On pointwise smooth dendroids

by

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Abstract. In this paper we study properties of pointwise smooth dendroids; in particular, we give some of their connections with the set function \( T \) and the concept of an \( H \)-continuum.

§ 1. Introduction Investigating smooth dendroids, defined by J. J. Charatonik and C. Eberhart in [3], we have observed that this concept is not good enough to characterize hereditarily contractible dendroids. In this paper some properties of another concept, namely that of pointwise smoothness, introduced by the author in [9] are studied. In particular, we give some characterization of pointwise smooth dendroids and we discuss relations between this concept and some other related notions known in the literature.

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§ 2. Definitions and preliminaries. A continuum means a compact, connected metric space. A property of a continuum \( X \) is said to be hereditary if each subcontinuum of \( X \) has that property. A continuum \( X \) is said to be arcwise connected if for every two points \( a \) and \( b \) of \( X \) there exists an arc \( ab \) joining \( a \) with \( b \) and contained in \( X \). A continuum \( X \) is called unicoherent if for each two subcontinua \( A \) and \( B \) of \( X \) such that \( X = A \cup B \) the intersection \( A \cap B \) is connected. A dendroid means an arcwise connected and hereditarily unicoherent continuum. A point \( p \) of an arcwise connected space \( X \) is called a ramification point of \( X \) provided there are three arcs, \( pa, pb \) and \( pc \), such that \( p \) is the only common point of every two of them. A dendroid which has only one ramification point is called a fan.

The closure of a set \( A \subset X \) is denoted by \( \overline{A} \), and we put \( \text{Fr} A = \overline{A} \cap X \backslash A \) for the boundary of \( A \) and \( \text{Int} A = X \backslash \overline{X \backslash A} \) for the interior of \( A \).

Given a sequence of subsets \( A_n \) of a topological space \( X \), we denote by \( \text{Ls} A_n (\text{Li} A_n) \) the set of all points \( x \in X \) for which every neighbourhood intersects \( A_n \) for arbitrarily large \( n \) (for infinitely many \( n \)). If \( \text{Li} A_n = \text{Ls} A_n \), then we say that a sequence \( A_n \) is convergent and we put \( \text{Li} A_n = \text{Li} A_n = \text{Ls} A_n \).

We say that a continuum \( X \) is connected in kleinen at a point \( x \in X \) if for each open neighbourhood of a point \( x \) there exists a subcontinuum \( K \) of \( X \) such that \( x \in \text{Int} K \subset K \subset U \).
Given a set $A$ contained in a continuum $X$, we define $TA$ as the set of all points $x$ of $X$ such that every subcontinuum of $X$ containing $x$ in its interior must intersect $A$. We write simply $T$ instead of $T(A)$.

(2.1) Definition (see [9], p. 298). A dendroid $X$ is said to be smooth if there exists a point $p$ in $X$, called an initial point of $X$, such that for every sequence of points $a_n$ of $X$ convergent to a point $a$ the sequence of arcs $pa_n$ is convergent and $L(p) = pa$.

(2.2) Definition (see [9], Definition 2, p. 216). A dendroid $X$ is said to be pointwise smooth if for each $x \in X$ there exists a point $p(x) \in X$ such that for each sequence $x_n$ convergent to a point $x$ we have $L(x_n, p(x)) = x_p(x)$. A point $p(x)$ will be called the initial point for $x$ in $X$.

It was proved in [9] that

(2.3) Proposition (see [9], Proposition 1, p. 216). If a dendroid $X$ is pointwise smooth, then every subdendroid of $X$ is also pointwise smooth (the heredity of pointwise smoothness for dendroids).

(2.4) Proposition (see [9], Theorem 1, p. 216). A fan is pointwise smooth if and only if it is smooth.

(2.5) Corollary (see [9], Corollary 1, p. 216). A fan is hereditarily contractible if and only if it is pointwise smooth.

The following theorem is an immediate consequence of Theorem 2 in [9], p. 217:

(2.6) Theorem. Let a point $x_0$ in a dendroid $X$ be given. The following statements are equivalent:
1) every point of $X$ is an initial point for $x_0$;
2) for each point $y \in X \setminus x_0$, we have $Ty = X \setminus x_0$;
3) $X$ is connected im kleinen at $x_0$.

