A class of spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf

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Abstract. We prove that if X is a Lindelöf space such that each closed subspace F of X contains a compact subset with non-empty interior (with respect to F), then the product $X^{\aleph_0} \times Y$ is Lindelöf for every hereditarily Lindelöf space Y.

Introduction. A general question is to characterize the class $\mathscr{L}$ of all spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf (1). It is well known that $\mathscr{L}$ is closed with respect to continuous images, perfect preimages and contains all separable metric spaces (see [E], Th. 3.8.6, Th. 3.8.8 and Problem 4.5.16.d), in particular, as was noticed by Z. Frolík [F], $\mathscr{L}$ contains Lindelöf spaces, complete in the sense of Čech, because they are perfect preimages of separable metric spaces.

E. Michael asked whether $\mathscr{L}$ is closed with respect to countable Cartesian products?

The main result of this paper is to prove that if X is a Lindelöf space such that each closed subspace F of X contains a compact subset with non-empty interior (with respect to F), then the product $Y \times X^{\aleph_0}$ is Lindelöf for every hereditarily Lindelöf space Y (2). This result exhibits a rather wide subclass of $\mathscr{L}$, closed under countable products, including, as was noticed in [AP], a class of function spaces (3).

Methods applied in this paper are related to [A].

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(1) From an example obtained by E. Michael ([M], Ex. 1.2), for some details see Remark 1, it follows that it is not reasonable to ask a similar question for the class of all Lindelöf spaces whose product with every Lindelöf space is Lindelöf.

(2) R. Telgársky noticed that X belongs to $\mathscr{L}$ [T].

(3) In [AP] it was proved that if K is a compact subspace of the Σ-product of $\mathfrak{m}$ copies of the real line, for a cardinal $\mathfrak{m}$ and R is the real line then the function space $C(K, R)$ endowed with the pointwise topology is a continuous image of a closed subspace of $X^{\aleph_0}$, where X is a Lindelöf space having only one non-isolated point.

Terminology and notation. Our topological terminology follows [E].

Let us recall that X is a P-space if every G_δ-subset of X is open.

We say that U is a basic open set in $\mathop{P}\limits_{i=1}^{\infty} X_i$, if it is of the following form $U = \mathop{P}\limits_{i=1}^{\infty} U_i \times \mathop{P}\limits_{i=n+1}^{\infty} X_i$, where U_i is open in X_i.

If $\mathscr{U}$ is a family of subsets of X then $\mathscr{U}^*$ denotes the family of all finite unions of elements of $\mathscr{U}$.

We write $\mathscr{U} \prec \mathscr{V}$ if $\mathscr{U}$ refines $\mathscr{V}$.

Put $X^{(0)} = X_c^{(0)} = X$. Denote by $X^{(1)}$ $(X_c^{(1)})$ the set of all non-isolated points of X (of all points of X at which X is not locally compact) (4). Put $X^{(\alpha)} = \bigcap \{X^{(\beta)}\colon \beta<\alpha\}$ $(X_c^{(\alpha)} = \bigcap \{X_c^{(\beta)}\colon \beta<\alpha\})$, if α is a limit ordinal number and $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$ $(X_c^{(\alpha+1)} = (X_c^{(\alpha)})_c^{(1)})$ (cf. [S]). One can prove by means of some standard reasoning, that X is a scattered space (5) (each closed subspace F of X contains a compact subset with non-empty interior, with respect to F) if and only if there exists α such that $X^{(\alpha)} = \emptyset$ $(X_c^{(\alpha)} = \emptyset)$ (see [S]).

The symbol $L(\aleph_1)$ stands for a space of cardinality $\aleph_1$ and having only one non isolated point p such that $p \in U \subset L(\aleph_1)$ is open, if $|L(\aleph_1) \setminus U| \leqslant \aleph_0$ (6).

Auxiliary constructions. If X is a zero-dimensional space such that each closed subspace F of X contains a compact subset with non-empty interior (with respect to F) then $\alpha(x)$ denotes the ordinal number and U_x an open and closed neigbourhood of x such that $x \in X_c^{(\alpha(x))} \setminus X_c^{(\alpha(x)+1)}$ and $F_x = U_x \cap X_c^{(\alpha(x))}$ is compact. Notice that if X is a Lindelöf space and U is a neighbourhood of F_x then

(0) there are $x_1, x_2, \ldots \in X$ such that $\alpha(x_n) < \alpha(x)$ and

$$U_x \setminus \bigcup \{U_{x_n}\colon n = 1, 2, \ldots\} \subset U.$$

Indeed, we can assume that U is open and closed so there are

$$x_1, x_2, \ldots, x_n, \ldots \in X \setminus X^{(\alpha(x))}$$

such that $U_x \setminus U \subset \bigcup \{U_{x_n}\colon n = 1, 2, \ldots\}$.

