A group automorphism is a factor of a direct product of a zero entropy automorphism and a Bernoulli automorphism.

by

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Abstract. Our aim is to show the title, that is, a compact metric abelian group $X$ with an automorphism $\sigma$ splits into a sum $X = X_1 + X_2$ of $\sigma$-invariant subgroups $X_1$ and $X_2$ such that $(X_1, \sigma)$ has zero entropy and $(X_2, \sigma)$ is Bernoullian.

§ 1. Introduction. The metrical and topological structures of affine transformations studied by many authors. It follows from Parry [12, 13] and Dani [6] that the metrical structure of an affine transformation of a compact abelian group has a relation to that of its automorphism part. For example, if an affine transformation of a compact abelian group is minimal, then its automorphism part has zero entropy. Further, an affine transformation is a Kolmogorov automorphism if and only if its automorphism part is so.


Let $X$ be a compact metric abelian group and $\sigma$ be an automorphism of $X$, then it is known (cf. [1]) that there exists a $\sigma$-invariant subgroup $X_2$ such that the factor of $\sigma$ on $X/X_2$ has zero entropy and the restriction of $\sigma$ on $X_2$ is Bernoullian. However, it is unknown yet whether $X$ contains a subgroup $X_1$ such that $(X_1, \sigma)$ has zero entropy and $X$ splits into a sum $X = X_1 + X_2$.

In this paper, we first prove that a compact metric abelian group $X$ splits into a sum $X = X_0 + F$ of the connected component $X_0$ of the identity and a totally disconnected subgroup $F$, invariant with respect to a given automorphism of $X$. Our aim is to show that every automorphism of a compact metric abelian group is an algebraic factor of a direct product of a Bernoulli automorphism and an automorphism with zero entropy. This characterizes a metrical and topological structure (in the ergodic theory) of every automorphism.
A group automorphism is a factor of a direct product

We shall give here a proof for completeness. Let \( \hat{g} = g + G, g \in G/G, \) and \( W \) be the \( \sigma \)-invariant subgroup of \( G/G \) generated by \( \hat{g} \). Under that action of \( \gamma \), we can consider \( W \) to be a \( Z[x, x^{-1}] \)-module \( Z[x, x^{-1}] \) denotes the ring of polynomials in \( x \) and \( x^{-1} \) with integer coefficients). Let \( \hat{Y} \) be the torsion subgroup of \( W \). Then \( \hat{Y} \) is clearly a \( Z[x, x^{-1}] \)-submodule of \( W \). Since \( Z[x, x^{-1}] \) is Noetherian, \( \hat{Y} \) is finitely generated under \( Z[x, x^{-1}] \) say by \( e_1, \ldots, e_n \). If \( d_j e_j \) is zero for non-zero integers \( d_j \) \((1 \leq j \leq n)\), and \( d \cdot \sum \sum d_j e_j \in 0 \) in \( G/G \) and so \( dW = \{0\} \) in \( G/G \). Let both \( dW = (dK_j + G)/G \), and \( G \) are torsion free, so \( dK_j \). We denote by \( G, \sigma \) the dual of \((X, \sigma)\) \( \langle \gamma(x) = g(x), \ g \in G \) and \( x \in X \).

Proof of Proposition 1. Denote by \( H \) \( \sigma \)-maximal torsion subgroup of \( G \). For a character \( \gamma \) \( \sigma \)-invariant, it follows that there is an integer \( d_0 > 0 \) such that \( d_0 K_j \) is torsion free. Since \( X \) is metrizable, \( G \) must be countable. Using Lemma 1 inductively, we see that there exist positive integers \( d_1, d_2, \ldots \) and characters \( \gamma_1, \gamma_2, \ldots \) \( \sigma \)-invariant such that \( G' = \sum \sum d_j K_j \) is torsion free and \( G/G' \) is a torsion group.

Let \( H \) denote the annihilator of \( G' \) in \( X \). Then \( H \) has the character group \( G' \). Thus \( H \) is totally disconnected and \( \sigma \)-invariant. On the other hand, since \( X/X_k \) is totally disconnected, \( \chi(X_k + H) \) must be connected and totally disconnected, that is, an identity group. The proof is completed.

§ 4. Proof of Theorem 2. In proving Theorem 2, we need the following propositions.

Proposition 2. Let \( X \) and \( \sigma \) be as in Theorem 2. If \( X \) is connected, then Theorem 2 is true for \((X, \sigma)\).