§ 3. Some characterizations of pointwise smooth dendroids. We say that a dendroid $X$ has property [7] if, for each $x$, $x \in X$, $x \neq x$, we have $Tx \cap xy = \{x\}$ or $Ty \cap xy = \{y\}$, or $Tx \cap Ty = \emptyset$.

(3.1) Theorem. A dendroid $X$ is pointwise smooth if and only if $X$ has property [7].

Proof. Let a dendroid $X$ be pointwise smooth. Suppose there are two points $x$ and $y$ in $X$ such that $x \cap xy = \{x\}$ and $Ty \cap xy = \{y\}$, and $Tx \cap Ty = \emptyset$. We conclude from the hereditary uniqueceness of $X$ that $Tx \cap xy = \emptyset$. So let $x \in Tx \cap Ty$. For each subcontinuum $K$ of $X$ with $x \in K$ we have $x, y \in K$ by the definition of the set function $T$, and thus $xy \subseteq K$. Let $s_n$ be a sequence of points of $X$ converging to $z$. Therefore $xs_n, z, xz = s_n, z, xz$. If follows from the pointwise smoothness of $X$ that the sequence of arcs $s_n(p(x))$ converges to the arc $x_p(x)$.

We claim that $x \in x_p(x)$.

Indeed, for each positive integer $n$ let $U_n$ be the open ball of radius $1/n$ with centre $x$. The smallest subcontinuum of $X$ which contains $U_n$ is the closure of the union of all arcs $zt$ with $t$ in $U_n$. Since $x$ is contained in every subcontinuum of $X$ which contains $x$ in its interior, for each positive integer $n$ there is a point $a_n$ of $U_n$ such that the arc $a_nz$ contains a point $b_n$ at a distance less than $1/n$ from $x$. The sequence $a_n$ converges to $x$ and $a_nz \in x_p(x) \cup x_p(x)$. It follows from the definition of $b_n$ that $x \in a_nz$. Since $a_nz \subseteq x_p(x) \cup x_p(x)$, the claim is proved.

Similarly $y \in x_p(x)$, and thus $xy \subseteq x_p(x)$. We conclude from $x \neq y$ that $x \neq x$ or $z \neq x$. We now have $Tx \cap Ty \cap xy = \{x\}$ or $Tx \cap Ty \cap xy = \{y\}$. Let $Tx \cap Ty \cap xy = \{y\}$. Since $x, y \in Tx \cap Ty$, we conclude that $xy \subseteq Ty \cap Ty$ and $x \cap (Ty \cap Ty) = Tx \cap x \neq \{x\}$. This is a contradiction.

Now, let $X$ have property [7]. We have to prove that $X$ is pointwise smooth. Let $x \in X$. If $X$ is connected in klein end, then by Theorem (2.6) we can put $p(x) = x$. If $X$ is not connected in klein end, then there exists a point $y \in X \setminus x$ such that $x \in Ty$ (Theorem (2.6)). Let us observe that

(i) for each $x, y \in X$ such that $x \in Tx \cap Ty$ we have either $x \in xy$ or $y \in xy$.

Indeed, let $x$ and $y$ be such that $x \in Tx \cap Ty$ and suppose that $x \notin xy$ and $y \notin xy$. The union $xy \cup xy$ is an arc or a triad. If $xy \cup xy$ is an arc, then $T(x \cup yz)$ and $T(x \cup yz)$, contrary to [7]. If $xy \cup xy$ is a triad with a point $t$ as its centre, then let us observe that $y \notin x, Ty \cup yz$, and $Ty \cup yz$, contrary to [7].

Therefore it follows from (i) that the set $Z = \{x \in Tx \cap Ty\}$ lies in an arc having $x$ as its end-point. Define $p(x)$ as the other end-point (different from $x$) of the minimal arc containing the set $Z$. It is easy to see that the set $Z$ is closed and that $x \in Ty(x)$. Let a sequence $x_n$ be convergent to a point $x$. Thus $x_p(x) \subseteq L(x_n, p(x)) \subseteq L(x_n, p(x))$.