Fix for any such U a countable set $A_x(U)$ consisting of points $x_1, \ldots, x_n, \ldots$ satisfying (0). By the definition of $A_x(U)$ we have

(1) $$\alpha(x) > \alpha(a) \quad \text{for every } a \in A_x(U).$$

Main results

THEOREM 1. *If X is a Lindelöf space such that each closed subspace F of X contains a compact subset with non-empty interior (with respect to F) and Y is such that $Y \times (L(\aleph_1)^{\aleph_0})$ is Lindelöf then $Y \times X^{\aleph_0}$ is Lindelöf.*

THEOREM 2. *If X is a Lindelöf space such that each closed subspace F of X contains a compact subset with non-empty interior (with respect to F) then the product $Y \times X^{\aleph_0}$ is Lindelöf for every hereditarily Lindelöf space Y.*

(4) X is locally compact at $x \in X$ if there is a compact neighbourhood of x in X.

(5) X is a scattered space if every closed subspace of X has an isolated point.

(6) The symbol $|A|$ stands for the cardinality of A.

The proof of Theorem 1 consists of two steps. In the first step we assign to an arbitrary open cover $\mathscr{U}$ of $Y \times X^{\aleph_0}$ a scattered Lindelöf P-space Z of weight not greater than $\aleph_1$ in such a way that if $Y \times Z^{\aleph_0}$ is a Lindelöf space then $\mathscr{U}$ has a countable refinement. In the second step we show that every Lindelöf, scattered, P-space Z of weight not greater than $\aleph_1$ can be embedded in $L(\aleph_1)^{\aleph_0}$ as a closed subset.

We prove Theorem 2 by showing that $Y \times (L(\aleph_1)^{\aleph_0})$ is Lindelöf for every hereditarily Lindelöf space Y and applying Theorem 1 to the product $Y \times X^{\aleph_0}$.

Notice that without loss of generality we can regard X as a subset of the product $I^{\mathfrak{m}}$, where I is the unit interval and $\mathfrak{m}$ a cardinal number. Let us take a continuous mapping $f\colon D^{\mathfrak{m}} \to I^{\mathfrak{m}}$ such that $f(D^{\mathfrak{m}}) = I^{\mathfrak{m}}$, where $D^{\mathfrak{m}}$ is a Cantor cube. Put $X' = f^{-1}(X)$. It is easy to see that X' satisfies assumptions of the theorems, is zero-dimensional and the product is $Y \times X'^{\aleph_0}$ is Lindelöf if and only if $Y \times X^{\aleph_0}$ is Lindelöf.

In the sequel Y stands for a space such that $Y \times (L(\aleph_1)^{\aleph_0})$ is Lindelöf, X for a zero-dimensional space satisfying the assumptions of the theorems and $\mathscr{U}$ for an open cover of $Y \times X^{\aleph_0}$.

Step 1.

LEMMA 1. *If $\mathscr{F} = \{F_n\colon n = 1, 2, \ldots\}$ is a family of compact subsets of X then there is a countable family $\mathscr{V}$ consisting of basic open sets in $Y \times X^{\aleph_0}$ and refining $\mathscr{U}^*$ such that if $(n_i)_{i=1}^{\infty}$ is a sequence of natural numbers and $y \in Y$ then there exists $V \in \mathscr{V}$ satisfying $\{y\} \times \mathop{P}\limits_{i=1}^{\infty} F_{n_i} \subset V$.*

Proof. If $\{F_n\colon n = 1, 2, \ldots\}$ is a discrete family in X then the lemma holds because of the following facts: $Y \times N^{\aleph_0}$ (7) is Lindelöf as a closed subset of $Y \times (L(\aleph_1)^{\aleph_0})$, a product of perfect mappings (8) is perfect and a perfect preimage of a Lindelöf space is Lindelöf.

