Noether lattices: Some constructions and decompositions

by

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Abstract. In this note we consider some constructions and decompositions for Noether lattices. Conditions are given for certain sublattices of a Noether lattice to be a Noether lattice. The decomposition of a Noether lattice into direct and local direct sums is also considered.

The summands associated with any decomposition theorem are normally susceptible to a variety of descriptions, as is the decomposition process itself.

In this paper we consider the construction processes associated with various points of view of decompositions of Noether lattices [2]. Our purpose is to determine when these alternative points of view lead to new, useful examples, and also to shed additional light on the decompositions themselves.

It is convenient to introduce the following notation: If $A_1, \ldots, A_n \in \mathcal{L}$, then $K(A_1, \ldots, A_n) = \bigvee_{i=1}^{n} A_i \wedge (\bigvee_{j \neq i} A_j)$. Using this notation, the decomposability of a Noether lattice $\mathcal{L}$ can be thought of as meaning, alternatively, that

(i) $\mathcal{L}$ can be represented as the Cartesian product of Noether lattices $\mathcal{L}_i$;

(ii) There exist elements $A_1, \ldots, A_n \in \mathcal{L}$ such that $A_1 \vee \ldots \vee A_n = I$ and $K(A_1, \ldots, A_n) = 0$;

(ii)′ There exist elements $A_1, \ldots, A_n \in \mathcal{L}$ such that each of the intervals $[A_i, 0]$ is a Noether lattice and $\mathcal{L}$ can be represented as the Cartesian product of the intervals $[A_i, 0]$;

(iii) There exist pairwise comaximal elements $B_1, \ldots, B_n \in \mathcal{L}$ such that $B_1 \wedge \ldots \wedge B_n = 0$;

(iii)′ There exist elements $B_1, \ldots, B_n \in \mathcal{L}$ such that $\mathcal{L}$ is representable as the Cartesian product of the Noether lattices $\mathcal{L}/B_i$;

(iv) There exists a Noether lattice $\mathcal{L}^*$ and elements $A_1, \ldots, A_n \in \mathcal{L}^*$ such that $K(A_1, \ldots, A_n) = 0$ and $\mathcal{L} = \{A \in \mathcal{L}^* \mid A = \bigvee_{i=1}^{n} (A \wedge A_i)\}$.

It is the conditions under which the choice of arbitrary elements A_i of a Noether lattice results in a Noether lattice by the processes described in (ii)′ and (iv) that concern us.

It is convenient to introduce the following notation: if $A_1, \ldots, A_n$ are elements of $\mathscr{L}$, then

$$\mathscr{L}^d(A_1, \ldots, A_n) = \{A \in \mathscr{L} \mid K(A_1, \ldots, A_n) \leqslant A \leqslant \bigvee_{i=1}^{n} (A \wedge A_i)\}.$$

When $n = 1$, we use the usual conventions so that $\mathscr{L}^d(A) = [A, 0]$.

LEMMA 1. *Let $\mathscr{L}$ be a Noether lattice and let $A_1, \ldots, A_n$ be elements of $\mathscr{L}$. Then $\mathscr{L}^d(A_1, \ldots, A_n)$ is a submultiplicative lattice of $\mathscr{L}/K(A_1, \ldots, A_n)$.*

Proof. Let C_1 and C_2 be elements of $\mathscr{L}^d(A_1, A_2)$, $(C_i \neq I)$. Then

$$\begin{aligned} C_1 \wedge C_2 &\leqslant C_1 \wedge \big((A_1 \vee (C_1 \wedge A_2)) \wedge ((C_2 \wedge A_1) \vee (C_2 \wedge A_2))\big) \\ &\leqslant C_1 \wedge \big((C_2 \wedge A_1) \vee ((A_1 \vee (C_1 \wedge A_2)) \wedge (C_2 \wedge A_2))\big) \\ &\leqslant C_1 \wedge \big((C_2 \wedge A_1) \vee (C_1 \wedge C_2 \wedge A_2)\big) \\ &\leqslant (C_1 \wedge C_2 \wedge A_1) \vee (C_1 \wedge C_2 \wedge A_2). \end{aligned}$$

We note that if $B_i = A_i \vee K(A_1, \ldots, A_n)$, then $\mathscr{L}^d(B_1, \ldots, B_n) = \mathscr{L}^d(A_1, \ldots, A_n)$. Hence, it may be assumed that $\{A_1, \ldots, A_n\}$ is an independent set of elements of $\mathscr{L}$. The induction is now straightforward, as is the verification of closure under multiplication and joins.