Now, let $q \in Lx_n, p(x)$. For each subcontinuum $K$ with property $x \in K$ we have $x_p(x) \subseteq K$ and $x_p(x) \subseteq K$ for almost all $n$. Therefore $L(x_n, p(x)) \subseteq K$, whence $x \in K$, and we have shown that $x \in Ty$. Thus $q \in x_p(x)$, and so we have $L(x_n, p(x)) \subseteq x_p(x)$. By the double inclusion above, this leads to $L(x_n, p(x)) = x_p(x)$, and the proof is complete.

Note in the proof above that if $X$ has property [7], then for each point $x \in X$ we can choose the point $p(x)$ in such a way that $x \in Ty(x)$. Hence we have the following.

(3.2) Corollary. A dendroid $X$ is pointwise smooth if and only if for each $x \in X$ there exists a point $p(x) \in X$ such that

(1) for each sequence $x_n$ convergent to a point $x$ we have $L(x_n, p(x)) = x_p(x)$

(2) $x \in Ty(x)$.

Given two disjoint subcontinua $A$ and $B$ of $X$, we denote by $I(A, B)$ a subcontinuum of $X$ which is irreducible with respect to containing $A \cup B$. It is quite easy to see that there are two points, $a \in A$ and $b \in B$, such that $I(a, B) = A \cup B \cup \{a, b\}$ and $a \circ b \cap A = \{a\}$, and $a \circ b \cap B = \{b\}$ (cf. e. g. [2], Theorem 21, p. 195 and Theorem 27, p. 197).
(3.3) **Corollary.** The following conditions are equivalent for a dendroid $X$:

1. $X$ is pointwise smooth,
2. for every two disjoint subcontinua $A$ and $B$ of $X$, either $TA \cap I(A, B) = A$ or $TB \cap I(A, B) = B$ or $TA \cap TB = \emptyset$.

**Proof.** The implication $(2) \implies (1)$ is immediate by Theorem (3.1). Let a dendroid $X$ be pointwise smooth and let disjoint subcontinua $A$ and $B$ of $X$ satisfy

$(*)$ \[ [TA \cap I(A, B)] \cap A \neq \emptyset \quad \text{and} \quad [TB \cap I(A, B)] \cap B \neq \emptyset. \]

Observe that

(i) \[ TA \cap a_b = T_a \cap a_b. \]

and

(ii) \[ TB \cap a_b = T_b \cap a_b. \]

Indeed, $T_{a_b} \cap a_b = TA \cap a_b$. Let $z \in TA \cap a_b$. Since $z \in TA \cap a_b$, this means, by the definition of set function $I$, that for each subcontinuum $K \subseteq X$ such that $z \in \text{Int } K$ we have $K \cap A \neq \emptyset$. Thus it follows from the hereditary unicoherence of $X$ that $a_b \in K$ and so $z \in T_{a_b}$. Hence $z \in T_{a_b} \cap a_b$. In the same way we can prove (i).

We claim that

$(ii)$ \[ (TA) \cap B = \emptyset = (TB) \cap A. \]

Indeed, if $(TA) \cap B \neq \emptyset$ then $(TB) \cap A \neq \emptyset$ since $(TA)(TB)$ is a continuum and $X$ is hereditarily unicoherent, then $a_b \in T_{a_b} \cap a_b$ (TB $\cap a_b = a_b$). Hence $z \in T_{a_b} \cap a_b$. Since (a) implies that $T_{a_b} \cap a_b \neq \emptyset$, and $TB \cap a_b \neq \emptyset$, we get a contradiction with the pointwise smoothness of $X$ by Theorem (3.1). So the claim is proved.

