The Axiom of Determinateness and canonical measures

by

James M. Henle * (Northampton, Mass.)

Abstract. In the mid-1960's R. M. Solovay proved that under the Axiom of Determinateness, $\aleph_1$ was a measurable cardinal. The method involved a map from 2^ω to $\aleph_1$ and an infinite game using this map.

In Section 1, this method is generalized to show under AD that every cardinal $\alpha<\theta$ of uncountable cofinality has an $\aleph_1$-additive, uniform ultrafilter, where θ is the least cardinal onto which the continuum cannot be mapped. An ultrafilter formed in this way depends on the choice of a map from 2^ω to α, but by defining a prewellordering on such maps, a canonical ultrafilter is found.

In Section 2, these results are used to show that either θ is regular, or else it carries an $\aleph_1$-additive, uniform ultrafilter.

In Section 3, we show that the length of the prewellordering in Section 1 has length at least $\aleph_2$.

The first major set-theoretic consequence of the Axiom of Determinateness was that proved by Solovay, that $\aleph_1$ is a measurable cardinal [12]. His technique consisted of playing the following game: given $A\subseteq\aleph_1$, players I and II each play integers forming real numbers r and s. These are broken up into ω-many reals each, $\{r_n\}_{n<\omega}$ and $\{s_n\}_{n<\omega}$, each coding an ordinal less than $\aleph_1$. Player I wins iff the supremum of all these ordinals lies in A. Solovay defined A to be measure one if player I had a strategy for the associated game. Solovay showed that this was an $\aleph_1$-additive, normal measure. In fact, he showed more: that every measure one set contained a closed, unbounded subset. Much of the power of this method is derived from the coding which has certain convenient definability properties.

In Section 1, we will extend Solovay's method to produce measures on larger cardinals. Since in general we will know next to nothing about our coding maps, we will have less success. Instead of showing cardinals measurable, for example, we will only be able to show their measures to be $\aleph_1$-additive. We will, however, be able to single out "canonical" measures on these cardinals. Looking at the situation under the stronger assumption, AD_R, we will see that given a map from the continuum to a cardinal α, we get a measure μ_α on α. With AD_R, this measure is unique in

* Most of the results in this paper first appeared in the author's Doctoral thesis, [5].

that it does not depend on the choice of the map f. Under just AD, however, the corresponding measure, μ_α^f does depend on f. In order to get a canonical measure, we will define a prewellordering on such maps. Then taking f from the least equivalence class, μ_α^f will be the canonical measure.

In Section 2, the previous results will be applied to the cardinal θ. The theorem here will be that under AD, either θ is regular, or else it possesses an $\aleph_1$-additive, uniform measure.

In Section 3, we consider the prewellordering of Section 1 and derive a few results concerning its length. In the case of $\aleph_1$, we will show that the length is at least $\aleph_2$.

§ 0. Definitions and facts. The Axiom of Determinateness states that winning strategies exist for certain kinds of infinite two-person games.

Given $A \subseteq {}^\omega 2$, the game G_A is defined by: Two players, I and II alternate playing 0's and 1's, creating an infinite sequence, r. I wins G_A if $r \in A$.

A *strategy* is a function from the set of finite sequences of 0's and 1's to $\{0, 1\}$. A *winning strategy* F for player I (II) is one which always guarantees a win for player I (II) whenever he uses it to determine his every move by applying it to his opponent's previous moves. A game is *determined* if one of the two players has a winning strategy.

The Axiom of Determinateness (AD) states that for all $A \subseteq {}^\omega 2$, G_A is determined.

AD has a rich history in modern set theory. In addition to the results mentioned above, AD has many detailed consequences about the $\aleph_n$, and higher cardinals. Early theorems were directed at analysis and include the startling fact that all sets of reals are Lebesgue-measurable.

Working with AD is complicated by its inconsistency with the full Axiom of Choice, but some limited choice is available. AD in fact implies countable choice for sets of reals. Also, as far as is known, AD is not inconsistent with DC. Throughout this paper, we will assume in addition to AD, DC.

Reference for the first definitions and results concerning the Axiom of Determinateness include: [4], [6], [10] and [11]. For later results see [8], [9] and [12].

AD_R, a more powerful version of AD involves games in which players play real numbers:

DEFINITION. For any $A \subseteq {}^\omega({}^\omega 2)$ let $\hat{G}_A$ be the game defined by:

Two players, I and II alternate playing elements of ${}^\omega 2$, creating a sequence S.

I wins $\hat{G}_A$ iff the sequence is in A. AD_R is the assertion that for all $A \subseteq {}^\omega({}^\omega 2)$, $\hat{G}_A$ is determined.

Clearly, AD_R implies AD. That AD_R is strictly stronger than AD was proved by A. Blass [1].