Obviously, a necessary condition that $\mathscr{L}^d(A_1, \ldots, A_n)$ be a Noether lattice is that $A = A_1 \vee \ldots \vee A_n$ be idempotent in $\mathscr{L}/K(A_1, \ldots, A_n)$.

THEOREM 2. *Let $\mathscr{L}$ be a Noether lattice and let $A_1, \ldots, A_n$ be elements of $\mathscr{L}$ such that $A = A_1 \vee \ldots \vee A_n$ is idempotent in $\mathscr{L}/K(A_1, \ldots, A_n)$. Then $\mathscr{L}^d(A_1, \ldots, A_n)$ is a Noether lattice. Moreover, in this case, $\mathscr{L}^d(A_1, \ldots, A_n) = A/K(A_1, \ldots, A_n)$. In particular, $\mathscr{L}^d(A) = [A, 0]$ is a Noether lattice if A is idempotent.*

Proof. Since $\mathscr{L}^d(A_1, \ldots, A_n)$ is clearly the direct sum of the quotients $(A_i \vee K(A_1, \ldots, A_n))/K(A_1, \ldots, A_n)$, it suffices to prove $\mathscr{L}^d(A)$ is a Noether lattice if A is idempotent. Hence, let $O = Q_1 \wedge \ldots \wedge Q_s$ be a normal decomposition of O with associated primes $P_1, \ldots, P_s$. Set

$$O_A = \bigwedge \{Q_i \mid P_i \vee A \neq I\} \quad \text{and} \quad O'_A = \bigwedge \{Q_i \mid P_i \vee A = I\}.$$

Let E be any principal element of $\mathscr{L}$ such that $E \leqslant A$. Then $E \leqslant \bigwedge_{n=1}^{\infty} A^n = O_A$, so $O:E = (O_A \wedge O'_A):E = O'_A:E$. It follows that $A \vee (O:E) = A \vee (O'_A:E) \geqslant A \vee O'_A = I$, and hence that $AE = E$. Therefore A is principal in $\mathscr{L}$, and $A^2 = A$ implies both $A \vee (O:A) = I$ and $A \wedge (O:A) = O$. It follows that $[A, 0] \approx \mathscr{L}/O:A$, and hence that $\mathscr{L}^d(A)$ is a Noether lattice. ∎

We now change the focus of our investigation slightly and ask when a Noether lattice is obtained by appending an identity to the top of either $[A, 0]$ or $\mathscr{L}^d(A_1, \ldots, A_n)$. We adopt the notation ${}_A\mathscr{L} = [A, 0] \cup \{I\}$, and $\mathscr{L}(A_1, \ldots, A_n) = \mathscr{L}^d(A_1, \ldots, A_n) \cup \{I\}$, so that ${}_A\mathscr{L} = \mathscr{L}(A)$ is something of a very simple localization of $\mathscr{L}$.

We note that if $A = A_1 \vee \ldots \vee A_n = I$, then ${}_A\mathscr{L} = \mathscr{L}^d(A)$ and $\mathscr{L}(A_1, \ldots, A_n) = \mathscr{L}^d(A_1, \ldots, A_n)$. Otherwise ${}_A\mathscr{L}$ and $\mathscr{L}(A_1, \ldots, A_n)$ are both local in structure. Since principal elements of a local Noether lattice are join-irreducible, it is convenient to introduce the following notation:

$\mathscr{I}(A) = \{E \leqslant A \mid E \neq 0$ and E is join-irreducible$\}$;
$\mathscr{G}(A) = \{E \mid E = 0,\ E = I,$ or E is the finite join of elements of $\mathscr{I}(A)\}$;
$m(E) = M$ if M is the unique maximal element containing $O:E$;
$\mathscr{M}(A) = \{m(E) \mid E \in \mathscr{I}(A)\}$; $\mathscr{J}(\mathscr{L}) = \bigwedge \{M \mid M$ is a maximal element of $\mathscr{L}\}$.

