On pointwise smooth dendroids

by

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Abstract. In this paper we study properties of pointwise smooth dendroids; in particular, we give some of their connections with the set function T and the concept of an R^3-continuum.

§ 1. Introduction Investigating smooth dendroids, defined by J. J. Charatonik and C. Eberhart in [3], we have observed that this concept is not good enough to characterize hereditarily contractible dendroids. In this paper some properties of another concept, namely that of pointwise smoothness, introduced by the author in [9] are studied. In particular, we give some characterization of pointwise smooth dendroids and we discuss relations between this concept and some other related notions known in the literature.

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§ 2. Definitions and preliminaries. A continuum means a compact, connected metric space. A property of a continuum X is said to be *hereditary* if each subcontinuum of X has that property. A continuum X is said to be *arcwise connected* if for every two points a and b of X there exists an arc ab joining a with b and contained in X. A continuum X is called *unicoherent* if for each two subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B$ is connected. A dendroid means an arcwise connected and hereditarily unicoherent continuum. A point p of an arcwise connected space X is called a *ramification point of* X provided there are three arcs, pa, pb and pc, such that p is the only common point of every two of them. A dendroid which has only one ramification point is called a *fan*.

The closure of a set $A \subset X$ is denoted by $\overline{A}$, and we put $\operatorname{Fr} A = \overline{A} \cap \overline{X \setminus A}$ for the boundary of A and $\operatorname{Int} A = X \setminus \overline{X \setminus A}$ for the interior of A.

Given a sequence of subsets A_n of a topological space X, we denote by $\operatorname{Ls} A_n (\operatorname{Li} A_n)$ the set of all points $x \in X$ for which every neighbourhood intersects A_n for arbitrarily large n (for infinitely many n). If $\operatorname{Li} A_n = \operatorname{Ls} A_n$, then we say that a sequence A_n is *convergent* and we put $\operatorname{Lt} A_n = \operatorname{Li} A_n = \operatorname{Ls} A_n$.

We say that a continuum X is *connected im kleinen at a point* $x \in X$ if for each open neighbourhood of a point x there exists a subcontinuum K of X such that $x \in \operatorname{Int} K \subset K \subset U$.

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Given a set A contained in a continuum X, we define TA as the set of all points x of X such that every subcontinuum of X which contains x in its interior must intersect A. We write simply Ta instead of $T\{a\}$.

(2.1) DEFINITION (see [3], p. 298). A dendroid X is said to be *smooth* if there exists a point $p \in X$, called an initial point of X, such that for every sequence of points a_n of X convergent to a point a the sequence of arcs pa_n is convergent and $\mathrm{Lt}\, pa_n = pa$.

(2.2) DEFINITION (see [9], Definition 2, p. 216). A dendroid X is said to be *pointwise smooth* if for each $x \in X$ there exists a point $p(x) \in X$ such that for each sequence x_n convergent to a point x we have $\mathrm{Lt}\, x_n p(x) = xp(x)$. A point $p(x)$ will be called the *initial point for* x *in* X.

It was proved in [9] that

(2.3) PROPOSITION (see [9], Proposition 1, p. 216). *If a dendroid X is pointwise smooth, then every subdendroid of X is also pointwise smooth (the heredity of pointwise smoothness for dendroids).*

(2.4) PROPOSITION (see [9], Theorem 1, p. 216). *A fan is pointwise smooth if and only if it is smooth.*

(2.5) COROLLARY (see [9], Corollary 1, p. 216). *A fan is hereditarily contractible if and only if it is pointwise smooth.*

The following theorem is an immediate consequence of Theorem 2 in [9], p. 217:

(2.6) THEOREM. *Let a point x_0 in a dendroid X be given. The following statements are equivalent:*

1) *every point of X is an initial point for x_0;*
2) *for each point $y \in X \setminus \{x_0\}$ we have $Ty \subset X \setminus \{x_0\}$;*
3) *X is connected im kleinen at x_0.*

§ 3. Some characterizations of pointwise smooth dendroids. We say that a *dendroid X has property* [T] if, for each $y, x \in X$, $x \neq y$, we have $Tx \cap xy = \{x\}$ or $Ty \cap xy = \{y\}$, or $Tx \cap Ty = \emptyset$.