For coding purposes, we will require the following notation: For $n \in \omega$ let p_n be the nth prime number. If $r \in {}^\omega 2$ is the sequence: $n_1, n_2, n_3, \dots$ and $k \in \omega$, then r^k is the sequence: $n_{p_k}, n_{p_k^2}, n_{p_k^3}, \dots$

§ 1. Canonical measures.

DEFINITION. θ is defined to be the least ordinal number onto which ${}^\omega 2$ cannot be mapped.

Under the assumption of the Axiom of Determinateness, θ has been shown to be very large. Moschovakis and Friedman [9] have proved that if $\alpha < \theta$, then $\alpha^+ < \theta$ and so θ is a limit cardinal. In addition, it is easy to see that $\mathrm{cf}(\theta) > \omega$, since if we can map ${}^\omega 2$ onto

$$\alpha_0 < \alpha_1 < \alpha_2 < \dots$$

then DC or countable choice would enable us to choose such maps and then glue them together to compose a map of ${}^\omega 2$ onto $\bigcup_{n<\omega} \alpha_n$. Limitation of choice prevents us from proving as well that θ is regular, although some partial results have been obtained:

(1) $\theta > \aleph_{\aleph_1}$ (H. Friedman),

(2) AD + Collection $\vdash \theta$ is regular (Moschovakis) (Collection is a choice-like axiom),

(3) θ is regular in $L[R]$ (Solovay).

The starting point of Solovay's theorem that $\aleph_1$ is measurable is a map from ${}^\omega 2$ to $\aleph_1$. Given any $\alpha < \theta$, however, we have a map $f\colon {}^\omega 2 \to \alpha$, and from this, we can derive a measure on α:

THEOREM 1.1. $\mathrm{ZF} + \mathrm{AD}_R \vdash$ "*for all* $\alpha < \theta$, $\mathrm{cf}(\alpha) > \omega$, α *has an* $\aleph_1$-*additive measure,* μ_α *such that for all* $\beta < \alpha$, $\mu_\alpha(\beta) = 0$".

Proof. Let $f\colon {}^\omega 2 \to \alpha$ be any onto map. For any $A \subseteq \alpha$, we play the following game G_A^f:

player I plays sequences r_1 r_2 r_3 ...

while player II plays sequences s_1 s_2

II wins G_A^f iff $\bigcup_{n<\omega} f(r_n) \cup \bigcup_{n<\omega} f(s_n) \in A$.

In the course of this section, we will introduce many games of this sort. It should be clear that they are all of the kind covered in the definitions of AD and AD_R. For example, if we let $B \subseteq {}^\omega({}^\omega 2)$ be defined by:

$$B = \{r_0 r_1 r_2 r_3 \dots \mid \bigcup_{n<\omega} f(r_n) \in A\}$$

then the game just described in $\hat{G}_B$. By AD_R, there exists a winning strategy for one of the two players, and so we may define μ_α on α by: $\mu_\alpha(A) = 1$ iff II has a winning strategy for G_A^f.

Note that if $\mu_\alpha(A) = 0$, then $\mu_\alpha(A^c) = 1$, for if F is a winning strategy for I in G_A^f, then the following is a winning strategy for II in $G_{A^c}^f$:

while I plays: r_1 r_2 r_3 ...

II plays: $F(\emptyset)$ $F(r_1)$ $F(r_1, r_2)$

Conversely, if $\mu_\alpha(A) = 1$, then $\mu_\alpha(A^c) = 0$, for if F is a winning strategy for II for G^f_A, then the following is a winning strategy for I for $G^f_{A^c}$:

I plays: t $F(s_1)$ $F(s_1, s_2)$ ...

while II plays: s_1 s_2 s_3

— where t is some real number such that $f(t) = 0$.

CLAIM. *μ_α is $\aleph_1$-additive.*

Proof of the claim. Suppose $\{A_n\}_{n<\omega}$ is a collection of measure 1 sets. Using countable choice, let F_n be a strategy for $G^f_{A_n}$. We will weld these strategies together to form a strategy for $G^f_{\bigcap_{n<\omega} A_n}$.

Imagine ω-many players playing reals. Player 0 is the actual player I, while the other players n for $n>0$ are all parts of player II. Each player n uses strategy F_{n-1} against all the rest of the players, so that in the end, the supremum of all the plays will be in each A_n. Specifically, if I plays $r_1, r_2, r_3, \ldots$, then II plays $s_1, s_2, s_3, \ldots$, where

$$s_n = \begin{cases} r_{n/2}, & \text{if } n \text{ is even,} \\ F_{b-1}(s_1, \ldots, s_a), & \text{if } n = p_b^a, \\ 0, & \text{otherwise.} \end{cases}$$

We first have:

$$\bigcup_{n<\omega} f(r_n) \cup \bigcup_{n<\omega} f(s_n) = \bigcup_{n<\omega} f(s_n).$$

But we also have:

$$\bigcup_{n<\omega} f(s_n) \in A_k \quad \text{for all } k,$$

since the sequence: $s_{p_{k+1}}, s_{p^2_{k+1}}, s_{p^3_{k+1}}, \ldots$ is a correct play of strategy F_k against $s_1, s_2, \ldots$ Thus, this describes a strategy for II in $G^f_{\bigcap_{n<\omega} A_n}$.