THEOREM 3. *Let $\mathscr{L}$ be a Noether lattice and let A be an element of $\mathscr{L}$, $A \neq 0, I$. Then ${}_A\mathscr{L}$ is a Noether lattice if, and only if,*

(i) $A \leqslant \mathscr{J}(\mathscr{L})$;
(ii) ${}_A\mathscr{L} \subseteq \mathscr{G}(A)$;
(iii) $m(E) = A \vee (O:E)$ *for all* $E \in \mathscr{I}(A)$.

Proof. Assume ${}_A\mathscr{L}$ is a Noether lattice. Since ${}_A\mathscr{L}$ is local, (ii) is immediate. On the other hand, if $E \in \mathscr{I}(A)$, then $I/(A \vee (O:E)) \approx E/AE$, which establishes (iii). (i) follows from the observation that $E \leqslant A \leqslant A \vee (O:E) = m(E)$, and $E \leqslant M$ for every maximal element $M \ngeqslant O:E$.

Now, assume $\mathscr{L}$ and A are as above. It is easily seen that if $E \leqslant A$ is principal in $\mathscr{L}$, then E is weak join principal in ${}_A\mathscr{L}$. Hence it suffices to show that if $E \in \mathscr{I}(A)$, then E is weak meet principal (see, for example [1], Proposition 1.1).

We denote the residual operation in ${}_A\mathscr{L}$ by $\mathbin{\S}$. If C is any element of ${}_A\mathscr{L}$, $C \leqslant E$, then $C = DE$ for some $D \in \mathscr{L}$. If $D \nleqslant AE:E$, then $D \vee (O:E) = I$ and $DE = E = IE$. If $D \leqslant AE:E$, then $C = DE \leqslant (A \vee (O:E))E = AE$, so $C = C \wedge AE = ((C:E) \wedge A)E$. It follows that in either case, $C = (C \mathbin{\S} E)E$, and hence that E is principal.

We further explore the relationship between ${}_A\mathscr{L}$ and $\mathscr{L}(A_1, \ldots, A_n)$.

THEOREM 4. *Let $\mathscr{L}$ be a Noether lattice and let $A \neq I$ be an element of $\mathscr{L}$ such that ${}_A\mathscr{L} \subseteq \mathscr{G}(A)$. Then $\mathscr{M}(A) = \{m(E) \mid E \leqslant A\}$ is a finite set. Moreover, if $m(E_1), \ldots, m(E_k)$ are the distinct elements of $\mathscr{M}(A)$, and if*

$$A_i = \bigvee \{E \in \mathscr{I}(A) \mid m(E) = m(E_i)\},$$

then

$${}_A\mathscr{L} = \mathscr{L}(A_1, \ldots, A_n).$$

Proof. Let $F_1, \ldots, F_s$ be elements of $\mathscr{I}(A)$ with join A. If F is any element of $\mathscr{I}(A)$, then $O:A \leqslant \bigwedge_{i=1}^{s} (O:F_i) \leqslant O:F \leqslant m(F)$, so $m(F) = m(F_i)$ for some i, $1 \leqslant i \leqslant s$.

Now, let $E_1, \ldots, E_k$ and $A_1, \ldots, A_k$ be as defined in the statement of the theorem. We note that $m(E_i)$ is the only maximal element containing $O:A_i$, so that any $F \in \mathscr{I}(A)$ such that $F \leqslant A_i$ necessarily satisfies $m(F) = m(E_i)$. Also, if

$$F \leqslant A_1 \vee \ldots \vee \hat{A}_i \vee \ldots \vee A_k,$$

then $\bigwedge_{j\neq i}(O:A_j)\leqslant O:F\leqslant m(F)$, so $m(F) = m(E_j)$ for some $j\neq i$, and therefore $F\leqslant A_j$ for some $j\neq i$. It follows that if $B\leqslant A$, then $O = K(A_1, \ldots, A_k)\leqslant B\leqslant \bigvee_{i=1}^{k}(B\wedge A_i)$, and hence that ${}_A\mathscr{L} = \mathscr{L}(A_1, \ldots, A_n)$.