(3.1) THEOREM. *A dendroid X is pointwise smooth if and only if X has property* [T].

Proof. Let a dendroid X be pointwise smooth. Suppose there are two points x and y in X such that $Tx \cap xy \neq \{x\}$ and $Ty \cap xy \neq \{y\}$, and $Tx \cap Ty \neq \emptyset$. We conclude from the hereditary unicoherence of X that $Tx \cap Ty \cap xy \neq \emptyset$. So let $z \in Tx \cap Ty \cap xy$. For each subcontinuum K of X with $z \in \mathrm{Int}\, K$ we have $\{x, y\} \subset K$ by the definition of the set function T, and thus $xy \subset K$. Let z_n be a sequence of points of X converging to z. Therefore $zx \subset \mathrm{Ls}\, z_n x$ and $zy \subset \mathrm{Ls}\, z_n y$. It follows from the pointwise smoothness of X that the sequence of arcs $z_n p(z)$ converges to the arc $zp(z)$.

We claim that $x \in zp(z)$.

Indeed, for each positive integer n let U_n be the open ball of radius $1/n$ with centre z. The smallest subcontinuum of X which contains U_n is the closure of the union of all arcs zt with t in U_n. Since x is contained in every subcontinuum of X which contains z in its interior, for each positive integer n there is a point a_n of U_n such that the arc za_n contains a point b_n at a distance less than $1/n$ from x. The sequence a_n converges to z and $za_n \subset a_n p(z) \cup zp(z)$. It follows from the definition of b_n that $x \in \mathrm{Li}\, za_n$. Since $\mathrm{Li}\, za_n \subset \mathrm{Lt}(a_n p(z) \cup zp(z)) = zp(z)$, the claim is proved. Similarly $y \in zp(z)$, and thus $xy \subset zp(z)$. We conclude from $z \in xy$ that either $z = x$ or $z = y$. We now have $Tx \cap Ty \cap xy = \{x\}$ or $Tx \cap Ty \cap xy = \{y\}$. Let $Tx \cap Ty \cap xy = \{x\}$. Since $x, y \in Ty \cap xy$, we conclude that $xy \subset Ty \cap xy$ and $Tx \cap (Ty \cap xy) \supset Tx \cap xy \neq \{x\}$. This is a contradiction.

Now, let X have property [T]. We have to prove that X is pointwise smooth. Let $x \in X$. If X is connected in kleinen at x, then by Theorem (2.6) we can put $p(x) = x$. If X is not connected im kleinen at x, then there exists a point $y \in X \setminus \{x\}$ such that $x \in Ty$ (Theorem (2.6)). Let us observe that

(i) for each $z, y \in X$ such that $x \in Tz \cap Ty$ we have either $z \in xy$ or $y \in xz$.

Indeed, let z and y be such that $x \in Tz \cap Ty$ and suppose that $z \notin xy$ and $y \notin xz$. The union $yx \cup xz$ is an arc or a triod. If $yx \cup xz$ is an arc, then $Tz \cap yz \supset xz$ and $Ty \cap yz \supset xy$, contrary to [T]. If $yx \cup xz$ is a triod with a point t as its centre, then let us observe that $t \notin \{y, z\}$, $Tz \cap yz \supset tz$ and $Ty \cap yz \supset ty$, contrary to [T].