If $\mathscr{F} = \{F_n\colon n = 1, 2, \ldots\}$ is not discrete in X then put $X' = X \times N$ and $F'_n = F_n \times \{n\}$, for $n \in N$. Notice that $\mathscr{F}' = \{F'_n\colon n = 1, 2, \ldots\}$ is discrete in X' so there is a suitable family $\mathscr{V}'$ for X' and $\mathscr{F}'$. Now it is enough to take $\mathscr{V} = n(\mathscr{V}') = \{n(V')\colon V' \in \mathscr{V}'\}$, where n is a natural mapping from $Y \times (X')^{\aleph_0}$ onto $Y \times X^{\aleph_0}$.

LEMMA 2. *There are a subset A of X of cardinality not greater than $\aleph_1$ and a cover $\mathscr{V}$ of $Y \times X^{\aleph_0}$ consisting of basic open sets such that*

(2) *there is a countable subset A_0 of A such that $\bigcup \{U_x\colon x \in A_0\} = X$,*

(3) *$\mathscr{V}$ refines $\mathscr{U}^*$ and $|\mathscr{V}| \leqslant \aleph_1$,*

(4) *for every $(a_n)_{n=1}^{\infty} \in A^{\aleph_0}$ and $y \in Y$ there are $V_0 \subset Y$, $V_n \subset X$, for $n = 1, 2, \ldots$, and $V \in \mathscr{V}$ such that $\{y\} \times \mathop{P}\limits_{n=1}^{\infty} F_{a_n} \subset \mathop{P}\limits_{n=0}^{\infty} V_n$ and $\bigcup \{A_{a_n}(V_n)\colon n = 1, 2, \ldots\} \subset A$ (see, Auxiliary constructions).*

(7) N stands for the set of natural numbers.

(8) A mapping $f\colon X \to Y$ is perfect if it is closed and $f^{-1}(y)$ is compact, for $y \in Y$.

Proof. For $\alpha<\omega_1$, we shall define $A_\alpha\subset X$ and an open family of basic sets $\mathscr{V}_\alpha$ in $Y\times X^{\aleph_0}$ such that

(5) $\bigcup\{U_x\colon x\in A_\alpha\} = X$ and $|A_\alpha|\leqslant\aleph_0$,

(6) $\mathscr{V}_\alpha$ is defined as Lemma 1 says, for $\mathscr{F} = \{F_a\colon a\in A_\alpha\}$,

(7) if $\alpha>0$ then

$$A_\alpha = \begin{cases}\bigcup\{A_\beta\colon \beta<\alpha\}, & \text{if } \alpha \text{ is a limit ordinal number,}\\ A_\beta\cup\bigcup\{A_x(H)\colon H\in\mathscr{H}_{\beta x},\, x\in A_\beta\} & \text{if } \alpha=\beta+1,\end{cases}$$

where $\mathscr{H}_{\beta x} = \{H\colon \exists n\geqslant 1\,\exists V = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} V_i\in\mathscr{V}_\beta \text{ and } H = V_n\supset F_x\}$.

If $\alpha = 0$ then there exists a countable set A_0 in X such that $\bigcup\{U_x\colon x\in A_0\} = X$. Let $\mathscr{V}_0$ be a family defined as (6) says.

From (6) and (7) it follows how to define A_α and $\mathscr{V}_\alpha$ for $\alpha>0$.

Notice that

(8) if $\{y\}\times\mathop{\mathbf{P}}\limits_{i=1}^{\infty} F_{a_i}\subset\mathop{\mathbf{P}}\limits_{i=0}^{\infty} V_i = V\in\mathscr{V}_\alpha$, where $a_i\in A_\alpha$, for $i=1,2,\ldots$, then $F_{a_i}\subset G_{\alpha a_i} = (U_{a_i}\setminus\bigcup\{U_z\colon z\in\bigcup\{A_{a_i}(H)\colon H\in\mathscr{H}_{\alpha a_i}\})\subset V_i$ for $i=1,2,\ldots$

Put $A = \bigcup\{A_\alpha\colon \alpha<\omega_1\}$.

We shall show that

(9) for every $x\in X$ there is $y\in A$ such that $x\in\bigcap\{G_{\alpha y}\colon y\in A_\alpha\}$.