Proposition 3. Let \( X \) and \( \sigma \) be as in Proposition 1. If \( X \) is totally disconnected, then Theorem 2 is true for \((X, \sigma)\).

The group X is expressed as \( X = X_k + H \) with the notations of Proposition 1. If we hold Propositions 2 and 3, then the subgroups \( X_k \) and \( H \) split into the sums \( X_k = X_k + H = H_1 + H_2 \) of subgroups such that \( H_1 \) and \( H_2 \) satisfy (i) \( H_1 \) and \( H_2 \) satisfy (ii). Denote \( X_1 = X_1 + H = X_1 \) and \( X_2 = X_1 + H = X_2 \) since \( X = X_k + H \). Since \( X_k, \sigma \) is an algebraic factor of the direct product system \( (X_1 \otimes H_1, \sigma \otimes \sigma), (X_k, \sigma) \) has zero entropy. Also \( (X_k, \sigma) \) is a factor of \((X_1 \otimes H_2, \sigma \otimes \sigma)\), so that \((X_k, \sigma) \) is ergodic (and hence Bernoullian).

It will be remain only to show Propositions 2 and 3.

(i) Proof of Proposition 2. For the proof we prepare some results that suffice for our needs. As before let \( G \) be a countable discrete torsion free abelian group and \( \gamma \) be an automorphism of \( G \). For \( g \in G \) we shall denote by \( p(x) \) a polynomial in \( x \) with minimal degree such that \( p(\gamma g) = 0 \) for some \( 0 \neq p(x) \in Z[x] \). We define subsets

\[ G_x = \{ g \in G : p(x) \} \]

(obviously \( G_x \) is a \( \gamma \)-invariant subgroup) and

\[ G_x = \{ g \in G : p(x) \} \] (has only roots of unity),

\[ G_x = \{ g \in G : p(x) \} \] (has not roots of unity).

\[ G_x = \{ g \in G : p(x) \} \]
**Lemma 2.** \( G_B \) and \( G_C \) are \( \gamma \)-invariant subgroups, and \( G_B \cap G_C = \{ 0 \} \).

**Proof.** It is obvious that \( G_B \) and \( G_C \) are \( \gamma \)-invariant. If \( f \) and \( g \) are in \( G_B \), then \( p_f(\gamma f) = p_g(\gamma g) = 0 \). Since \( p_f(p_g(\gamma f)) = 0 \), a polynomial \( p_f(\gamma f) \) divides \( p_f(\gamma g) \) over \( Q \) (denoting the rational field). Thus \( p_f(\gamma f) \) has only roots of unity, whence \( f + g \in G_B \). It is easy to see that \( G_C \) is a subgroup and \( G_B \cap G_C = \{ 0 \} \).

**Lemma 3.** \( G_B(G_B \cap G_C) \) is a torsion group.

**Proof.** If \( \text{rank}(G_B) < r < \infty \), then \( G_B \) is imbedded in \( Q^r \) where \( Q^r \) is the \( r \)-dimensional vector space over \( Q \). We can consider \( G_B \) to be a subset of \( Q^r \). Then there is an extension of \( \gamma \) on \( Q^r \) (we denote it by the same symbol \( \gamma \)). The space \( Q^r \) has a direct sum splitting \( Q^r = Q_B \oplus Q_C \) of \( \gamma \)-invariant subspaces where all the eigenvalues of \( \gamma_B \) are roots of unity and those of \( \gamma_C \) are not roots of unity. Obviously, \( G_B \subset Q_B \) and \( G_C \subset Q_C \). Let \( f \in G_B \), then \( f = f_B + f_C \) with some \( f_B \in Q_B \) and some \( f_C \in Q_C \). Since \( Q^r \) is a divisible extension of \( G_B \), the \( \gamma_B \) and \( \gamma_C \) are for some \( n > 0 \), hence \( f_B = \gamma_B^n f = \gamma_C^n f \in G_B \oplus G_C \). This shows that \( G_B(G_B \cap G_C) \) is a torsion group.

If \( \text{rank}(G_B) = \infty \), it is easy to see that \( G_B \) contains a sequence
\[
G_B^{(1)} \supseteq G_B^{(2)} \supseteq \ldots \supseteq G_B^{(n)} = G_B
\]
of \( \gamma \)-invariant subgroups such that \( \text{rank}(G_B^{(n)}) < \infty \) for all \( n > 1 \). Hence there exist subgroups \( G_B^{(n)} \) and \( G_B^{(n)} \) such that \( G_B^{(n)} = G_B^{(n-1)} \cap G_B^{(n)} \), and \( G_B^{(n)}(G_B^{(n)} \cap G_B^{(n)}) \) is a torsion group. We obtain at once the conclusion.

