\sigma)(-, \ldots, -\infty) = \sigma$.

(iv) if the diagram

$$\begin{array}{ccc} \sigma & \xrightarrow{\varphi} & \sigma' \\ {\scriptstyle \overline{\varphi}}\uparrow & & \uparrow{\scriptstyle \overline{\varphi}'} \\ \hat{\sigma} & \xrightarrow{\hat{\varphi}} & \hat{\sigma}' \end{array}$$

is commutative, then the diagrams

$$\begin{array}{ccc} \varrho_k(\sigma) & \xrightarrow{\varrho_k(\varphi)} & \varrho_k(\sigma') \\ {\scriptstyle \varrho_k(\bar\varphi)}\uparrow & & \uparrow{\scriptstyle \varrho_k(\bar\varphi')} \\ \varrho_k(\hat\sigma) & \xrightarrow{\varrho_k(\hat\varphi)} & \varrho_k(\hat\sigma') \end{array} \qquad \begin{array}{ccc} \nu_k(\sigma) & \xrightarrow{\nu_k(\varphi)} & \nu_k(\sigma') \\ {\scriptstyle \nu_k(\bar\varphi)}\uparrow & & \uparrow{\scriptstyle \nu_k(\bar\varphi')} \\ \nu_k(\hat\sigma) & \xrightarrow{\nu_k(\hat\varphi)} & \nu_k(\hat\sigma') \end{array}$$

are also commutative.

$(2', k)$ — if a functor $\gamma\colon \mathscr{Z}^{k+1} \to \mathscr{E}$ satisfies the condition that for some functor $\sigma\colon \mathscr{Z}^k \to \mathscr{E}$, $p\gamma = p\nu_k\varrho_k(\sigma)$, $\gamma(-\infty, m_1, \ldots, m_k) = d_1^1\nu_k\varrho_k(\sigma)(m_1, \ldots, m_k)$ $= \varrho_k(\sigma)(-m_k, m_1, \ldots, m_{k-1})$, then γ induces functors $\tilde\gamma, \tilde{\tilde\gamma}\colon \mathscr{Z}^{k+1} \to \mathscr{E}$; any natural transformation $\varphi\colon \gamma \to \gamma'$ of the functors satisfying the above conditions induces natural transformations $\tilde\varphi\colon \tilde\gamma \to \tilde\gamma'$, $\tilde{\tilde\varphi}\colon \tilde{\tilde\gamma} \to \tilde{\tilde\gamma}'$ in such a way that:

(i) $d_{k+1}^1\tilde\gamma = \varrho_k(\sigma)$, $d_{k+1}^0\tilde\gamma = d_{k+1}^1\tilde{\tilde\gamma}$ and $d_{k+1}^0\tilde{\tilde\gamma} = d_{k+1}^0\gamma$;

(ii) for any $m \in \mathrm{ob}_{-n_0}\mathscr{Z}_{n_0}$ we have $\tilde\gamma(n_0, \ldots, m) = \tilde{\tilde\gamma}(n_0, \ldots, m) = \tau_k(\Sigma_k(\sigma))$, $p\tilde\gamma(\ldots, m) = p\tilde{\tilde\gamma}(\ldots, m) = p\varrho_k(\sigma)$ and for each morphism $m \to m'$ of the category ${}_{-n_0}\mathscr{Z}_{n_0}$ we have $p\tilde\gamma(\ldots, m \to m') = p\tilde{\tilde\gamma}(\ldots, m \to m') = \mathbf{1}_{p\varrho_k(\sigma)}$;