Suppose that (9) does not hold. There exists $y_1\in A_0$ such that $x\in U_{y_1}$. If $y_1,\ldots,y_n$ are defined in such a way that $\{y_1,\ldots,y_n\}\subset A$, $x\in\bigcap\{U_{y_i}\colon i=1,2,\ldots,n\}$ and $\alpha(y_1)>\alpha(y_2)>\ldots>\alpha(y_n)$, then by the assumption there is α such that $y_n\in A_\alpha$, $x\notin G_{\alpha y_n}$ and $x\in U_{y_n}$, so there exists $y_{n+1}\in\bigcup\{A_{y_n}(H)\colon H\in\mathscr{H}_{\alpha y_n}\}$ and $x\in U_{y_{n+1}}$. By (7) $y_{n+1}\in A_{\alpha+1}$ and by (1) $\alpha(y_{n+1})<\alpha(y_n)$. This way we would obtain an infinite decreasing sequence of ordinals, a contradiction.

Put $\mathscr{V} = \bigcup\{\mathscr{V}_\alpha\colon\alpha<\omega_1\}$. Notice that (4) follows from (7) and (6). We shall finish the proof of Lemma 2 by showing that $\mathscr{V}$ covers $Y\times X^{\aleph_0}$. Let $(y, x_1, x_2, \ldots)$ be an arbitrary point of $Y\times X^{\aleph_0}$. By (9) there are $y_1,\ldots,y_n,\ldots\in A$ such that

(10) $$x_n\in\bigcap\{G_{\alpha y_n}\colon y_n\in A_\alpha\} = G_{y_n}.$$

Notice that $\{A_\alpha\colon\alpha<\omega_1\}$ is an increasing family so there are γ and $V\in\mathscr{V}_\gamma$ such that $\{y_1,\ldots,y_n,\ldots\}\subset A_\gamma$ and $\{y\}\times\mathop{\mathbf{P}}\limits_{n=1}^{\infty} F_{y_n}\subset V$. By (8) and (10)

$$(y, x_1,\ldots,x_n,\ldots)\in\{y\}\times\mathop{\mathbf{P}}\limits_{n=1}^{\infty} G_{y_n}\subset\{y\}\times\mathop{\mathbf{P}}\limits_{n=1}^{\infty} G_{\gamma y_n}\subset V.$$

LEMMA 3. *There is a scattered, Lindelöf, P-space Z of weight not greater than $\aleph_1$ such that if $Y\times Z^{\aleph_0}$ is a Lindelöf space then $\mathscr{V}$, see Lemma 2, has a countable refinement.*

Proof. Let us order A, see Lemma 2, in the type of ω_1 and put $Z = \{(\beta_1,\ldots,\beta_n)\colon n\in N,\ \beta_1\in A_0$ (see (2)) and $\alpha(a_{\beta_i})>\alpha(a_{\beta_{i+1}})$, for $i=1,2,\ldots,n-1\}$. The base at the point $z = (\beta_1,\ldots,\beta_n)\in Z$ consists of the sets of the form

(11) $B_\gamma(z) = \{(\gamma_1,\ldots,\gamma_m)\in Z\colon m\geqslant n,\ \gamma_i=\beta_i$, for $i=1,2,\ldots,n$, and $\gamma_{n+1}\geqslant\gamma\}$, for $\gamma<\omega_1$.

Notice that $|Z|\leqslant\aleph_1$ so by (11) we infer that the weight of Z is not greater than $\aleph_1$.

If $S\subset Z$ then $z = (\gamma_1,\ldots,\gamma_m)\in S$, with minimal $\alpha(a_{\gamma_m})$, is an isolated point in S, therefore Z is a scattered space.

Z is a P-space by (11).

Let us notice that $B_\gamma(z)$ is an open and closed subset of Z, for $z\in Z$ and $\gamma<\omega_1$.

If $z = (\gamma_1,\ldots,\gamma_n)$ and $\alpha(a_{\gamma_n}) = 0$ then $B_0(z) = \{z\}$. Assume that $B_0(z)$ is a Lindelöf space for every $z = (\gamma_1,\ldots,\gamma_n)$ such that $\alpha(a_{\gamma_n})<\beta>0$ and suppose that $\alpha(a_{\gamma_n}) = \beta$. If γ is an arbitrary countable ordinal then $B_0(z)\setminus B_\gamma(z) = \bigcup\{B_0(z_i)\colon$ where $z_i = (\gamma_1,\ldots,\gamma_n; \beta_i)\}$ for $i=1,2,\ldots$ By the definition of Z $\alpha(a_{\beta_i})<\alpha(a_{\gamma_n}) = \beta$ for $i=1,2,\ldots$ so from the inductive assumption it follows that $\bigcup\{B_0(z_i)\colon i=1,2,\ldots\}$ is Lindelöf and we conclude that $B_0(z)$ is Lindelöf. Notice that $Z = \bigcup\{B_0(z)\colon z\in A_0\}$ therefore it is Lindelöf.