THEOREM 5. *Let $\mathscr{L}$ be a Noether lattice and let $A_1, \ldots, A_n$ be elements of $\mathscr{L}$. Let $\overline{\mathscr{L}} = \mathscr{L}/K(A_1, \ldots, A_n)$ and $\overline{A}_i = A_i\vee K(A_1, \ldots, A_n)$. Then $\mathscr{L}(A_1, \ldots, A_n)$ is the local direct sum* [3] *of the multiplicative lattices ${}_{\overline{A}_i}\overline{\mathscr{L}}$. In particular, $\mathscr{L}(A_1, \ldots, A_n)$ is a Noether lattice if, and only if, each of the lattices ${}_{\overline{A}_i}\overline{\mathscr{L}}$ is.*

Proof. Since $\mathscr{L}(A_1, \ldots, A_n) = (\mathscr{L}/K(A_1, \ldots, A_n))(\overline{A}_1, \ldots, \overline{A}_n)$, and since $\{\overline{A}_1, \ldots, \overline{A}_n\}$ is an independent set of distributive elements of $\mathscr{L}/K(A_1, \ldots, A_n)$, it is immediate that the map φ defined by $\varphi(B) = (B\wedge A_1, \ldots, B\wedge A_n)$ is an isomorphism of the quotient $(A_1\vee\ldots\vee A_n)/O$ in $\mathscr{L}(A_1, \ldots, A_n)$ onto the quotient $(\overline{A}_1, \ldots, \overline{A}_n)/O$ of ${}_{\overline{A}_1}\overline{\mathscr{L}}\oplus\ldots\oplus{}_{\overline{A}_n}\overline{\mathscr{L}}$.

We summarize the relationships of ${}_A\mathscr{L}$ and $\mathscr{L}(A_1, \ldots, A_n)$ to local direct sums:

THEOREM 6. *Let $(\mathscr{L}, M)$ be a local Noether lattice. Then the following are equivalent:*

(i) *$\mathscr{L}$ is decomposable as a local direct sum of local Noether lattices;*

(ii) *There exist elements $A_1, \ldots, A_n$ in $\mathscr{L}$ such that $\mathscr{L} = \mathscr{L}(A_1, \ldots, A_n)$;*

(iii) *There exists an element $A\in\mathscr{L}$ such that the map φ: $\mathscr{L}\to{}_A\mathscr{L}$ defined by $\varphi(I) = I$ and $\varphi(X) = X\wedge A$ for $X\neq I$ is a multiplicative lattice homomorphism;*

(iv) *There exists a distributive element $A\in\mathscr{L}$ such that*

a) *${}_A\mathscr{L}$ is a Noether lattice and*

b) *If E and F are principal elements of $\mathscr{L}$ such that $EF\leqslant A$, then either $E\leqslant A$ or $F\leqslant A$ or $EF = 0$.*

Proof. We show (iii) implies (iv) implies (i). Hence, assume (iii) holds. If E and F are principal elements of $\mathscr{L}$ such that $E\nleqslant A$, $F\nleqslant A$, and $EF\leqslant A$, then $EF = \varphi(EF) = \varphi(E)\varphi(F) = (E\wedge A)(F\wedge A) = (A:E)(A:F)EF$, so $EF = 0$.

Now, assume (iv) holds. Let $E_1, \ldots, E_k$ be a minimal collection of principal elements such that $A\vee E_1\vee\ldots\vee E_k = M$. Then $A:E_i = A\vee(O:E_i)$, so $A\wedge E_i = AE_i$. Let F be any principal element of ${}_A\mathscr{L}$ $(F\neq 0)$ and note that $MF<F$, so that $MF\leqslant AF\leqslant MF$, and therefore $M = A\vee(O:F)$. Since A is distributive and the join of principal elements of ${}_A\mathscr{L}$, it follows that $M = A\vee(O:A)$ and that $E_i\leqslant O:A$. Hence $A\wedge E_i = AE_i = 0$. It follows that $D_1 = D$ and $D_2 = E_1\vee\ldots\vee E_k$ are independent distributive elements with join M, and hence that $\mathscr{L}$ is the local direct sum of ${}_{D_1}\mathscr{L}$ and ${}_{D_2}\mathscr{L}$.