To complete the proof of this theorem, let $\beta<\alpha$, and let $r \in 2^\omega$ be such that $f(r)>\beta$. Then the strategy F which is constantly equal to r is a winning strategy for I in G^f_β, hence $\mu_\alpha(\beta) = 0$. Note that if α is regular, this then implies that μ_α is uniform. ■

LEMMA 1.2. *The measures μ_α defined in the previous theorem do not depend on the choice of a function f: ${}^\omega 2 \to \alpha$.*

Proof. Let f: ${}^\omega 2 \to \alpha$ and g: ${}^\omega 2 \to \alpha$ be onto maps, and let μ_α and μ'_α, be the measures on α defined as in Theorem 1.1 from f and g respectively. Suppose that $\mu_\alpha(A) = 1$, for some $A \subseteq \alpha$. We will show that $\mu'_\alpha(A) = 1$, proving the lemma. Consider two games: In G_1, I and II each play only a single real number:

I: r

II: s

and II wins iff $f(s) = g(r)$. Clearly, I cannot have a winning strategy, and so II must.

Let F_1 be a winning strategy for II. Similarly, we define G_2 in almost the same way, but so that II wins iff $g(s) = f(r)$. Again, II must have the winning strategy, and let F_2 be such a strategy. Now, let F be a winning strategy for II in G^f_A. Then the winning strategy for II in G^g_A is described by:

while I plays: r_1 r_2 r_3

II plays: $F_2(F(F_1(r_1)))$ $F_2(F(F_1(r_1), F_1(r_2)))$ ... ■

We will now attempt to follow the same path under the weaker assumption of AD. The analog of Theorem 1.1 will not be difficult. The analog of Lemma 1.2, however is another story. It is, in fact, false, and circumventing it will not be easy. Since the proofs of many theorems here will be technical, we will often preface them with an outline of the main ideas. In general, the principles can be stated simply, and the details which show combinatorially that the appropriate coding can be carried out do not contribute to understanding the proof.

THEOREM 1.3. ZF+AD $\vdash$ "*For all $\alpha<\theta$, $\mathrm{cf}(\alpha)>\omega$, α has a $\aleph_1$-additive measure μ_α such that for all $\beta<\alpha$, $\mu_\beta(\beta) = 0$*".

Proof. Given $\alpha<\theta$, let f: ${}^\omega 2 \to \alpha$, such that $f''{}^\omega 2$ is unbounded in α and such that $f(0,0,0,0,0,\ldots)$ is least in $f''{}^\omega 2$. For any $A \subseteq \alpha$, we define the following game $G_{f,A}$:

I plays: n_1 n_2 n_3

II plays: m_1 m_2 m_3

— where $n_i, m_i = 0$ or 1, for all i. At the conclusion of the game, I has formed the real r while II has formed s. II wins $G_{f,A}$ iff

$$\bigcup_n f(r^n) \cup \bigcup_n f(s^n) \in A.$$

Note that for both players, the kth move is irrelevant for k not a prime power. As before, this is a game of the sort covered in the definition of AD, and hence one player has a winning strategy. We define a measure μ^f_α on α by: $\mu^f_\alpha(A) = 1$ iff II has a winning strategy for $G_{f,A}$.

As in the proof of Theorem 1.1, I has a winning strategy for $G_{f,A}$ iff II has a winning strategy for G_{f,A^c}. Suppose, for example, F is a winning strategy for II in G_{f,A^c}. Mimicing the proof from AD_R, I plays "0" at all places which form the first real r^1 of his play, that is, if his plays are $n_0, n_1, n_2, \ldots$, then n_k is automatically 0 if k is a power of 2. For the rest of his moves, he plays F against II, i.e., I's strategy F^* is:

$$F^*(m_0, \ldots, m_k) = \begin{cases} F(m_0, \ldots, m_l),\ l = p^b_a & \text{if } k = p^b_{a+1}, \quad a>1, \\ 0 & \text{otherwise.} \end{cases}$$

Playing in this manner, I wins $G_{f,A}$. The other way is easier: if I has a winning strategy F for $G_{f,A}$, then the following is a strategy for II for

G_{f,A^c}: $F^*(n_1, \ldots, n_k) = F(n_1, \ldots, n_{k-1})$ and $F^*(n_1) = F(\emptyset)$.