In order to finish the proof of the lemma it is enough to show that there exists an open cover $\mathscr{H}$ of $Y\times Z^{\aleph_0}$ such that if $\mathscr{H}$ has a countable refinement than $\mathscr{V}$ has also.

If $(y, z_1,\ldots,z_n,\ldots) = p\in Y\times Z^{\aleph_0}$, where $z_i = (\gamma_1^i,\ldots,\gamma_{m_i}^i)$, for $i=1,2,\ldots$ then by (4) there is $V(p) = \mathop{\mathbf{P}}\limits_{n=0}^{\infty} V_n(p)\in\mathscr{V}$ such that $\{y\}\times\mathop{\mathbf{P}}\limits_{i=1}^{\infty} F_{a_{\gamma_{m_i}^i}}\subset V(p)$. Put

(12) $$H_i(V(p)) = \begin{cases} Z, & \text{if } V_i(p) = X,\\ V_0, & \text{if } i=0,\\ B_\gamma(z_i), & \text{where } \gamma = \sup\{\beta\colon a_\beta\in A_{a_{\gamma_{m_i}^i}}(V_i(p))\}+1, \text{ otherwise,}\end{cases}$$

$H(V(p)) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} H_i(V(p))$ and $\mathscr{H} = \{H(V(p))\colon p\in Y\times Z^{\aleph_0}\}$. Notice that $H(V(p))$ is an open neighbourhood of p so $\mathscr{H}$ covers $Y\times Z^{\aleph_0}$.

Let us attach to $x\in X$ an element $(\beta_1(x),\ldots,\beta_{n(x)}(x)) = z(x)\in Z$ in such a way that $\beta_1(x)$ is the first number of A_0 such that $x\in U_{a_{\beta_1(x)}}$. If $\beta_1(x),\ldots,\beta_n(x)$ are defined then $\beta_{n+1}(x)$ is the first ordinal number such that $x\in U_{a_{\beta_{n+1}(x)}}$ and $\alpha(a_{\beta_{n+1}(x)})<\alpha(a_{\beta_n(x)})$. We continue the induction as long as it is possible.

We shall show that

(13) if $(\beta_1(x),\ldots,\beta_{n(x)}(x))\in H_i(V(p))$ for $p\in Y\times Z^{\aleph_0}$ and $i\geqslant 1$ then $x\in V_i(p)$, where $V(p) = \mathop{\mathbf{P}}\limits_{n=0}^{\infty} V_n(p)$.

If $H_i(V(p)) = Z$ then $V_i(p) = X$. If $H_i(V(p)) = B_\gamma(z_i)$, where

$$p = (y, z_1, \ldots, z_n, \ldots), \qquad z_n = (\gamma_1^n, \ldots, \gamma_{m_n}^n)$$

and $\gamma = \sup\{\beta\colon a_\beta \in A_{a_{\gamma^i_{m_i}}}(V_i(p))\}+1$ then

$$(14)\qquad \begin{aligned} V_i(p) &\supset U_{a_{\gamma^i_{m_i}}} \setminus \bigcup \{U_z\colon z \in A_{a_{\gamma^i_{m_i}}}(V_i(p))\} \\ &\supset U_{a_{\gamma^i_{m_i}}} \setminus \bigcup \{U_{a_\beta}\colon \beta<\gamma, \text{ where } \alpha(a_\beta)<\alpha(a_{\gamma^i_{m_i}})\}. \end{aligned}$$

From (11) and (13) it follows that $n(x) \geqslant m_i$, $\gamma^i_{m_i} = \beta_{m_i}(x)$, so $x \in U_{a_{\gamma^i_{m_i}}}$ and that $n(x) = m_i$ or $\gamma \leqslant \beta_{m_i+1}(x)$. By the definition of $(\beta_1(x), \ldots, \beta_{n(x)}(x))$ we infer that $x \notin \{U_{a_\beta}\colon \beta<\gamma$, where $\alpha(a_\beta)<\alpha(a_{\gamma^i_{m_i}})\}$ and finally we obtain $x \in V_i(p)$.