**Lemma 4.** If \( \gamma \notin G_B \), then the subgroup \( K_B = \bigoplus_{n=1}^\infty \gamma^n G_B \) splits into a (restricted) direct sum \( K_B = \bigoplus K_B^n \).

**Proof.** This follows from the fact that \( p(\gamma) \gamma \neq 0 \) for all \( 0 \neq p(\gamma) \in Z[x] \).

The set \( \Gamma = \{ \gamma \in G \neq \gamma \in G \} \) is countable and \( G_B + \Gamma = G \) holds. For \( \gamma_B \in \Gamma \), we denote by \( \gamma_B \) a character \( f \in \Gamma \) such that \( K_B \cap K_B \neq \{ 0 \} \), and by \( \gamma_B \) a character \( f \in \Gamma \) such that \( K_B \cap K_B \neq \{ 0 \} \). Repeating this step, we get a sequence \( \{ K_B^n \} \) of subgroups and obviously
\[
G_B = \bigoplus_{n=1}^\infty K_B^n.
\]

Then \( G_B \) is \( \gamma \)-invariant and \( G_B \cap G_B = \{ 0 \} \). We claim that \( G_B \cap G_B = \{ 0 \} \) is not necessarily torsion free.

**Lemma 5.** For any \( \gamma \in G_B \), there exists \( 0 \neq p(\gamma) \in Z[x] \) such that \( p(\gamma) = 0 \) (the identity of \( G_B \)).

**Proof.** This follows from the definition of \( G_B \).

**Lemma 6.** Let \( X \) be a compact connected metric abelian group, \( \gamma \) be an automorphism of \( X \) and \( (G, \gamma) \) be the dual of \( (X, \sigma) \). If \( G = G_B \) (that is, for any \( f \in G \), \( p_f(\gamma f) = 0 \) and \( p_f(x) \) has only roots of unity), then \( X, \sigma \) has zero entropy.

**Proposition 2.** We notice that \( G \) is torsion free. Let \( G_B \) denote a subgroup generated by \( \{ \gamma \gamma : -1 < \gamma < 1 \} \) for any \( \gamma \in G \). Since \( p_f(\gamma) = 0 \), the rank of \( G_B \) is finite. We have that \( p_f(\gamma) \) is monic and its constant term is \( \pm 1 \) since it has only roots of unity and \( p_f(\gamma) \in Z[x] \). Therefore it follows that \( G_B \) is finitely generated. Since \( G \) is countable, \( G \) contains a sequence \( G_B \subset G_B \subset \ldots \) of \( \gamma \)-invariant subgroups such that \( G \subset G_B \setminus G \) and for any \( n > 1 \), \( G \subset G_B \setminus G \) is finitely generated. If \( X, \sigma \) is the annihilator of \( G_B \), then \( X, \sigma \) is \( \sigma \)-invariant and \( X, \sigma \) is a finite-dimensional torus. Since every polynomial \( p_f(x) \) has only roots of unity, \( (X, \sigma) \) has zero entropy (cf. p. 168 of [17]). Hence \( (X, \sigma) \) has also zero entropy since \( X, \sigma \).

**Theorem A (§ 6 of [16]).** Let \( \sigma \) be an automorphism of a compact metric abelian group \( X \). If \( N \) is a \( \sigma \)-invariant subgroup of \( X \), then \( h(\sigma) = h(\sigma_N) + h(\sigma) \) (the notation \( h(\sigma) \) means the Kolmogorov entropy).

From now on we shall prove Proposition 2. For the proof we must use Proposition 3 that is proved later on. As before let \((G, \gamma)\) be the dual of \((X, \sigma)\). Since \( X \) is connected and metrizable, \( G \) is a countable discrete torsion free abelian group. We have

\[
G = G_B \oplus G_B \oplus G_B \oplus G_B \oplus G_B
\]

Denote by \( \text{ann}(X, G) \) the annihilator of a subgroup \( G \) in \( X \), and put
\[
Y_1 = \text{ann}(X, G_B \oplus G_B) \quad \text{and} \quad Y_2 = \text{ann}(X, G_B \oplus G_B)
\]

Then we get
\[
Y_1 + Y_2 = \text{ann}(X, G_B \oplus G_B) = \text{ann}(X, (0)) = X.
\]