Therefore it follows from (i) that the set $Z = \{y \in X\colon\ x \in Ty\}$ lies in an arc having x as its end-point. Define $p(x)$ as the other end-point (different from x) of the minimal arc containing the set Z. It is easy to see that the set Z is closed and that $x \in Tp(x)$. Let a sequence x_n be convergent to a point x. Thus

$$xp(x) \subset \mathrm{Li}\, x_n p(x) \subset \mathrm{Ls}\, x_n p(x)\,.$$

Now, let $q \in \mathrm{Ls}\, x_n p(x)$. For each subcontinuum K with property $x \in \mathrm{Int}\, K$ we have $p(x) \in K$ and $x_n p(x) \subset K$ for almost all n. Therefore $\mathrm{Ls}\, x_n p(x) \subset K$, whence $q \in K$, and we have shown that $x \in Tq$. Thus $q \in Z \subset xp(x)$, and so we have $\mathrm{Ls}\, x_n p(x) \subset xp(x)$. By the double inclusion above, this leads to $\mathrm{Lt}\, x_n p(x) = xp(x)$, and the proof is complete.

Note in the proof above that if X has property [T], then for each point $x \in X$ we can choose the point $p(x)$ in such a way that $x \in Tp(x)$. Hence we have the following

(3.2) COROLLARY. *A dendroid X is pointwise smooth if and only if for each $x \in X$ there exists a point $p(x) \in X$ such that*

(1) *for each sequence x_n convergent to a point x we have $\mathrm{Lt}\, x_n p(x) = xp(x)$ and*

(2) $x \in Tp(x)$.

Given two disjoint subcontinua A and B of X, we denote by $I(A, B)$ a subcontinuum of X which is irreducible with respect to containing $A \cup B$. It is quite easy to see that there are two points, $a_0 \in A$ and $b_0 \in B$, such that $I(A, B) = A \cup B \cup a_0 b_0$ and $a_0 b_0 \cap A = \{a_0\}$, and $a_0 b_0 \cap B = \{b_0\}$ (cf. e.g. [2], Theorem 21, p. 195 and Theorem 27, p. 197).

(3.3) Corollary. *The following conditions are equivalent for a dendroid* X:

(1) X *is pointwise smooth,*

(2) *for every two disjoint subcontinua* A *and* B *of* X, *either* $TA\cap I(A,B)=A$ *or* $TB\cap I(A,B)=B$ *or* $TA\cap TB=\emptyset$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate by Theorem (3.1). Let a dendroid X be pointwise smooth and let disjoint subcontinua A and B of X satisfy

(*) $$[TA\cap I(A,B)]\setminus A\neq\emptyset \quad\text{and}\quad [TB\cap I(A,B)]\setminus B\neq\emptyset .$$

Observe that

(i) $$TA\cap a_0b_0=Ta_0\cap a_0b_0$$

and

(i′) $$TB\cap a_0b_0=Tb_0\cap a_0b_0 .$$

Indeed, $Ta_0\cap a_0b_0\subset TA\cap a_0b_0$. Let $z\in TA\cap a_0b_0$; this means, by the definition of set function T, that for each subcontinuum $K\subset X$ such that $z\in\operatorname{Int}K$ we have $K\cap A\neq\emptyset$. Thus it follows from the hereditary unicoherence of X that $a_0\in K$, and so $z\in Ta_0$. Hence $z\in Ta_0\cap a_0b_0$. In the same way we can prove (i′).

We claim that

(ii) $$(TA)\cap B=\emptyset=(TB)\cap A .$$

Indeed, if $(TA)\cap B\neq\emptyset$ $((TB)\cap A\neq\emptyset)$ since $TA(TB)$ is a continuum and X is hereditarily unicoherent, then $a_0b_0\subset TA$ $(a_0b_0\subset TB)$, and so $TA\cap a_0b_0=a_0b_0$ $(TB\cap a_0b_0=a_0b_0)$. Thereby from (i) ((i′)) we have $b_0\in Ta_0\cap Tb_0\neq\emptyset$. Since (*) implies that $Ta_0\cap a_0b_0\neq\{a_0\}$ and $Tb_0\cap a_0b_0\neq\{b_0\}$, we get a contradiction with the pointwise smoothness of X by Theorem (3.1). So the claim is proved.