(iii) if the diagram

$$\begin{array}{ccc} \gamma & \xrightarrow{\varphi} & \gamma' \\ {\scriptstyle \bar\varphi}\uparrow & & \uparrow{\scriptstyle \bar\varphi'} \\ \hat\gamma & \xrightarrow{\hat\varphi} & \hat\gamma' \end{array}$$

is commutative, then the diagrams

$$\begin{array}{ccc} \tilde\gamma & \xrightarrow{\tilde\varphi} & \tilde\gamma' \\ {\scriptstyle \tilde{\bar\varphi}}\uparrow & & \uparrow{\scriptstyle \tilde{\bar\varphi}'} \\ \tilde{\hat\gamma} & \xrightarrow{\tilde{\hat\varphi}} & \tilde{\hat\gamma}' \end{array} \qquad \begin{array}{ccc} \tilde{\tilde\gamma} & \xrightarrow{\tilde{\tilde\varphi}} & \tilde{\tilde\gamma}' \\ {\scriptstyle \tilde{\tilde{\bar\varphi}}}\uparrow & & \uparrow{\scriptstyle \tilde{\tilde{\bar\varphi}}'} \\ \tilde{\tilde{\hat\gamma}} & \xrightarrow{\tilde{\tilde{\hat\varphi}}} & \tilde{\tilde{\hat\gamma}}' \end{array}$$

are also commutative.

Applying $(1', k)$ we see that the map $\Sigma_k\colon \sqcap^{(0,1)}([1]^k) \to Q(\mathscr{E})$ determines the functor $\tau_k(\Sigma_k)\colon \mathscr{Z}^{k-1} \to \mathscr{E}$, while the functor $\omega\colon \mathscr{Z}^k \to \mathscr{B}$, corresponding to the map $\Omega_k\colon \square([1]^k) \to Q(\mathscr{B})$, induces the functor $\varrho_k(\omega)\colon \mathscr{Z}^k \to \mathscr{B}$.

From the commutativity of the diagram $(*)$ and from the naturalness of the construction of the functor $\tau_k(\Sigma_k)$ it follows that $d_1^1\varrho_k(\omega) = \tau_k(\Sigma_k(\omega)) = p\tau_k(\Sigma_k)$.

The functor $p\colon \mathscr{E} \to \mathscr{B}$ is a Serre fibration, and so there exists a functor $\gamma\colon \mathscr{Z}^k \to \mathscr{E}$ such that $p\gamma = \varrho_k(\omega)$ and $d_1^1\gamma = \tau_k(\Sigma_k)$. Observe that if $\mathscr{B} = *$, then the functor $\gamma = s_1(\tau_k(\Sigma_n))$ satisfies conditions $(2', k)$.

$(3', k)$ — the functor $\gamma\colon \mathscr{Z}^k \to \mathscr{E}$ satisfying the condition $p\gamma = \varrho_k(\omega)$, $d_1^1\gamma = \tau_k(\Sigma_k)$ determines the functor $\bar\rho_k(\gamma)\colon \mathscr{Z}^k \to \mathscr{E}$ and the natural transformation $\varphi\colon \gamma \to \gamma'$ determines the natural transformation $\bar\rho_k(\varphi)\colon \bar\rho_k(\gamma) \to \bar\rho_k(\gamma')$ in such a way that:

if $\omega = p\sigma$, for certain functors $\sigma\colon \mathscr{Z}^k \to \mathscr{E}$ and $\gamma = \varrho_k(\sigma)$, then $\bar\rho_k(\gamma) = \sigma$;

the map of cubical sets $\Gamma_k\colon \square([1]^k) \to Q(\mathscr{E})$ determined by the functor $\bar\rho_k(\gamma)\colon \mathscr{Z}^k \to \mathscr{E}$ diagonalizes the diagram $(*)$;

if the diagram

$$\begin{array}{ccc} \gamma & \xrightarrow{\varphi} & \gamma' \\ {\scriptstyle \psi}\uparrow & & \uparrow{\scriptstyle \psi'} \\ \hat\gamma & \xrightarrow{\hat\varphi} & \hat\gamma' \end{array}$$

is commutative, then the diagram

$$\begin{array}{ccc} \bar\rho_k(\gamma) & \xrightarrow{\bar\varrho_k(\varphi)} & \bar\rho_k(\gamma') \\ {\scriptstyle \bar\varrho_k(\psi)}\uparrow & & \uparrow{\scriptstyle \bar\varrho_k(\psi')} \\ \bar\rho_k(\hat\gamma) & \xrightarrow{\bar\varrho_k(\hat\varphi)} & \bar\rho_k(\hat\gamma') \end{array}$$

is also commutative. ■

Observe that for any category $\mathscr{C} \in \mathrm{ob}\,\mathscr{C}at$ the functor $\mathscr{C} \to *$ is a Serre fibration; by the above theorem it follows that the map $Q(\mathscr{C}) \to Q(*) = *$ is a Kan fibration.