LEMMA 7. *Let $\mathscr{L}$ be a Noether lattice. Let M be a maximal element of $\mathscr{L}$ and let $D\neq I$ be an element such that $m(D) = M$. If $D\leqslant M$, then ${}_D\mathscr{L}\approx{}_{D_M}(\mathscr{L}_M)$. If $D\nleqslant M$, then ${}_{D_M}(\mathscr{L}_M)\approx\mathscr{L}^d(D) = [D, 0]$.*

Proof. Define φ: $[D, 0]\to{}_{D_M}(\mathscr{L}_M)$ by $\varphi(B) = B_M$. Note that if M' is any other maximal element of $\mathscr{L}$, then $A_{M'} = O_{M'}$ for all $A\leqslant D$. Since $(A_M\wedge D)_M = A_M$ for all $A_M\leqslant D_M$, it follows that φ is an isomorphism if $D_M = I$. If $D_M\neq I$, then the obvious extension of φ to ${}_D\mathscr{L}$ is an isomorphism.

COROLLARY 8. *Let $\mathscr{L}$ be a Noether lattice and let A be an element of $\mathscr{L}$ such that ${}_A\mathscr{L}\subseteq\mathscr{G}(A)$. Then, in the notation of Theorem 4, ${}_A\mathscr{L}$ is a Noether lattice, if and only if, ${}_{(A_i)_{m(E_i)}}(\mathscr{L}_{m(E_i)})$ is a Noether lattice for all i.*

Since Corollary 8 gives special importance to the local case, we rephrase Theorem 3 in that special case.

THEOREM 9. *Let $(\mathscr{L}, M)$ be a local Noether lattice and let $A\neq I$ be an element of $\mathscr{L}$. Then ${}_A\mathscr{L}$ is a Noether lattice if, and only if, $M = A\vee(O:E)$ for every non-zero principal element $E\leqslant A$. In particular, ${}_A\mathscr{L}$ is a Noether lattice if $M = A\vee(O:A)$, and if A is a distributive element of $\mathscr{L}_1$ then this condition is necessary as well as sufficient.*

We now shift our attention to join-irreducible elements.

THEOREM 10. *Let $\mathscr{L}$ be a Noether lattice and $E\in\mathscr{J}(I)$ such that $E\nleqslant\mathscr{J}(\mathscr{L})$. Then E is idempotent and $\mathscr{L} = [E, 0]\oplus[(O:E), D]$.*

Proof. Let M be a maximal element such that $E\nleqslant M$. Then $M\vee E = I$, so $ME\vee E^2 = E$. Since E is join-irreducible, and since $E\nleqslant M$, whereas $ME\leqslant M$, it follows that $E = E^2$.

COROLLARY 11. *Let $\mathscr{L}$ be a Noether lattice such that every element is the join of join-irreducible elements. Then $\mathscr{L}$ is the direct sum of local Noether lattices.*

THEOREM 12. *Let $(\mathscr{L}, M_1, \ldots, M_n)$ be a semi-local Noether lattice. Let $\mathscr{J}(\mathscr{L}) = J$. Then ${}_J\mathscr{L}$ is a Noether lattice, if and only if, $\mathscr{L}$ is the direct sum of the Noether lattices $\mathscr{L}_{M_i}$.*

Proof. Assume ${}_J\mathscr{L}$ is a Noether lattice. Note that if Q is any primary element of $\mathscr{L}$ which is contained in more than one maximal element, say $Q\leqslant M_1\wedge M_2$ $(M_1\neq M_2)$, then necessarily $Q\geqslant J$, since $O:E\nleqslant\sqrt{Q}$ for all join-irreducible elements $E\leqslant J$. Since J is the finite meet of maximal elements, it follows that no primary is contained in more than one maximal element. Since every maximal contains a prime of O, it follows that $\mathscr{L}$ is the direct sum of the Noether lattices $\mathscr{L}/A_i\approx\mathscr{L}_{M_i}$, where A_i is the meet of the primaries of O contained in M_i.

THEOREM 13. *Let $(\mathscr{L}, M_1, \ldots, M_n)$ be a semilocal Noether lattice such that ${}_J\mathscr{L}\subseteq\mathscr{G}(J)$. Then ${}_J\mathscr{L}$ is a Noether lattice.*

Proof. Let E be an element of $\mathscr{J}(J)$ and $A\leqslant E$. Since E is principal in $\mathscr{L}$, necessarily $A = BE$, for some $B\in\mathscr{L}$. If $A = E$, then $A = IE$. Hence, assume $A<E$. Then $A\leqslant ME$, where $M = m(E)$. Since $M'E = E$ for every maximal element $M'\neq m(E)$, it follows that $A\leqslant JE$, and hence that $A = A\wedge JE = ((A:E)\wedge J)E$. It is immediate that E is join principal in ${}_J\mathscr{L}$, so ${}_J\mathscr{L}$ is principally generated and a Noether lattice.