CLAIM. *μ_α^f is $\aleph_1$-additive.*

Proof of the claim. Let $\{A_n\}_{n<\omega}$ be a collection of measure one sets and let F_n be a winning strategy for II for G_{f,A_n} for $n<\omega$. Fundamentally, our proof will be the same as the one with AD_R, except that we must code quite a bit more carefully. Essentially, II pretends to be ω-many players, each player n using F_n against everyone else. These are the combinatorial details:

Against I's plays $n_1, n_2, n_3, \ldots$ II plays $m_1, m_2, \ldots$ where

$$m_k = \begin{cases} n_l,\ l = p_a^b, & \text{if } k = p_{2^a}^b, \\ F_{n-1}(m_1, \ldots, m_l),\ l = p_a, & \text{if } k = p_{p_n^a}^b,\ n>0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that since k is always greater than l, this strategy is well-defined.

Let r and s be the sequences created by I and II respectively. Since $r^a = s^{(2^a)}$ for all a, it follows as before that

$$\bigcup_n f(r^n) \cup \bigcup_n f(s^n) = \bigcup_n f(s^n).$$

Furthermore, suppose for some n, the strategy F_n were used by some player against s to produce t, so

$$\bigcup_a f(s^a) \cup \bigcup_a f(t^a) \in A_n.$$

But

$$t^a = s^k \text{ where } k = p_{n+1}^a \quad \text{so} \quad \bigcup_a f(s^a) \in A_n.$$

Thus we have described a winning strategy for player II in $G_{f,\bigcap_n A_n}$.

To complete the proof of the theorem, suppose $\beta<\alpha$. Let $t \in {}^\omega 2$ be such that $f(t)>\beta$, then for I to win $G_{f,\beta}$ he merely plays a sequence s such that $s^k = t$ for all k. ∎

As we warned, the measures μ_α^f are not independent of f:

THEOREM 1.4 (AD). *For any $\alpha<\theta$, there exist functions $f, g\colon {}^\omega 2 \to \alpha$ such that $f''{}^\omega 2$ and $g''{}^\omega 2$ are unbounded in α, such that $\mu_\alpha^f \neq \mu_\alpha^g$.*

Proof. Let $h = {}^\omega 2 \to \alpha$ be any onto map. Let h^* be the map: $h^*(r) = h(r^*)$ where r^* is the subsequence of r obtained as follows:

$r^*(0) =$ the first digit following the 1st "1" in r, and if $r^*(n) = r(m)$ has been defined,

$r^*(n+1) =$ the first digit following the 1st "1" following $r(m)$ in r. If r^* is a finite sequence, let $h^*(r) = 0$.

In the strategy below, II will want to copy some of I's moves. I will produce these moves too slowly for II to copy, but the introduction of h^* allows II to delay.

We now define:

$$f(r) = \bigcup_n h^*(r^n)+1 \quad \text{and} \quad g(r) = \bigcup_n h^*(r^n)+\omega.$$

Let $A = \{\text{all limit ordinals} <\alpha\}$. Clearly $\mu_\alpha^g(A) = 1$, since for all r, $g(r)$ is a limit ordinal. On the other hand, $\mu_\alpha^f(A) = 0$ as follows: II can win G_{f,A^c} by playing the sequence r to I's s so that each r^n is identical, and the subsequences $\{(r^n)^{m^*}\}_m$ include all the subsequences $\{(s^a)^{b^*}\}_{a,b}$. This is done quite mechanically — the use of h^* allows II to delay if he gets ahead of I. The result is that

$$\bigcup_a f(s^n) \cup \bigcup_a f(r^n) = \bigcup_m h^*((r^0)^m)+1 \in A^c. \ ∎$$

We wish now to construct for each $\alpha<\theta$ a "canonical" measure. In our present situation, for each $\alpha<\theta$, we have many different measures μ_α^f. We will proceed by defining an equivalence relation on functions $f\colon {}^\omega 2 \to \alpha$, unbounded and an ordering on the equivalence classes, and prove:

(1) that the ordering is a well-ordering, and

(2) that if f and g are equivalent, then $\mu_\alpha^f = \mu_\alpha^g$.

This will yield us our canonical measures, for we then can choose f from the least equivalence class, and the measure μ_α^f will be independent of that choice.

DEFINITION. For any $f, g\colon {}^\omega 2 \to \{\text{the ordinals}\}$, the game $G_{f,h}$ is defined as follows: I plays the sequence r while II plays integers forming a real s. II wins $G_{f,g}$ iff

$$\bigcup_n f(r^n) \leqslant \bigcup_n g(s^n).$$

The game $G'_{f,g}$ is defined similarly so that II wins iff

$$\bigcup_n f(r^n) < \bigcup_n g(s^n).$$

DEFINITION. Given $f, g\colon {}^\omega 2 \to \{\text{the ordinals}\}$, we define

$f \leqslant g$ iff II has a winning strategy for $G_{f,g}$ and

$f < g$ iff II has a winning strategy for $G'_{f,g}$.