Corollary 1.3. *For any small category $\mathscr{C} \in \mathrm{ob}\,\mathscr{C}at$ the cubical set $Q(\mathscr{C})$ satisfies the Kan extension condition.*

Thus we obtain the functor $\pi\colon \mathscr{C}at \to \mathscr{G}r$, which is defined by the formula $\pi(\mathscr{C}) = c(Q(\mathscr{C}))$, where $c\colon \mathscr{S}et^{\square^{op}} \to \mathscr{C}at$ denotes the functor defined in paper [2], while $\mathscr{G}r$ is the category of groupoids.

On the other hand, for any small category $\mathscr{C} \in \mathrm{ob}\,\mathscr{C}at$ the localization of $\mathscr{C}$ with respect to any set of morphisms $\Sigma \subset \mathrm{Ar}\,\mathscr{C}$ (see [1], ch. I) is defined as a colimit (in $\mathscr{C}at$) of the diagram:

$$\begin{array}{ccc} \coprod_{\alpha\in\Sigma}[1]_\alpha & \longrightarrow & \coprod_{\alpha\in\Sigma}[1]'_\alpha \\ \downarrow & & \downarrow{\scriptstyle i_1} \\ \mathscr{C} & \xrightarrow{i_0} & \mathscr{C}[\Sigma^{-1}], \end{array}$$

where $[1]_\alpha = [1]$, whereas $[1]'_\alpha = [1]'$ denotes the groupoid with two objects $0, 1$ and two nonidentity morphisms, $0 \to 1$, $1 \to 0$, mutually inverse.

Theorem 1.4. *For any $\mathscr{C} \in \mathrm{ob}\,\mathscr{C}at$ there exists an equivalence of groupoids* $\pi(\mathscr{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{C}[\mathrm{Ar}\,\mathscr{C}^{-1}]$.

Proof. Observe that $\mathrm{ob}\,\pi(\mathscr{C}) = \mathrm{ob}\,\mathscr{C}[\mathrm{Ar}\,\mathscr{C}^{-1}] = \mathrm{ob}\,\mathscr{C}$ and we define $F(C)$ $= G(C) = C$ for every $C \in \mathrm{ob}\,\mathscr{C}$.

In order to define the functor $G\colon \mathscr{C}[\mathrm{Ar}\,\mathscr{C}^{-1}] \to \mathscr{C}$ it is enough to define the pair of functors $G_0\colon \mathscr{C} \to \pi(\mathscr{C})$, $G_1\colon \coprod_{\alpha\in\mathrm{Ar}\,\mathscr{C}}[1]_\alpha \to \mathscr{C}$ satisfying the appropriate conditions of consistence. In order to define G_0, G_1 let us consider any morphism $\alpha\colon C \to C'$

of the category $\mathscr{C}$ and the functor $\sigma_\alpha\colon \mathscr{X} \to \mathscr{C}$ defined in the following way: $\sigma_\alpha(0 \to 1) = \alpha$, $\sigma_\alpha(-\infty) = \sigma_\alpha(0) = C$ and $\sigma_\alpha(+\infty) = \sigma_\alpha(1) = C'$.

The morphism of the groupoid $\pi(\mathscr{C})$ determined by this functor is denoted by $[\sigma_\alpha]_\sim$.

We then put $G_0(\alpha) = G_1((0 \to 1)_\alpha) = [\sigma_\alpha]_\sim$, (the morphism $[\sigma_\alpha]_\sim$ is inversible in $\pi(\mathscr{C})$).