COROLLARY 14. *Let $(\mathscr{L}, M_1, \ldots, M_n)$ be a semilocal Noether lattice such that ${}_J\mathscr{L}\subseteq\mathscr{G}(J)$. Then $\mathscr{L}$ is the direct sum of the localizations $\mathscr{L}_{M_i}$.*

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Quotients of reflexive modules

by

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Abstract. Ist M ein R-Rechtsmodul und S ein fester R-Bimodul, so heißt M in Verallgemeinerung einer wohlbekannten Begriffsbildung S-reflexiv, wenn die kanonische Abbildung $\sigma\colon M \to M^{**}$ ein Isomorphismus ist, wobei $M^* = \mathrm{Hom}_R(M, S)$ ist. Beispiele hierfür sind endlich erzeugte K-Vektorräume mit $R = S = K$ oder $M = S^I$ (I eine nicht meßbare Indexmenge) mit $S \subsetneq \boldsymbol{Q}$ und $R = \boldsymbol{Z}$. Es werden Moduln G der Form $0 \to K \to Y \to X \to G \to 0$ mit S-reflexiven X, Y untersucht. Ist ferner jeder Teilmodul von X^* projektiv, so gilt der allgemeine Satz: $G \cong D \oplus \mathrm{Ext}^1_R(A, S)$, wobei D ein direkter Summand von Y ist und $K \cong \mathrm{Hom}_R(A, S)$. Aus diesem Resultat lassen sich viele interessante Spezialfälle herleiten: Ist R ein schlanker Ring (z.B. ein abzählbarer Dedekindring, der kein Körper ist), so erfüllen die cartesischen Potenzen von R die Voraussetzungen des Satzes. Sind X, Y, S abelsche Gruppen, so heiße (X, Y) BEŁZ-Paar bezüglich S, falls X und Y S-reflexiv sind, $\mathrm{Ext}(X^*, S) = 0$ ist und ferner für jeden Homomorphismus $f\colon X \to Y$ der Annihilator $f(X)^{\perp}$ von $f(X)$ in Y^* ein direkter Summand von Y^* ist. Obiges Resultat läßt sich nun sofort auf BEŁZ-Paare anwenden. Ist G schlank und $\mathrm{End}(G) \cong \boldsymbol{Z}$, so ist (G^I, G^J) ein BEŁZ-Paar bezüglich G. Ebenso ist $(\boldsymbol{Z}^I, R)$ ein BEŁZ-Paar bezüglich $\boldsymbol{Z}$, falls R aus der Reid-Klasse (Kleinste Klasse, die $\boldsymbol{Z}$ enthält und unter $\oplus$ und $\prod$ abgeschlossen ist) ist.

Ferner wird gezeigt: Ist R ein schlanker Dedekindring und $A \subset R^J$, so sind im Universum $V = L$ äquivalent: (1) $A \cong R^I$ oder A projektiv und endlich erzeugt. (2) A ist direkter Summand von R^J (3) $R^J/A \cong R^I$ oder R^J/A ist projektiv und endlich erzeugt.

§ 1. Introduction. Let M^I be the cartesian power (over I) of some right R-module $M = M_R$, i.e. the set of all functions on the set I with values in M and the scalar multiplication and addition defined by components. The aim of this paper is, to obtain informations about the structure of quotient R-modules G of the form

$$(*) \qquad 0 \to K \to X \to Y \to G \to 0$$

with certain conditions posed on X and Y as explained in the following. In M. Dugas and R. Göbel [6; Satz p. 15] the structure of G was determined explicitly in a model of ZFC+V = L where $K = 0$, $R = \boldsymbol{Z}$ and X, Y are isomorphic to cartesian powers of $\boldsymbol{Z}$ over any non measurable sets; cf. Remark after (3.5). In particular we get

(**) $G \cong D \oplus \mathrm{Ext}_R(A, S)$ where D is a direct summand of Y and $K = \mathrm{Hom}_R(A, S)$ for a R-bimodule S and some left R-module A.

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