Now, by (i), (ii) and (ii) we have

\[ TA \cap I(A, B) = TA \cap (a_b \cup A \cup B) = (TA \cap a_b) \cup A = (TA \cap a_b) \cup A \]

and

\[ TB \cap I(A, B) = (TB \cap a_b) \cup B, \]

whence $TA \cap TB \cap I(A, B) = T_{a_b} \cap a_b \cap a_b = \emptyset$. Recall that

$(TA \cap I(A, B)) \cap A \neq \emptyset$

implies $T_{a_b} \cap a_b \neq \emptyset$, and $[TB \cap I(A, B)] \cap B \neq \emptyset$ implies $T_{a_b} \cap a_b \neq \emptyset$. Thus it follows from the pointwise smoothness of $X$ by Theorem (3.1) that

\[ T_{a_b} \cap a_b = TA \cap TB \cap I(A, B) = \emptyset. \]

By the hereditary unicoherence of $X$ this last condition implies $TA \cap TB = \emptyset$. The proof is complete.

(3.4) **Definition.** A non-empty subcontinuum $K \subseteq X$ of a dendroid $X$ is called an $R^2$-continuum in $X$ if there exist an open set $U$ such that $X \subseteq U$, and a sequence $C_n$ of components of $U$ such that $\text{Li } C_n = K$. We say that a dendroid $X$ has property $[R]$ if there exist two subcontinua $A$ and $Y$ of $X$ such that $A \subseteq Y \subseteq X$ and $A$ is an $R^2$-continuum in $Y$.

The following two propositions are proved in [8] and [6]:

(3.5) **Proposition.** If a dendroid $X$ contains an $R^2$-continuum, then it is not contractible.

(3.6) **Proposition.** If a dendroid $X$ has property $[R]$, then it is not hereditarily contractible.

(3.7) **Theorem.** If a dendroid is not pointwise smooth, then it has property $[R]$. **Proof.** If a dendroid $X$ is not pointwise smooth, then by Theorem (3.1) there exist two different points $x$ and $y$ of $X$ such that $Tx \cap xy \neq \{x\}$, $Ty \cap xy \neq \{y\}$ and $Tx \cap Ty \neq \emptyset$.

If $x \neq Ty$ and $y \neq Tx$, then the subcontinuum $Tx \cap Ty$ contains an $R^2$-continuum (see [7], Theorem 7, p. 305). Let $x \in Tx$. Then $Ty \cap xy = x$. Take $z \in (xy \setminus \{x, y\}) \cap Tx$, and let $U$ be an open ball with centre at $z$ such that $z \in U \cap X \neq \emptyset$. Observe that $z \notin Tx \cap Ty \in X$. Let $U_z$ be the open ball with centre $z$ and radius $1/n$. The smallest subcontinuum of $X$ which contains $U_z$ is the closure of the union of all arcs $z_i$ with $z_i \in U_z$. Since $x$ and $y$ are contained in every subcontinuum of $X$ which contains $z$, and for each positive integer $n$ there are points $z_1^x$ and $z_2^y$ such that the arc $z_1^x$ contains a point $a$ at a distance less than $1/n$ from $y$ and the arc $z_2^y$ contains a point $b$ at a distance less than $1/n$ from $x$. We can choose the sequences $z_1^x$ and $z_2^y$ in such a way that the following conditions hold:

(i) the sequences $z_1^x$ and $z_2^y$ are convergent to $z$ and

(ii) $L_{x^1} z_1 < z < z_2 < L_{y^1} z < y = y^1.$

Let $Y = \bigcup_{i=1}^{\infty} (z_1^x \cup z_2^y) \cup L_{x^1} z_2 \cup L_{y^1} z_1$. It is easy to see that $Y$ is a subcontinuum of $X$ and that $x \notin Ty \cup y \notin Tx$ and $Tx \cap Ty \neq \emptyset$, and so $Tx \cap Ty \in X$ contains an $R^2$-continuum (see [7] and [8]). The proof is complete.