LEMMA 1.5. *For all $f, g\colon {}^\omega 2 \to \{\text{the ordinals}\}$,*

$$f \leqslant g \quad \text{iff} \quad g \nless f.$$

Proof. Suppose $f \leqslant g$. Let F be a winning strategy for II for $G_{f,g}$. As in earlier proofs, player I can fashion a winning strategy for $G'_{g,f}$ from F by playing a real r such that r^1 is entirely arbitrary, and so that the rest of his plays are according to F against II.

Conversely, if $g \nless f$, a winning strategy for I for $G'_{g,f}$ is easily converted into a winning strategy for II for $G_{f,g}$. ∎

DEFINITION. For any $f, g\colon {}^\omega 2 \to \{\text{the ordinals}\}$, $f \sim g$ iff $f \geqslant g$ and $g \geqslant f$.

LEMMA 1.6. *The relations $\geqslant$, $>$ and $\sim$ are all transitive.*

Proof. In view of Lemma 1.5, it is sufficient to show the transitivity of $\leqslant$. Suppose $f \leqslant g \leqslant h$. Let F, G be winning strategies for II for $G_{f,g}$ and $G_{g,h}$ respectively. II can then win by comparing these strategies. Essentially, II plays the strategy G

against an imaginary player who is using F against I. If I plays $n_1, n_2, \ldots$, then on his kth move, II plays $G(F(n_1), F(n_1, n_2), \ldots, F(n_1, \ldots, n_k))$. ■

LEMMA 1.7 (DC). *The relation* $>$ *is a well-ordering on the equivalence classes of* $\sim$.

Proof. The Axiom of Determinateness guarantees that $<$ is a total ordering. We need only prove well-foundedness. Suppose $<$ is not well-founded. Let us use DC to pick a descending chain of functions:

$$f_0 > f_1 > f_2 > f_3 > \ldots$$

Using choice again, but only countable choice, choose for each $k<\omega$, a winning strategy F_k for I for $G_{f_k, f_{k+1}}$. Using these strategies, we set ω-many players to work manufacturing infinitely many reals $\{r_k\}_{k<\omega}$ as follows:

r_0 is any sequence,

r_{k+1} is the result of applying strategy F_k to r_k.

It follows that

$$\bigcup_n f_0(r_0^n) > \bigcup_n f_1(r_1^n) > \ldots$$

an infinite descending chain of ordinals, an impossibility, therefore $>$ is well-founded. ■

LEMMA 1.8. *If* f, g: ${}^{\omega}2 \to \alpha$, $\mathrm{cf}(\alpha)>\omega$, *are both onto maps and* $f\sim g$, *then* $\mu_\alpha^f = \mu_\alpha^g$.

Proof. Let $X\subseteq\alpha$ be any set such that $\mu_\alpha^f(X) = 1$. By symmetry, it will be enough to show that $\mu_\alpha^g(X) = 1$. Let F be a winning strategy for II for $G_{f,H}$, and let F_1 and F_2 be winning strategies respectively for II for $G_{f,g}$ and $G_{g,f}$. We will construct a winning strategy for II for $G_{g,X}$ as follows:

II pretends there are two auxiliary players A and B. Let the sequences formed by I, A, B, and II be respectively, r, s, t and u, with

$$\alpha = \bigcup_n g(r^n), \quad \beta = \bigcup_n f(s^n), \quad \gamma = \bigcup_n f(t^n), \quad \text{and} \quad \delta = \bigcup_n g(u^n).$$

In II's imagination, A uses strategy F_2 against *both* I and II, so that $\beta\geqslant\alpha, \delta$; B uses F against A so that $\beta \cup \gamma \in X$; and II uses F_1 against *both* A and B so that $\delta\geqslant\beta, \gamma$. The result is that $\alpha \cup \delta \in X$, since

$$\beta \cup \gamma \geqslant (\alpha \cup \delta) \cup \gamma = \alpha \cup \delta \geqslant \alpha \cup (\beta \cup \gamma) = \beta \cup \gamma .$$

So $\alpha \cup \delta = \beta \cup \gamma \in X$ and we have a winning strategy for $G_{g,X}$.

To calculate A's moves, II applies the strategy F_2 to a sequence v composed so that $v^{2a} = r^a$ and $v^{2a+1} = u^a$. To calculate B's moves, II applies F to s and to calculate his own moves, II applies F_1 to the sequence w composed so that $w^{2a} = s^a$ and $w^{2a+1} = t^a$. It is routine to check that II's kth move can be computed in this way knowing only the first k moves of I. This completes the proof. ■

This work is summarized by:

THEOREM 1.9. *For each* $\alpha<\theta$, $\mathrm{cf}(\alpha)>\omega$, *there is a canonically chosen measure* μ_α *on* α *with the property that if* $\beta<\alpha$, $\mu_\alpha(\beta)<0$.