Now, by (i), (i′) and (ii) we have

$$TA\cap I(A,B)=TA\cap(a_0b_0\cup A\cup B)=(TA\cap a_0b_0)\cup A=(Ta_0\cap a_0b_0)\cup A$$

and

$$TB\cap I(A,B)=(Tb_0\cap a_0b_0)\cup B ,$$

whence $TA\cap TB\cap I(A,B)=Ta_0\cap Tb_0\cap a_0b_0$. Recall that

$$[TA\cap I(A,B)]\setminus A\neq\emptyset$$

implies $Ta_0\cap a_0b_0\neq\{a_0\}$ and $[TB\cap I(A,B)]\setminus B\neq\emptyset$ implies $Tb_0\cap a_0b_0\neq\{b_0\}$. Thus it follows from the pointwise smoothness of X by Theorem (3.1) that

$$Ta_0\cap Tb_0\cap a_0b_0=TA\cap TB\cap I(A,B)=\emptyset .$$

By the hereditary unicoherence of X this last condition implies $TA\cap TB=\emptyset$. The proof is complete.

(3.4) Definition (see [8], Definition 1.3, p. 75). A non-empty subcontinuum $K\neq X$ of a dendroid X is called an R^3-*continuum in* X if there exist an open set U such that $K\subset U$, and a sequence C_n of components of U such that $\operatorname{Li}C_n=K$.

We say that a dendroid X has *property* [R] if there exist two subcontinua A and Y of X such that $A\subset Y\subset X$ and A is an R^3-continuum in Y.

The following two propositions are proved in [8] and [6]:

(3.5) Proposition. *If a dendroid* X *contains an* R^3-*continuum, then it is not contractible.*

(3.6) Proposition. *If a dendroid* X *has property* [R], *then it is not hereditarily contractible.*

(3.7) Theorem. *If a dendroid is not pointwise smooth, then it has property* [R].

Proof. If a dendroid X is not pointwise smooth, then by Theorem (3.1) there exist two different points x and y of X such that $Tx\cap xy\neq\{x\}$, $Ty\cap xy\neq\{y\}$ and $Tx\cap Ty\neq\emptyset$.

If $x\notin Ty$ and $y\notin Tx$, then the subcontinuum $Tx\cap Ty$ contains an R^3-continuum (see [7], Theorem 7, p. 305). Let $x\in Ty$. Then $Ty\cap xy=xy$. Take $z\in(xy\setminus\{x,y\})\cap Tx$, and let U be an open ball with centre at z such that $\{x,y\}\cap\overline{U}=\emptyset$. Observe that $z\in Tx\cap Ty\cap xy$. Let U_n denote the open ball with centre z and radius $1/n$. The smallest subcontinuum of X which contains U_n is the closure of the union of all arcs zt with t in U_n. Since x and y are contained in every subcontinuum of X which contains z in its interior, for each positive integer n there are points z_n^1 and z_n^2 of U_n such that the arc zz_n^1 contains a point a_n at a distance less than $1/n$ from y and the arc zz_n^2 contains a point b_n at a distance less than $1/n$ from x. We can choose the sequences z_n^1 and z_n^2 in a such way that the following conditions hold:

(i) the sequences z_n^1 and z_n^2 are convergent to z and

(ii) $\operatorname{Ls}z_n^1z\cap zx\subset\overline{U}$, $\operatorname{Ls}z_n^2z\cap zy\subset\overline{U}$.

Let $Y=\bigcup_{n=1}^{\infty}(z_n^1z\cup z_n^2z)\cup\operatorname{Ls}z_n^1z\cup\operatorname{Ls}z_n^2z$. It is easy to see that Y is a subcontinuum of X and that $x\notin T_Yy$, $y\notin T_Yx$ and $T_Yx\cap T_Yy\neq\emptyset$, and so $T_Yx\cap T_Yy$ contains an R^3-continuum (see [7] and [8]). The proof is complete.