(3.8) **Theorem.** If a dendroid $X$ has property $[R]$, then it is not pointwise smooth. **Proof.** Let $Y$ be a subcontinuum of $X$ and $A$ be an $R^2$-continuum in $Y$, and let $C_n$ be a sequence of components of the open set $U$ such that $A \subseteq U$ in $Y$ with $\text{Li } C_n = A$. Let $x \in A$ and $x_n \in C_n$ where $x_n$ is convergent to a point $x$. If $X$ is pointwise smooth, then it is also pointwise smooth by Proposition (2.3), and let $p(x)$ denote the initial point for $x$ in $Y$. If $p(x)$ belongs to the same component of $U$ as $x$, then $x \notin p(x)$ and thus $x \notin p(x)$ in $Fr U = \emptyset$, but by the definition of $x_n$, it is quite easy to see that $L_{x^1} p(x) \notin Fr U = \emptyset$. This contradiction shows us that $p(x)$ does not belong to the same component of $U$ as $x$. Let $q$ denote the first point of the arc $x_{n+1}$ (ordered from $x$ to $p(x)$) which is in Fr $U$ and let $B_n$ denote the component of $x_{n+1} p(x)$ which contains $x_{n+1}$. We claim that $q \notin Li B_n$.

Indeed, let denote the first point of the arc $x_{n+1} p(x)$ from $x_n$ to $p(x)$ which...
is in $Fr U$. Let $q'$ be the limit point of a subsequence $q_n$ of the sequence $q$. Since $xq' \in LiB_q$, and $q' \in Fr U$, we find, by definition of $q$, that $q \in xq'$. Consequently $q \in LiB_q$.

Therefore $LiB_q \cap Fr U \neq \emptyset$, but $B_q \subseteq C_x$, and from this $LiB_q \subseteq LiC_x = A \cap X \setminus Fr U$; so $LiB_q \cap Fr U = \emptyset$. This contradiction finishes the proof.

(3.9) COROLLARY. A dendroid $X$ is pointwise smooth if and only if it does not have property $[X]$.

Hence it follows by a result from [8] that

(3.10) COROLLARY. If a dendroid $X$ is hereditarily contractible, then it is pointwise smooth.

The following question seems to be natural:

(3.11) QUESTION. Does pointwise smoothness imply hereditary contractibility, of a dendroid $X$?

For fans Corollary (2.5) gives a positive answer to this question.

§ 4. Some remarks on other weaker forms of the concept of smoothness. A dendroid $X$ is said to be semismooth (see [3], p. 306) provided there exists in $X$ a point $p$ such that whenever $a_n$ converges to $a$, then $Lap_0 a_n$ is an arc.

A dendroid $X$ is said to be weakly smooth (see [10], p. 111, and Theorem 4, p. 114) provided there exists in $X$ a point $p$ such that, whenever $x_n$ converges to $x$, then $Lap_0 x_n = px$ for a certain $y$ from $X$.

In this section we discuss all relations between the concept of pointwise smoothness and semismoothness, and weak smoothness.

Let $\overline{x y}$ denote a straight line segment with $x$ and $y$ as its end-points.

In the following three examples let $(x, y)$ denote a point of the Euclidean plane endowed with the ordinary rectangular coordinate system $\overline{xy}$.

(4.1) EXAMPLE. Let $(x, y)$ denote a point of the Euclidean plane.

Let $q = (0, 0), k = (1, 0), t = (2, 0), z = (3, 0), a_1 = (2, 1/n), b_2 = (2 - 1/n, 0), c_3 = (2, -1/n), d_4 = (-2, -1/n), e_5 = (-2 - 1/n, 0), f_6 = (-2, 1/n), g_7 = (1, 1/n)$.

Let $A_n = \overline{z t}, A_\alpha = \overline{a_1 b_2} \cup \overline{b_2 c_3} \cup \overline{c_3 d_4} \cup \overline{a_1 b_2} \cup \overline{c_3 d_4} \cup \overline{a_1 b_2} \cup \overline{f_6 g_7}, X = \bigcup_{n=1}^{\infty} A_n$ (see Fig. 1).

We can easily see that a dendroid $X$ is not pointwise smooth and it is weakly smooth if we take as a point $p$, in the definition of weak smoothness, point $z$ or point $p$ only.

Let us take a map $\psi$ of $X \setminus \{0\}$ onto $X \setminus \{1\}$, which identifies points $(q, 0)$ and $(q, 1)$. Put $Y = \psi(X \setminus \{0\}) \cup X \setminus \{1\}$.