From (13) it follows that if $\bigcup\{H(V(p_i))\colon i = 1, 2, \ldots\} = Y \times Z^{\aleph_0}$ then $\bigcup\{V(p_i)\colon i = 1, 2, \ldots\} = Y \times X^{\aleph_0}$.

Step 2.

LEMMA 4. *If Z is a scattered, Lindelöf, P-space such that weight of Z is not greater than $\aleph_1$ then Z can be embedded in $L(\aleph_1)^{\aleph_0}$ as a closed subset.*

Proof. If $Z^{(1)} = \emptyset$ then the lemma is trivial. Let us assume that the lemma holds for every $\beta<\alpha$ such that $Z^{(\beta)} = \emptyset$.

Suppose that $Z^{(\alpha)} = \emptyset$. If α is a limit number then $Z = \bigcup\{Z_n\colon n = 1, 2, \ldots\}$, where $Z_1, Z_2, \ldots$ are pairwise disjoint open and closed subsets of Z such that $Z_n^{(\beta_n)} = \emptyset$ and $\beta_n<\alpha$, for $n = 1, 2, \ldots$ From the inductive assumption it follows that for $n = 1, 2, \ldots$ there is an embedding h_n of Z_n onto a closed subset of $\{n\} \times (L(\aleph_1)^{\aleph_0})$. Put $h(z) = h_n(z)$, if $z \in Z_n$. Then h is a desired embedding.

Let us identify the set of all isolated points of $L(\aleph_1)$ with the set of countable ordinal numbers and let p be the unique non-isolated point of $L(\aleph_1)$.

Now let us consider the case $\alpha = \beta+1$. Then $Z^{(\beta)} = \{z_n\colon n = 1, 2, \ldots\}$ and there is a family $\{Z_n\colon n = 1, 2, \ldots\}$ consisting of pairwise disjoint, open and closed sets covering Z and such that $Z_n \cap Z^{(\beta)} = \{z_n\}$. Let $\{V_{n\gamma}\colon \gamma<\omega_1\}$ be a decreasing open and closed base at z_n. Put $Z_{n0} = Z_n \setminus V_{n0}$ and $Z_{n\gamma} = (V_{n\gamma} \setminus V_{n\gamma+1}) \cap Z_n$, for $n \in N$ and $\gamma<\omega_1$. By the inductive assumption, for $n \in N$ and $\gamma<\omega_1$, there is an embedding $h_{n\gamma}$ of $Z_{n\gamma}$ onto a closed subset $\{n\} \times \{\gamma\} \times Y_\gamma^{\aleph_0}$, where $Y_\gamma = \{x \in L(\aleph_1)\colon x = p$ or $x \geqslant \gamma\}$. Put

$$h(z) = \begin{cases} (n, p, \ldots, p, \ldots), & \text{if } z = z_n, \\ h_{n\gamma}(z), & \text{if } z \in Z_{n\gamma}. \end{cases}$$

It is easy to see that h is an embedding of Z in $(L(\aleph_1))^{\aleph_0}$. Let

$$y = (y_1, \ldots, y_n, \ldots) \in L(\aleph_1)^{\aleph_0} \setminus h(X).$$

Put

$$U_y = \begin{cases} \pi_1^{-1}(y_1), & \text{if } y_1 \notin \{n\colon n = 1, 2, \ldots\}, \\ \pi_2^{-1}(y) \setminus h_{n\gamma}(Z_{n\gamma}), & \text{if } y_2 = \gamma, \\ \pi_i^{-1}(y) \cap \pi_2^{-1}(Y_{\gamma+1}), & \text{if } y_2 = p \text{ and } y_i = \gamma, \end{cases}$$

where π_i is the projection of $L(\aleph_1)^{\aleph_0}$ onto the ith coordinate. U_y is an open neighbourhood of y disjoint with $h(X)$ so $h(X)$ is a closed subset of $L(\aleph_1)^{\aleph_0}$.

Proof of Theorem 1. Theorem 1 follows from Lemmas 1, 2, 3 and 4.

The next lemma, which will complete the proof of Theorem 2, was proved in [AP]. We give a sketch of the proof of it for the sake of completeness.