(3.8) Theorem. *If a dendroid* X *has property* [R], *then it is not pointwise smooth.*

Proof. Let Y be a subcontinuum of X and A be an R^3-continuum in Y, and let C_n be a sequence of components of the open set U such that $A\subset U$ in Y with $\operatorname{Li}C_n=A$. Let $x\in A$ and $x_n\in C_n$ where x_n is convergent to a point x. If X is pointwise smooth, then Y is also pointwise smooth by Proposition (2.3), and let $p(x)\in Y$ denote the initial point for x in Y. If $p(x)$ belongs to the same component of U as x, then $xp(x)\subset U$ and thus $xp(x)\cap\operatorname{Fr}U=\emptyset$, but by the definition of x_n it is quite easy to see that $\operatorname{Ls}x_np(x)\cap\operatorname{Fr}U\neq\emptyset$. This contradiction shows us that $p(x)$ does not belong to the same component of U as x. Let q denote the first point of the arc $xp(x)$ (ordered from x to $p(x)$) which is in $\operatorname{Fr}U$ and let B_n denote the component of $x_np(x)\cap C_n$ which contains x_n.

We claim that $q\in\operatorname{Li}B_n$.

Indeed, let q_n denote the first point of the arc $x_np(x)$ from x_n to $p(x)$ which is in $\operatorname{Fr} U$. Let q' be the limit point of a subsequence q_{n_k} of the sequence q_n. Since $xq' \subset \operatorname{Li} B_{n_k}$ and $q' \in \operatorname{Fr} U$, we find, by definition of q, that $q \in xq'$. Consequently $q \in \operatorname{Li} B_n$.

Therefore $\operatorname{Li} B_n \cap \operatorname{Fr} U \neq \emptyset$, but $B_n \subset C_n$, and from this $\operatorname{Li} B_n \subset \operatorname{Li} C_n = A \subset X \setminus \operatorname{Fr} U$; so $\operatorname{Li} B_n \cap \operatorname{Fr} U = \emptyset$. This contradiction finishes the proof.

(3.9) COROLLARY. *A dendroid X is pointwise smooth if and only if it does not have property* [*R*].

Hence it follows by a result from [8] that

(3.10) COROLLARY. *If a dendroid X is hereditarily contractible, then it is pointwise smooth.*

The following question seems to be natural:

(3.11) QUESTION. Does pointwise smoothness imply hereditary contractibility, of a dendroid X?

For fans Corollary (2.5) gives a positive answer to this question.

§ 4. Some remarks on other weaker forms of the concept of smoothness. A dendroid X is said to be *semismooth* (see [3], p. 306) provided there exists in X a point p such that whenever a_n converges to a, then $\operatorname{Ls} pa_n$ is an arc.

A dendroid X is said to be *weakly smooth* (see [10], p. 111, and Theorem 4, p. 114) provided there exists in X a point p such that, whenever x_n converges to x, then $\operatorname{Li} px_n = py$ for a certain y from X.

In this section we discuss all relations between the concept of pointwise smoothness and semismoothness, and weak smoothness.

Let $\overline{xy}$ denote a straight line segment with x and y as its end-points.

In the following three examples let (x, y) denote a point of the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy.

(4.1) EXAMPLE. Put

$$q = (0,0),\ k = (1,0),\ t = (2,0),\ z = (-2,0),$$
$$a_n = (2, 1/n),\ b_n = (2+1/n, 0),\ c_n = (2, -1/n),\ d_n = (-2, -1/n),$$
$$e_n = (-2-1/n, 0),\ f_n = (-2, 1/n),\ g_n = (-1, 1/n).$$

Let $A_0 = \overline{zt}$, $A_n = \overline{ka_n} \cup \overline{a_nb_n} \cup \overline{b_nc_n} \cup \overline{c_nd_n} \cup \overline{d_ne_n} \cup \overline{e_nf_n} \cup \overline{f_ng_n}$, $X = \bigcup_{n=0}^{\infty} A_n$ (see Fig. 1).