It is easy to see that $Y$ is semismooth but it is not weakly smooth and not pointwise smooth.

(4.2) EXAMPLE. We define

$A = \{(x, y): 0 \leq x \leq 2, y = 1\}$,

$A' = \{(x, y): 0 \leq x < 2, y = 1\}$, $B = \{(x, y): x = 1, -1 \leq y \leq 1\}$,

$a = (0, 1), a' = (0, -1), a_\alpha = (2, 1 + 1/n), a'_\beta = (2, -1 - 1/n)$.

Let $X = A \cup A' \cup B \cup \bigcup_{n=1}^{\infty} \overline{a_\alpha a'_\beta} \cup \bigcup_{n=1}^{\infty} \overline{a_\alpha a'_\beta}$ (see Fig. 2).

It is easy to see that $X$ is pointwise smooth but it is not semismooth and it is not weakly smooth.

(4.3) EXAMPLE. Put $p = (-2, 0), q = (2, 0), a_\alpha = (-1, 1/n), b_\beta = (1, 1/n)$.

Let $A_\alpha = \overline{a_\alpha p} \cup \overline{pq} \cup \overline{q_\beta}, X = \bigcup_{n=1}^{\infty} A_\alpha$.

The dendroid $X$ is semismooth, weakly smooth and pointwise smooth but it is not smooth.

(4.4) EXAMPLE. Let $Y$ be a dendroid described in [5] (Proposition 12, pp. 234–235) (see Fig. 3). It is quite easy to see that $Y$ is semismooth and weakly smooth, but $Y$ contains an $R^3$-continuum (namely the arc $ab$), and so it is not pointwise smooth.
If $\text{Ls}_{p_0}x_\alpha$ is an arc for each sequence $x_\alpha$ converging to a point $x$, then $X$ is semismooth.

If there exist a point $x_0$ and a sequence of points $x_\alpha$ converging to a point $x_\alpha$, such that $\text{Ls}_{p_0}x_\alpha$ is not an arc, then let $t$ denote the centre of the trio $p_0(x_\alpha) \cup x_\alpha$, $p(x_\alpha)$, and $\text{l}p(x_\alpha) < t$ (this means that $t$ is a such point that $p_0(x_\alpha) \cup x_\alpha = p(x_\alpha)$ and $x_\alpha \neq p(x_\alpha)$ and $p_0(x_\alpha) \cup x_\alpha)$ and $p_0(x_\alpha) \cap x_\alpha = p(x_\alpha)$). Let $k$ be a point of $\text{l}p(x_\alpha) \cup x_\alpha$.

We can choose a subsequence $x_\beta$ from the sequence $x_\alpha$ such that on the arcs $p_0x_\beta$, there exist points $k_i$ converging to a point $k$. For this subsequence we have $\text{Ls}_{p_0}x_\alpha \cap x_\alpha$, and so $\text{Ls}_{p_0}x_\alpha$ is not an arc. This contradicts the assumption that $X$ is weakly smooth with the initial point $p_0$. So $\text{Ls}_{p_0}x_\alpha$ is always an arc, and thus $X$ is semismooth. The proof is complete.

(4.7) Theorem. If a dendroid is pointwise smooth and semismooth, then there exists a point $p \in X$ such that $X$ is connected in the sense that for each $x$ and $y$ in $X$ there exists a sequence $x_\alpha$ converging to a point $x$ and there exists a sequence $y_\alpha$ converging to a point $y$. So $\text{Ls}_{p_0}x_\alpha$ holds for certain $y \in X$.

Proof. Let a dendroid $X$ be pointwise smooth and semismooth. It follows from the pointwise smoothness of $X$ that

$$\text{Ls}_{x_\alpha} = \text{Ls}_{x_\alpha,p(x_\alpha)} \cup p(x_\alpha),$$

so, for each $x$ and each sequence $x_\alpha$ converging to a point $x$, the set $\text{Ls}_{x_\alpha}x$ is an arc with $x$ as one of its end-points. Let $p'$ be an initial point of $X$ in the definition of the semismoothness of $X$. This means that $\text{Ls}_{x_\alpha,p'}$ is an arc for each sequence $x_\alpha$ converging to a point $x$.