Proof. By the previous lemma, let f: ${}^{\omega}2 \to \alpha$ be a member of the least equivalence class, and let $\mu_\alpha = \mu_\alpha^f$. ■

§ 2. **A consequence for** θ. As we have seen, under certain additional assumptions, θ is regular. It seems possible that θ is provably regular under AD alone, but that is yet to be proved. The following theorem can be thought of as evidence in this direction.

THEOREM 2.1. ZF+AD ⊢ "*Either* θ *is regular or else there is an* $\aleph_1$*-additive, uniform* (¹) *measure on* θ".

We first require:

LEMMA 2.2. *The set of regular cardinals below* θ *is unbounded.*

This lemma was probably proved by Moschovakis. In any event, a stronger result, that the set of measurable cardinals less than θ is unbounded, is mentioned in a paper of Martin [8], as having been noticed by Moschovakis.

Proof of Theorem 2.1. For each $\beta<\theta$, $\mathrm{cf}(\beta)>\omega$ let μ_β be as defined in Theorem 1.9. If $\mathrm{cf}(\theta) = \alpha<\theta$, then let f: $\alpha \to \theta$ be a cofinal sequence of regular cardinals (²). We can define a non-uniform measure on θ from μ_α by:

$$\text{for all } A\subseteq\theta, \quad \mu_1(A) = \mu_\alpha(f^{-1}A).$$

It is routine to check that μ_1 is $\aleph_1$-additive. μ_1 is not uniform, however, since $\mu_1(f''\alpha) = 1$. To get a uniform measure, we "glue together" measures on regular cardinals below θ. We define:

$$\text{for all } A\subseteq\theta, \quad \mu_2(A) = \mu_1\{\beta|\ \mu_\beta(A\cap\beta) = 1\}.$$

Again, it is easy to check that this measure is $\aleph_1$-additive, for suppose $\{A_n\}_{n<\omega}$ is a collection of subsets of θ such that $\mu_2(A_n) = 1$ for all n, and let $A = \bigcap_{n<\omega} A_n$. For each n, let $B_n = \{\beta|\ \mu_\beta(A_n\cap\beta) = 1\}$. By the additivity of μ_1, $\mu_1(B) = 1$, where $B = \bigcap_{n<\omega} B_n$. Further, by the additivity of μ_β, if $\beta\in B$, then $\mu_\beta(A\cap\beta) = 1$, hence $\mu_2(A) = 1$.

Finally, μ_2 is uniform, that is, if $\mu_2(A) = 1$, then $\bar{A} = \theta$. To see this, let δ be any cardinal less than θ. Let $\beta>\delta$ be any element of $f''\alpha\cap\{\beta|\ \mu_\beta(A\cap\beta) = 1\}$. Since $\mu_\beta(A\cap\beta) = 1$, $A\cap\beta$ is unbounded in β and since β is regular, $\overline{\overline{A\cap\beta}} = \beta>\delta$, thus $\bar{A}>\delta$ for all $\delta<\theta$ so $\bar{A} = \theta$. ■

Note that it was crucial in the above proof that the measures μ_β be canonical, as there is no way otherwise we can choose a measure μ_β^f for each $\beta<\theta$. AD is inconsistent even with $\mathrm{AC}_{\aleph_1}$.

(¹) μ is *uniform* on θ iff $\mu(A) = 1 \to \bar{A} = \theta$ for all $A\subseteq\theta$.

(²) Note that $\alpha>\omega$.

§ 3. **The length of the prewellorderings.** A prewellordering of a set A is a map of A into a well-ordered set. The length of the prewellordering is the order-type of the range of this map, for example, a map f from ${}^{\omega}2$ onto α is a prewellordering of ${}^{\omega}2$ of length α. In proving Theorem 1.9, we essentially defined a prewellordering on the set of these prewellorderings, and a very interesting problem is to find its length.

Let us designate by Ξ_β the set of all the equivalence classes of prewellorderings of 2^ω of length β, and let T_β denote its order-type under our well-ordering. By Theorem 1.4, for all $\beta<\theta$, $T_\beta\geqslant 2$.

THEOREM 3.1. *For all* $\beta<\theta$, $\mathrm{cf}(\beta)>\omega$, $T_\beta\geqslant\beta$.

Proof. We use the technique introduced in the proof of 1.4. Given $h\colon {}^{\omega}2\to\beta$, unbounded, let h^* be as defined in that proof, and for all $\alpha<\beta$, let

$$g_\alpha(r)=\bigcup_n h^*(r^n)+\alpha \quad \text{for all } r\in{}^{\omega}2.$$

CLAIM. $\alpha<\gamma$ *implies* $g_\alpha<g_\gamma$.