To define the functor $F\colon \pi(\mathscr{C}) \to \mathscr{C}[\mathrm{Ar}\mathscr{C}^{-1}]$, let us first remark that $\pi(\mathscr{C}) = \mathrm{Pa}\,\mathfrak{X}/\sim$ (see [2]), where $\mathrm{Pa}\,\mathfrak{X}$ is the category of paths determined by the diagram scheme $\mathfrak{X} = (Q_0(\mathscr{C}), Q_1(\mathscr{C}); d_1^0, d_1^1)$, whereas $\sim$ is the smallest congruence satisfying the conditions:

1) $s_1\sigma_0 \sim \mathbf{1}\sigma_0$, for $\sigma_0 \in Q_0(\mathscr{C})$,

2) $(d_1^0\sigma, d_2^1\sigma) \sim (d_2^0\sigma, d_1^1\sigma)$, for $\sigma \in Q_2(\mathscr{C})$.

In order to define the functor $F\colon \pi(\mathscr{C}) \to \mathscr{C}[\mathrm{Ar}\mathscr{C}^{-1}]$ it is enough to define the functor $\bar{F}\colon \mathrm{Pa}\,\mathfrak{X} \to \mathscr{C}[\mathrm{Ar}\mathscr{C}^{-1}]$ preserving the congruence $\sim$. Observe that, for any element $\sigma \in Q_1(\mathscr{C})$, there exist $n_0 \operatorname{ob} \mathscr{Z}$ and the morphisms α_k of the category $\mathscr{C}$, for $k = -n_0, \ldots, n_0$ such that $\sigma \sim (\sigma_{\alpha_{-n_0}}, \ldots, \sigma_{\alpha_{n_0}})$.

There exists a unique functor $\bar{F}\colon \mathrm{Pa}\,\mathfrak{X} \to \mathscr{C}[\mathrm{Ar}\mathscr{C}^{-1}]$ such that $\bar{F}(\sigma_\alpha) = i_0(\alpha)$, and for any element $\sigma \in Q_2(\mathscr{C})$ we have $\bar{F}(d_1^0\sigma, d_2^1\sigma) = \bar{F}(d_2^1\sigma, d_1^1\sigma)$. It follows from the definition of functors $\pi(\mathscr{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{C}[\mathrm{Ar}\mathscr{C}^{-1}]$ that $FG = \mathbf{1}_{\mathscr{C}[\mathrm{Ar}\mathscr{C}^{-1}]}$, $GF = \mathbf{1}_{\pi(\mathscr{C})}$.

Remark 1.5. *The functor $\pi\colon \mathscr{C}at \to \mathscr{G}r$ is left adjoint to the functor of inclusion $i\colon \mathscr{G}r \to \mathscr{C}at$.*

D. M. Kan defines the nth homotopy group $\pi_n(X, *)$, for $n \geqslant 0$ of any pointed cubical set satisfying the Kan extension condition (see [6]);

the nth homotopy group of the category $\mathscr{C} \in \operatorname{ob}\mathscr{C}at^*$ is called the *n-th homotopy group of the cubical set* $Q(\mathscr{C})$; i.e. $\pi_n(\mathscr{C}, *) = \pi_n(Q(\mathscr{C}), *)$.

The functor $\pi_0\colon \mathscr{C}at \to \mathscr{S}et$ is left adjoint to the functor of inclusion dis: $\mathscr{S}et \to \mathscr{C}$at, assigning to every set a discrete category.

Denoting by $\mathscr{G}rp$ the category of groups and by $\mathscr{C}at_c^*$ a full subcategory of the category $\mathscr{C}at^*$ determined by the connected categories, it is not difficult to show that the functor $\pi_1\colon \mathscr{C}at_c^* \to \mathscr{G}rp$ is left adjoint to the functor of inclusion $i\colon \mathscr{G}rp \to \mathscr{C}at_c^*$.