Case 1. If $X$ is connected in the sense that $p' \neq x$, then $\text{Ls}_{x_\alpha,p'} = p_0x_\alpha$ for each $x$ and for each sequence $x_\alpha$ converging to a point $x$, and for a certain $p_0$.

Indeed, otherwise we would have for a certain $x$ and a certain sequence $x_\alpha$ converging to a point $x$, $\text{Ls}_{x_\alpha,p'} = p_0x_\alpha$, where $p' \neq x$ and $x_\alpha$. Obviously either $p' \neq x_\alpha$ or $p' \neq x$. If $p' \neq x_\alpha$, then by the definition of the set function $T_x$, we would have $x \in x_\alpha$, and thus $x' \neq x_\alpha$. This contradicts Theorem (2.6).

If $p' \neq x$, the argumentation is the same.

Case 2. If $X$ is not connected in the sense that $p'$, then by Theorem (2.6) there exists a point $y \neq x'$ such that $p' \in T_y$. We shall now define by induction a sequence of points $p_n$, where $n$ is a non-negative integer. Put $p_0 = p'$. By Corollary (3.2) we can define $p_1 = p_0$, such that $p_1 \in T_1$. In the same way we define $p_{n+1} = p_{p_n}$. We observe that

$$p_{n+1} \cap p_{n+1} = \{p_n\}.$$

Indeed, we can see by the definition of $p_n$ that $p_{n+1} \in T_{p_n}$ and $p_{n+1} \in T_{p_{n+1}}$. Since $T_{p_n} \cap T_{p_{n+1}} = \emptyset$ and $p_{n+1} \in T_{p_{n+1}}$, it follows from Theorem (3.1) that $p_{n+1} \cap p_{n+1} = \{p_n\}$.
It follows from (1) that there exists an arc $K$ with $p_0$ as one of its end-points such that all points $p_x$ belong to $K$, and from this it is easy to see that

(2) for each $i = 0, 1, 2, \ldots$ and for each sequence $x_a$ converging to a point $x$ the set $L_{x_a} p_x$ is an arc (see [10], Lemma p. 18).

Let us now extend the definition of $p_x$ to that of $p_{x_a}$, where $x$ is an arbitrary ordinal number, putting $p_{x+1} = p(p_x)$ with $p_x \in D_{p_x+1}$ (as before, see Corollary (3.2)), and $p_1 = \lim \{p_x; x < \lambda\}$, $\lambda$ being a limit ordinal.

From (1), (2) and the definition of $p_x$ we have

(1') if $a < b < \lambda$, then $p_a p_b \cap p_x p_b = \{p_b\}$,
(2') for each ordinal number $x$ and each sequence $x_a$ convergent to a point $x$ the set $L_{x_a} p_x$ is an arc (see [10], Lemma p. 18).

Let now define $p''$ as the other end-point (different from $p_0$) of the minimal arc containing the set $P = \{p_x; x$ is an ordinal number}. It is quite easy to see that (cf. Theorem (2.6))

(3) $X$ is connected in the sense at $p''$,

and thus, by (2'), we conclude that (see Case 1)

(4) for each sequence $x_a$ converging to a point $x$ we have $L_{x_a} p = p_x p''$ for a certain $y_{x}$.

The proof of theorem is complete.

Note in the proof above that if $X$ is pointwise smooth and in the sequence $p_x$ we start from $p_0 = x$, where $x \in X$, then, using the same arguments, we have the following

(4.8) Corollary. A dendroid $X$ is pointwise smooth if and only if for each $x \in X$ there exists a point $p(x) \in X$ such that

(1) for each sequence $x_a$ converging to a point $x$ we have $L_{x_a} p(x) = x p(x)$

and

(2) $X$ is connected in the sense at $p(x)$.

As an easy consequence of Theorem (4.7) we have the following

(4.9) Corollary. If a dendroid $X$ is pointwise smooth and semismooth, then it is also weakly smooth.

References


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