Proof of the claim. The following is a strategy for player II in the game G'_{g_α,g_γ}: While I plays the sequence r, II plays s so that each s^n is identical and $\{(s^n)^{m^*}\}_m$ includes $\{(r^a)^{b^*}\}_{a,b}$. Thus

$$\bigcup_{i<\omega} g_\alpha(r^i)\leqslant\bigcup_{n<\omega} g_\alpha(s^n)=g_\alpha(s^0)<g_\gamma(s^0)=\bigcup_{n<\omega} g_\gamma(s^n).\ \blacksquare$$

For each $\alpha<\theta$ and $f\colon {}^{\omega}2\to\alpha$ μ^f_α is $\aleph_1$-additive, so the ultrapower $\alpha^\alpha/\mu^f_\alpha$ is well-founded.

THEOREM 3.2. *For all* $\beta<\theta$, $\mathrm{cf}(\beta)>\omega$ *and* $h\colon {}^{\omega}2\to\beta$, T_β *is greater than or equal to the order-type of the ultrapower* β^β/μ^h_β.

Proof. Let h^* be as before, and for any $p\in{}^{\beta}\beta$, we define

$$g_p(r)=p\Big(\bigcup_n h^*(r^n)\Big)\quad \text{for all } r\in{}^{\omega}2.$$

CLAIM. *If* $p<q \bmod \mu^h_\beta$ *then* $g_p<g_q$, *and if* $p\sim q\bmod\mu^h_\beta$ *then* $g_p\sim g_q$.

Proof of the claim. Suppose $p<q$. Let $A=\{\alpha|\ p(\alpha)<q(\alpha)\}$. Since $\mu^h_\beta(A)=1$, $\mu^{h^*}_\beta(A)=1$. Let F be a winning strategy for II in the game $G^*_{h,A}$. Using F, we define a strategy for II for $G'_{g_p g_q}$ as follows:

While I plays the sequence r, II plays s in such a way that each s^n is identical and $\{(s^n)^{m^*}\}$ includes representatives of all the sequences played by both players of a game in which one player uses the strategy F while the other player plays *all* the sequences $(r^a)^b$ for all a and b. (Again, this is the same trick used in 1.4).

The result is that for any a, n,

$$\alpha_a=\bigcup_b h^*((r^a)^b)\leqslant\beta_n=\bigcup_m h^*((s^n)^m)\in A$$

hence $g_p(r^a)=p(\alpha_a)\leqslant p(\beta_n)<q(\beta_n)$. Since the $\{\beta_n\}$ are identical,

$$\bigcup_a g_p(r^a)\leqslant p(\beta_0)<q(\beta_0)=\bigcup_n g_q(s^n)$$

so $g_p<g_q$. That $p\sim q$ implies $g_p\sim g_q$ is proved similarly. $\blacksquare$

In the case of $\aleph_1$, Theorem 3.2 has interesting consequences. Kunen has shown that for any measure μ

$\aleph_1^{\aleph_1}/\mu$ has order type $\aleph_n$ for some n.

Thus, $T_{\aleph_1}\geqslant\aleph_2$. On the other hand, while measures have been found on $\aleph_1$ such that

$\aleph_1^{\aleph_1}/\mu$ has order type $\aleph_n$ for any particular n,

no such measure has been shown to be of the type μ^f for some f.

THEOREM 3.3. *For all* $\alpha<\theta$, $\mathrm{cf}(\alpha)>\omega$, *implies* $\mathrm{cf}(T_\alpha)>\omega$.

Proof. First note that T_α cannot be a successor ordinal by the proof of Theorem 3.1. Next, suppose that $\alpha_0<\alpha_1<\ldots$ is an increasing sequence of ordinals below T_α of length ω. Using choice, choose g_n, an element of the α_nth equivalence class. Define $g\colon {}^{\omega}2\to\alpha$ by:

$$g(r)=\bigcup_{n<\omega} g_n(r)\quad \text{for all } r\in{}^{\omega}2.$$

It is routine to show that $g>g_n$ for all n. $\blacksquare$

The techniques of this paper can also be used to tackle the compactness of $\aleph_1$. The first move in this direction was made by E. M. Kleinberg who used exactly the same methods to find a new proof that $\aleph_1$ is α-strongly compact for all $\alpha<\theta$. The procedure is simple. By a theorem of Moschovakis, if $\alpha<\theta$, then there is a map of the continuum onto 2^α. Using this fact, we choose a map $F\colon {}^{\omega}2\to P_{\aleph_1}(\alpha)$, onto, and for every $A\subseteq P_{\aleph_1}(\alpha)$, we define the game $H_{F,A}$: I forms the sequence r while II forms s. II wins iff

$$\bigcup_{n<\omega} F(r_n)\cup\bigcup_{n<\omega} F(s_n)\in A.$$

As in Section 1, we define the measure v^F_α by $v^F_\alpha(A)=1$ iff II has a winning strategy for $H_{F,A}$. Exactly as in the proof of Theorem 1.3, it can be proved that v^F_α is $\aleph_1$-additive and fine. It is not known if any of these measures are normal, however. If we define the comparable game with AD_R, the measures obtained are normal, as Kleinberg has shown. In general, the measure v^F_α we get with AD will depend on the coding map F, as in Theorem 1.4.