By applying Theorem 1.2 and referring to the appropriate considerations in the simplicial theory (see [1], ch. I), it is not difficult to prove that the following theorem is true:

THEOREM 1.6. *The Serre fibration $p\colon \mathscr{E} \to \mathscr{B}$ induces a long exact sequence of the homotopy groups*

$$\ldots \to \pi_1(\mathscr{E}, *) \to \pi_1(\mathscr{B}, *) \to \pi_0(\mathscr{F}, *) \to \pi_0(\mathscr{E}, *) \to \pi_0(\mathscr{B}, *), \quad \text{where} \quad \mathscr{F} = p^{-1}(*).$$

2. Small categories and a model category. By a *model category* we mean a category $\mathscr{C}$ together with three classes of morphisms in $\mathscr{C}$: F(*fibrations*), C(*cofibrations*), W(*weak equivalence*) satisfying the axioms M0–M5 (see [7]).

Define in the category $\mathscr{C}at$ the above classes of morphisms:

1) $p\colon \mathscr{E} \to \mathscr{B}$ is a fibration iff it is a Serre fibration;

2) $f\colon \mathscr{C} \to \mathscr{D}$ is a weak equivalence iff $\pi_n(f, *)\colon \pi_n(\mathscr{C}, *) \to \pi_n(\mathscr{D}, f(*))$ is an isomorphism for $n \geqslant 0$, $* \in \operatorname{ob}\mathscr{C}$;

3) $i\colon \mathscr{A} \to \mathscr{B}'$ is a cofibration iff the commutative square

$$\begin{array}{ccc} \mathscr{A} & \to & \mathscr{E} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{B}' & \to & \mathscr{B} \end{array}$$

is diagonalizable, where p is a fibration and a weak equivalence;

($p\colon \mathscr{E} \to \mathscr{B}$ is called a *trivial* (co)*fibration* if it is a (co)fibration and a weak equivalence).

M0 is satisfied in the category $\mathscr{C}at$ (see [1]); M1 is a consequence of the definition of fibration and of the following lemma, the proof of which is an adaption of the proof of the appropriate fact in the category of topological spaces (see [7]).

LEMMA 2.1. *The following conditions are equivalent for any functor $i\colon \mathscr{A} \to \mathscr{B}'$:*

(i) *i is a trivial cofibration,*

(ii) *i is a cofibration and a strong deformation retract,*

(iii) *the commutative square*

$$\begin{array}{ccc} \mathscr{A} & \to & \mathscr{E} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{B}' & \to & \mathscr{B} \end{array}$$

where p is a fibration, is diagonalizable.

By applying this lemma it is not difficult to prove axioms M3, M4, M5.

The attempt to prove axiom M2 (i.e. that any map $f\colon \mathscr{C} \to \mathscr{D}$ may be factored $f = pi$, where i is a cofibration and a weak equivalence and p is a fibration; also $f = p'i'$, where i' is a cofibration and p' is a fibration and a weak equivalence) and the fact that the functor $P\colon \mathscr{C}at \to \mathscr{C}at$ is prorepresentable suggest that both factorizations should be found in the category *pro-$\mathscr{C}at$* (see [0]).

References

[0] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Math. 100 (1969).

[1] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, Berlin 1967.

[2] M. Golasiński, *The category of cubical sets and the category of small categories*, Bull. Acad. Polon. Sci. 27 (1979), pp. 941–945.

[3] G. Hoff, *Categories fibreés et homotopie*, C.R. Acad. Sci. Paris 278 (1974), pp. 223–225.

[4] — *Categories cofibreés, fibrations faibles*, C. R. Acad. Sci. Paris 278 (1974), pp. 1077–1080.

[5] K. H. Kamps, *Kan-Bedingungen und abstrakte Homotopietheorie*, Math. Z. 124 (1972), pp. 215–236.

[6] D. M. Kan, *Abstract homotopy* I; Proc. Nat. Acad. Sci. USA, 41 (1955), p. 1092–1096.

[7] D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Math. 43 (1967).

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