We can easily see that a dendroid X is not pointwise smooth and it is weakly smooth if we take as a point p, in the definition of weak smoothness, point z or point p only.

Let us take a map ψ of $X \times \{0\} \cup X \times \{1\}$, which identifies points $(q, 0)$ and $(q, 1)$. Put $Y = \psi(X \times \{0\} \cup X \times \{1\})$.

It is easy to see that Y is semismooth but it is not weakly smooth and not pointwise smooth.

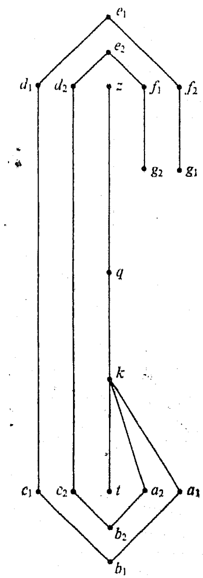

Fig. 1

(4.2) EXAMPLE. We define

$$A = \{(x, y)\colon 0 \leqslant x \leqslant 2, y = 1\},$$
$$A' = \{(x, y)\colon 0 \leqslant x \leqslant 2, y = -1\},\ B = \{(x, y)\colon x = 1, -1 \leqslant y \leqslant 1\},$$
$$a = (0, 1),\ a' = (0, -1),\ a_n = (2, 1+1/n),\ a'_n = (2, -1-1/n).$$

Let $X = A \cup A' \cup B \cup \bigcup_{n=1}^{\infty} \overline{aa_n} \cup \bigcup_{n=1}^{\infty} \overline{a'a'_n}$ (see Fig. 2).

It is easy to see that X is pointwise smooth but it is not semismooth and it is not weakly smooth.

(4.3) EXAMPLE. Put $p = (-2, 0)$, $q = (2, 0)$, $a_n = (-1, 1/n)$, $b_n = (1, 1/n)$.

Let $A_n = \overline{a_np} \cup \overline{pq} \cup \overline{qb_n}$, $X = \bigcup_{n=1}^{\infty} A_n$.

The dendroid X is semismooth, weakly smooth and pointwise smooth but it is not smooth.

(4.4) EXAMPLE. Let Y be a dendroid described in [5] (Proposition 12, pp. 234–235) (see Fig. 3). It is quite easy to see that Y is semismooth and weakly smooth, but Y contains an R^3-continuum (namely the arc ab), and so it is not pointwise smooth.

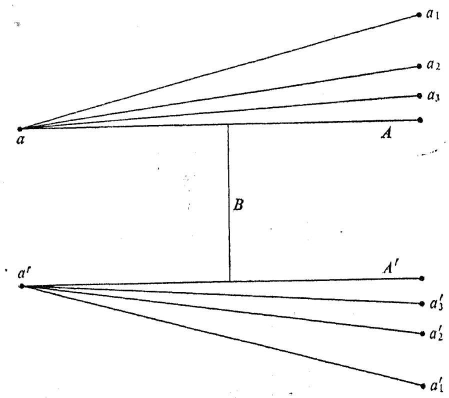

Fig. 2

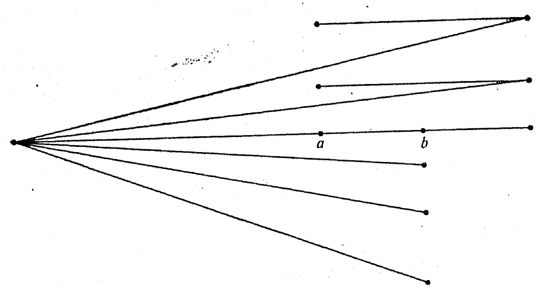

Fig. 3

(4.5) EXAMPLE. Let X denote a fan with the top a described in [4], (p. 95).