Proceeding with this analysis, we can prove a result about θ corresponding to Theorem 2.1: $\mathrm{ZF}+\mathrm{AD}_R\vdash$ "Either θ is regular or $\aleph_1$ is θ-strongly compact". This is proved in exactly the same manner. It can be shown that the measures AD_R produces on $P_{\aleph_1}(\alpha)$ are independent of the coding maps, and that these measures can be glued together by a measure on a sequence cofinal in θ. With only AD, though, we are lost. There does not seem to be any way to pick a canonical measure on $P_{\aleph_1}(\alpha)$ [1].

---

[1] A totally different application of AD to super compactness is found in [2]. References for the theory of compact cardinals include: [3] and [7].

References

[1] A. Blass, *The equivalence of* 2 *strong forms of determinacy*, Proc. Amer. Math. Soc. 52, (1975), pp. 373–376.

[2] C. di Prisco and J. Henle, *On the compactness of* $\aleph_1$ *and* $\aleph_2$, J. Symb. Logic 43 (3) (1978), pp. 394–401.

[3] F. Drake, *Set Theory an Introduction to Large Cardinals*, North-Holland Publishing Co., 1974.

[4] D. Gale and F. Stewart, *Infinite games with perfect information*, Ann. Math. Studies 28 (153), pp. 245–266.

[5] J. Henle, *Aspects of choiceless combinatorial set theory*, Doctoral dissertation, M. I. T. (1976).

[6] A. Kechris, *Notes prepared for the M. I. T. Logic Seminar* 1972–73.

[7] M. Magidor, *Combinatorial characterization of supercompact cardinals*, Proc. Amer. Math. Soc. 42 (1) (1974), pp. 279–285.

[8] D. A. Martin, *Determinateness implies many cardinals are measurable*, mimeographed.

[9] Y. N. Moschovakis, *Determinacy and prewellorderings of the continuum*, Math. Logic and Found. of Set Theory (1968), pp. 24–62.

[10] J. Mycielski, *On the axiom of determinateness*, Fund. Math. 53 (1963), pp. 205–224.

[11] — and Świerczkowski, *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. 54 (1964), pp. 67–71.

[12] R. M. Solovay, *Measurable cardinals and the Axiom of Determinateness*, Lecture notes prepared in connection with the Summer Institute of Axiomatic Set Theory held at U.C.L.A. Summer, 1967.

Accepté par la Rédaction le 10. 7. 1978

On a certain prewellordering

by

J. M. Henle (Northampton, Mass.) and **W. Zwicker** (Schenectady, N. Y.)

In [1], a certain prewellordering of functions from ${}^{\omega}2$ of $\aleph_1$ is defined under the Axiom of Determinateness, and is shown to have length at least $\aleph_2$. We will show that this length is in fact at least θ, the least cardinal onto which the continuum cannot be mapped. We use throughout the notation and techniques of [1], particularly those of Theorem 1.4.

THEOREM (AD). $T_{\aleph_1} \geqslant \theta$.

Proof. Given $\gamma < \theta$, let f map ${}^{\omega}2$ onto $P_{\aleph_1}(\gamma)$, and let $f_\alpha\colon {}^{\omega}2 \to \aleph_1$, $\alpha < \gamma$, be defined by: $f_\alpha(r) =$ the order type of $\bigcup_{n<\omega} f^*(r^n) \cap \alpha$. For any $\alpha < \beta < \gamma$, we can show $f_\alpha < f_\beta$ by describing a winning strategy for player II in G'_{f_α, f_β}. Such a strategy consists of playing the real s to player I's real r so that each s^n is identical and all the reals $\{(r^m)^{k^*}\}_{m,k}$ are included in $\{(s^n)^{k^*}\}_k$ as well as a real t such that $\alpha \in f^*(t)$.

This establishes the functions $\{f_\alpha\}_{\alpha<\gamma}$ as a sequence of length γ in the prewellordering. ■

References

[1] J. M. Henle, *The Axiom of Determinateness and canonical measures*, Fund. Math. 114.3 (1981), pp. 171–182.

Accepté par la Rédaction le 4. 2. 1980