LEMMA 5. *If Y is a hereditarily Lindelöf space then the product $Y \times (L(\aleph_1)^{\aleph_0})$ is a Lindelöf space.*

Proof. Put $X_n = L(\aleph_1)$, for $n = 1, 2, \ldots$ Let p_n be the projection of $Y \times \mathop{P}_{i=1}^{\infty} X_i$ onto $Y \times \mathop{P}_{i=1}^{n} X_i$ and p_0 the projection of $Y \times \mathop{P}_{n=1}^{\infty} X_n$ onto Y.

The weight of $\mathop{P}_{n=1}^{\infty} X_n$ is not greater than $\aleph_1$, Y is a hereditarily Lindelöf space, so every open cover of $Y \times \mathop{P}_{n=1}^{\infty} X_n$ has a refinement of cardinality not greater than $\aleph_1$. In order to finish the proof of the lemma it is enough to show that every uncountable subset A of $Y \times \mathop{P}_{n=1}^{\infty} X_n$ has a point of condensation ([9]).

Case 1. There is $y \in Y$ such that $|p_0^{-1}(y) \cap A| > \aleph_0$. Then there exists a point of condensation, by a Noble's result [N], which says that a countable Cartesian product of Lindelöf, P-spaces is a Lindelöf space.

Case 2. Let us assume that for every $y \in Y$ $|p_0^{-1}(y) \cap A| \leqslant \aleph_0$. Without loss of generality we can assume that

$$(15)\qquad A = \{(y, a_y)\colon y \in Y \text{ and } a_y \in \mathop{P}_{n=1}^{\infty} X_n\} \quad\text{and}\quad a_y \neq a_{y'}, \text{ if } y \neq y'.$$

For every $n \in N$ and $(x(1), \ldots, x(n)) = x \in \mathop{P}_{n=1}^{n} X_i$ put

$$(16)\qquad W_x^i = \begin{cases} X & \text{if } i>n \text{ or } x(i) \doteq p, \\ \{x(i)\} & \text{otherwise}, \end{cases}$$

$$W_x = \mathop{P}_{i=1}^{\infty} W_x^i$$

and

$$(17)\qquad A_x = \{y \in Y\colon a_y \in W_x \text{ and } (y, x) \in Y \times \mathop{P}_{i=1}^{n} X_i \text{ is a condensation point of } p_n(A)\}.$$

(9) A point x of X is called a *condensation point of a set* $A \subset X$ if every neighbourhood of x contains uncountably many points of A.

Notice, that

(18) if $i \leqslant n \leqslant m$, $x \in \mathop{P}_{i=1}^{n} X_i$, $x' \in \mathop{P}_{i=1}^{m} X_i$, $A_x \cap A_{x'} \neq \emptyset$ and $x(i) \neq p \neq x'(i)$ then $x(i) = x'(i)$,

as in the opposite case we would have $W_x \cap W_{x'} = \emptyset$.

We shall prove that

(19) if $T \subset Y$ is uncountable and $n \in N$, then there exists $x \in \mathop{P}_{i=1}^{n} X_i$ such that $A_x \cap T \neq \emptyset$.

Let $H = \{a_t|n\colon t \in T\} \subset \mathop{P}_{i=1}^{n} X_i$, where $a_t = (a_t(1), a_t(2), \ldots)$ and $a_t|n = (a_t(1), \ldots, a_t(n))$. If H is a countable set then there is $x \in \mathop{P}_{i=1}^{\infty} X_i$ such that $S = \{t \in T\colon a_t|\ n = x\}$ is uncountable. Without loss of generality we can assume that S is locally uncountable ([10]), Notice, that $S \subset A_x$. If H is uncountable then there is $x \in \mathop{P}_{i=1}^{n} X_i$ which is a point of condensation of H. X is a P-space and the weight of it is not greater than $\aleph_1$ so there is an uncountable subset H' of H such that for every neighbourhood U of x $|H' \setminus U| \leqslant \aleph_0$. We can assume that $\{t\colon a_t|\ n \in H'\} \subset T$ is locally uncountable. It is easy to see that $S \subset A_x$.

By (19) we infer, that for $n \in N$, $|Y \setminus \bigcup \{A_x\colon x \in \mathop{P}_{i=1}^{n} X_i\}| \leqslant \aleph_0$. Y is an uncountable set so there is $y \in Y$ and $x_i \in \mathop{P}_{j=1}^{i} X_j$, for $i = 1, 2, \ldots$, such that

$$y \in \bigcap \{A_{x_i}\colon i = 1, 2, \ldots\}. \tag{20}$$

From (18) it follows that there exist $c_1, c_2, \ldots$ such that for $n \in N$

(21) if $i \leqslant n$ then $x_n(i) = c_i$ or $x_n(i) = p$.