Let $X_i = X \times \{i\}$, for $i = 0, 1, 2$. Let us take a map ψ of $X_1 \cup X_2 \cup X_3$ which identifies points $(a, 0)$, $(a, 1)$ and $(a, 2)$. Put $Y = \psi(X_1 \cup X_2 \cup X_3)$.

It is easy to see that Y is a weakly smooth fan but it is not semismooth and not pointwise smooth.

(4.6) THEOREM. *If a dendroid X is pointwise smooth and weakly smooth, then it is also semismooth.*

Proof. The dendroid X is weakly smooth, and so there exists a point p_0 such that for each point x and each sequence of points x_n converging to a point x there exists a point y such that $\mathrm{Li}\, p_0 x_n$ is an arc $p_0 y$. Let $p(x)$ denote an initial point for x in X. For each x from X we have

$$\begin{aligned} p_0 x \subset \mathrm{Li}\, p_0 x_n = p_0 y \subset \mathrm{Ls}\, p_0 x_n \subset \mathrm{Ls}\big(p_0 p(x) \cup p(x) x_n\big) \\ = \mathrm{Ls}\, p_0 p(x) \cup \mathrm{Ls}\, p(x) x_n = p_0 p(x) \cup p(x) x\,. \end{aligned}$$

If $\mathrm{Ls}\, p_0 x_n$ is an arc for each sequence x_n converging to a point x, then X is semismooth.

If there exist a point x_0 and a sequence of points x_n converging to a point x_0, such that $\mathrm{Ls}\, p_0 x_n$ is not an arc, then let t denote the centre of the triod $p_0 p(x_0) \cup p(x_0) x_0$ (this means that t is a such point that $p_0 p(x_0) \cup p(x_0) x_0 = p_0 t \cup p(x_0) t \cup x_0 t$ and $p_0 t \cap p(x_0) t = p_0 t \cap x_0 t = p(x_0) t \cap x_0 t = \{t\}$) and let k be a point of $[\mathrm{Ls}\, p_0 x_n \cap t p(x_0)] \setminus \{t\}$.

We can choose a subsequence x_{n_i} from the sequence x_n such that on the arcs $p_0 x_{n_i}$ there exist points k_i converging to a point k. For this subsequence we have $\mathrm{Li}\, p_0 x_{n_i} \supset p_0 x_0 \cup tk$, and so $\mathrm{Li}\, p_0 x_{n_i}$ is not an arc. This contradicts the assumption that X is weakly smooth with the initial point p_0. So $\mathrm{Ls}\, p_0 x_n$ is always an arc, and thus X is semismooth. The proof is complete.

(4.7) THEOREM. *If a dendroid X is pointwise smooth and semismooth, then there exists a point $p \in X$ such that X is connected im kleinen at p and for each x and each sequence x_n convergent to a point x the equality $\mathrm{Ls}\, x_n p = py$ holds for a certain $y \in X$.*

Proof. Let a dendroid X be pointwise smooth and semismooth. It follows from the pointwise smoothness of X that

$$\mathrm{Ls}\, x_n x \subset \mathrm{Ls}\big(x_n p(x) \cup p(x) x\big) = \mathrm{Ls}\, x_n p(x) \cup p(x) x = x p(x)\,;$$

so, for each x and each sequence x_n converging to a point x, the set $\mathrm{Ls}\, x_n x$ is an arc with x as one of its end-points. Let p' be an initial point of X in the definition of the semismoothness of X. This means that $\mathrm{Ls}\, x_n p'$ is an arc for each sequence x_n converging to a point x.