The space C given by $C = \mathop{P}_{i=1}^{\infty} \{p, c_i\}$ is a compact subset of $\mathop{P}_{i=1}^{\infty} X_i$. Put $\bar{x}_n(i) = x_n(i)$, if $i \leqslant n$ and $\bar{x}_n(i) = p$ for $i > n$. Notice that $\bar{x}_n \in C$, for $n \in N$, so there is an accumulation point c of $(\bar{x}_n)_{n=1}^{\infty}$. We shall show that (y, c) is a point of condensation of A. Indeed, if $U = U_0 \times \mathop{P}_{i=1}^{n} U_i \times \mathop{P}_{i=n+1}^{\infty} X_i$ is a neighbourhood of (y, c) then there exists $n' \geqslant n$ such that $\bar{x}_{n'} \in U$. From $\bar{x}_{n'}|n' = x_{n'}$, (20) and the definition of $A_{x_{n'}}$, it follows that $A \cap U$ is an uncountable set.

([10]) X is locally uncountable if for every $x \in X$ an arbitrary neighbourhood of x is uncountable.

Remark 1. *E. Michael proved* ([M], Ex. 1.2), *that, under Continuum Hypothesis, there is an uncountable subset K of the real line containing the set Q of rational numbers such that K_Q ([11]) is Lindelöf but $K_Q \times P$ ([12]) is not, so it is not enough to assume, in Theorem 2, that Y is only a Lindelöf space.*

Notice that the set of non-isolated points of K_Q is equal to Q, so it is metric and countable.

Remark 2. *If Z is an element of $\mathscr{L}$ and A its Lindelöf subset then A does not have to belong to $\mathscr{L}$. Indeed, it is enough to observe that K_Q can be embedded in $L(\aleph_1)^{\aleph_0}$.*

Remark 3. *Let $\mathscr{C}$ be a minimal class of spaces satisfying the following conditions:*

(a) *if X is Lindelöf and satisfies the assumptions of theorems then $X \in \mathscr{C}$.*

(b) *if $X_1, X_2, \ldots \in \mathscr{C}$ then $\bigcup_{n=1}^{\infty} X_n$ and $\mathop{P}_{n=1}^{\infty} X_n$ belong to $\mathscr{C}$.*

(c) *$X \in \mathscr{C}$ and F is a closed subset of X then $F \in \mathscr{C}$.*

(d) *$Y \in \mathscr{C}$ then every perfect preimage and continuous image of Y belongs to $\mathscr{C}$.*

One can prove that $\mathscr{L}$ contains $\mathscr{C}$.

Added in proof. Recently I have proved that under the assumption of an existence of an uncountable coanalytic set of reals without uncountable compact subsets (in Gödel's constructible universum such a set exists, K. Gödel and P. S. Novikov) there exists a separable metric space M and a Lindelöf space X such that for every hereditarily Lindelöf space Y and every natural number n the products $Y \times X^n$ and $X^{\aleph_0}$ are Lindelöf but $M \times X^{\aleph_0}$ is not Lindelöf.

References

[A] K. Alster, *Some remarks on Eberlein compacts*, Fund. Math. 104 (1979), pp. 43–46.

[AP] — and R. Pol, *On function spaces of compact subspaces of Σ-products of the real line*, Fund. Math. 107 (1980), pp. 135–143.

[E] R. Engelking, *General Topology*, Warszawa 1977.

[F] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), pp. 747–750.

[M] E. Michael, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Comp. Math. 23 (1971), pp. 199–244.

[N] N. Noble, *Products with closed projection II*, Trans. Amer. Math. Soc. 160 (1971), pp. 169–183.

[S] A. H. Stone, *Kernel constructions and Borel sets*, Trans. Amer. Math. Soc. 107 (1963), pp. 58–70.

[T] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. 73 (1971), pp. 59–74.

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([11]) If $A \subset X$ then X_A stands for the space such that every point of $X \setminus A$ is isolated and the base at $x \in A$ in X_A is the same as in X.

([12]) P stands for the set of irrational numbers which is topologically equal, as it is known, to the product $N^{\aleph_0}$.