Case 1. If X is connected im kleinen at p', then $\mathrm{Ls}\, x_n p' = y_0 p'$ for each x and each sequence x_n convergent to a point x, and for a certain y_0.

Indeed, otherwise we would have for a certain x and for a certain sequence x_n converging to a point x, $\mathrm{Ls}\, x_n p' = ab$, where $p' \notin \{a, b\}$ and thus $p' \neq x \in xa$. Obviously either $p' \in xa$ or $p' \in xb$. If $p' \in xa$, then by the definition of the set function T, we would have $x \in Ta$ and thus $p' \in Ta$. This contradicts Theorem (2.6). If $p' \in xb$, the argumentation is the same.

Case 2. If X is not connected im kleinen at p', then by Theorem (2.6) there exists a point $y \neq p'$ such that $p' \in Ty$. We shall now define by induction a sequence of points p_n where n is a non-negative integer. Put $p_0 = p'$. By Corollary (3.2) we can define $p_1 = p(p_0)$ such that $p_0 \in Tp_1$. In the same way we define $p_{n+1} = p(p_n)$.

Observe that

$$p_{i-1} p_i \cap p_i p_{i+1} = \{p_i\}\,. \tag{1}$$

Indeed, we can see by the definition of p_n that $p_{i-1} p_i \subset Tp_i$ and $p_i p_{i+1} \subset Tp_{i+1}$. Since $Tp_i \cap Tp_{i+1} \neq \emptyset$ and $p_i p_{i+1} \cap Tp_{i+1} \neq \{p_{i+1}\}$, it follows from Theorem (3.1) that $p_{i-1} p_i \cap p_i p_{i+1} \subset Tp_i \cap p_i p_{i+1} = \{p_i\}$.

It follows from (1) that there exists an arc K with p_0 as one of its end-points such that all points p_n belong to K, and from this it is easy to see that

(2) for each $i = 0, 1, 2, \ldots$ and for each sequence x_n converging to a point x the set $\mathrm{Ls}\, x_n p_i$ is an arc (see [10], Lemma, p. 18).

Let as now extend the definition of p_n to that of p_α, where α is an arbitrary ordinal number, putting $p_{\alpha+1} = p(p_\alpha)$ with $p_\alpha \in Tp_{\alpha+1}$ (as before, see Corollary (3.2)), and $p_\lambda = \lim\{p_\alpha\colon \alpha<\lambda\}$, λ being a limit ordinal.

From (1), (2) and the definition of p_α we have

(1'') if $\alpha<\beta<\lambda$, then $p_\alpha p_\beta \cap p_\beta p_\lambda = \{p_\beta\}$,

(2'') for each ordinal number α and each sequence x_n convergent to a point x the set $\mathrm{Ls}\, x_n p_\alpha$ is an arc (see [10], Lemma p. 18).

Let now define p'' as the other end-point (different from p_0) of the minimal arc containing the set $P = \{p_\alpha\colon \alpha \text{ is an ordinal number}\}$. It is quite easy to see (cf. Theorem (2.6)) that

(3) X is connected im kleinen at p'',

and thus, by (2'') we conclude that (see Case 1)

(4) for each sequence x_n converging to a point x we have $\mathrm{Ls}\, x_n p = y_1 p''$ for a certain y_1.

The proof of theorem is complete.

Note in the proof above that if X is pointwise smooth and in the sequence p_α we start from $p_0 = x$, where $x \in X$, then, using the same arguments, we have the following

(4.8) Corollary. *A dendroid X is pointwise smooth if and only if for each $x \in X$ there exists a point $p(x) \in X$ such that*

(1) *for each sequence x_n converging to a point x we have $\mathrm{Lt}\, x_n p(x) = xp(x)$ and*

(2) *X is connected im kleinen at $p(x)$.*

As an easy consequence of Theorem (4.7) we have the following

(4.9) Corollary. *If a dendroid X is pointwise smooth and semismooth, then it is also weakly smooth.*

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