It follows that \((X, \sigma)\) is ergodic and \((X, \sigma)\) has zero entropy. For, the character group of \( X, \sigma \) is \( G_B \) and \( G_B \) has no finite orbits except the identity. If \( \gamma \neq \gamma \in G_B \), then \( \gamma(\gamma f) = \gamma(\gamma f) \), for some \( k \neq 0 \), then \( (\gamma - 1) f \in G_B \) and so \( p_{\gamma - 1}(\gamma - 1) f = 0 \). Since \( p_{\gamma - 1}(\gamma - 1) f \) has only roots of unity, so is \( p_{\gamma - 1}(\gamma - 1) f \). Hence \( G_B \) and \( G_B \) is the identity of \( G_B \). Therefore \((X, \sigma)\) is ergodic. The character group of \( X, \sigma \) is \( G_B \). Hence by Lemma 6, \((X, \sigma)\) has zero entropy. It is easy to see that \( X, \sigma \) is the maximal subgroup satisfying the above conditions (by using Theorem A).

Using Proposition 1, we see that there are \( \sigma \)-invariant subgroups \( Z \) and \( H \) such that \( Z \) is connected, \( H \) is totally disconnected and
\[
Y_1 = Z + H.
\]

If we hold Proposition 3, then \( H \) contains \( \sigma \)-invariant subgroups \( H_1 \) and \( H_2 \) such that \((H_1, \sigma)\) has zero entropy, \((H_2, \sigma)\) is ergodic and
\[
H = H_1 + H_2.
\]

To get the splitting as in Theorem 2, let \((G^{(1)}, \gamma)\) be the dual of \((Z, \sigma)\), then \((G^{(1)})\) is the factor group of \((G_B \oplus G_B)\), which is the character group of \( Y_1 \). Hence by
Lemma 5, for any $x \in G_i^0_\gamma$ there is $x \ne p(x) \in \mathbb{A}_x$ such that $p(x) = 0$. As before, define $\gamma$-invariant subgroups
\[ G_i^0 = \{ x \in G_i : p(x) \text{ has only roots of unity} \}, \]
\[ G_0^0 = \{ x \in G_i : p(x) \} \text{ has not roots of unity} \} \].
Then $G_0^0 \cap (G_0^0 \oplus G_0^0)$ is a torsion group (Lemma 3). Since $G_0^0$ is torsion free, there exists the minimal divisible extension $G_0^0$ of $G_0^0$ (Lemma 7), so $G_0^0$ (or $G_0^0$ (e)) (see § 2 of [11] and [1]).
It is easy to see that $G_0^0 = G_0^0 \oplus G_0^0$, where $G_0^0 = \{ x \in G_i : mG_i \text{ for some } m \ne 0 \}$ and $G_0^0$ (a) is defined in the same way. Obviously, for $x \in G_0^0$ a polynomial $p(x)$ has only roots of unity and for $x \in G_0^0$ a polynomial $p(x)$ has not roots of unity.

Let $(Z, \sigma)$ be the dual of $(G_i^0, \gamma)$, then $Z$ splits into a direct sum
\[ Z = Z_i \oplus Z_0 \]
of $\sigma$-invariant subgroups where $Z_0 = \text{ann}(Z, G_0^0)$ and $Z_0 = \text{ann}(Z, G_0^0)$ (Lemma 7). Since $Z_0$ has the character group $G_0^0$, from Lemma 6 it follows that $(Z_0, \sigma)$ has zero entropy.
On the other hand, the character group $G_0^0$ of $Z_0$ has no periodic points under $\gamma$ except the identity, so that $(Z_0, \sigma)$ is ergodic.

Let $K = \text{ann}(Z, G_0^0)$, then $K$ is $\sigma$-invariant and $Z/K$ is isomorphic to $Z_0$. Since $Z/K$ and $Z_0$ have the same character group $G_0^0$ on which actions of $\gamma$ coincide, $(Z, \sigma)$ is isomorphic to $(Z/K, \sigma)$. Therefore $Z$ is expressed as
\[ Z = Z_1 \oplus Z_2 \]
of $\sigma$-invariant subgroups where $Z_1$ and $Z_2$ are factors of $Z_0$ and $Z_0$, respectively.
From the maximality of $X_1$, $Z_1 \not\subset Z_0$, let us put $X_1 = Z_1 \cup X_0$, then $(X_1, \sigma)$ has zero entropy, and $X = X_1 \cup X_0$.
In order to conclude the proof of Proposition 2 and get Theorem 2, we have to show Proposition 3.

(II) (Proposition 3). For the proof we need the following results.

Theorem B (11.1 of [16]). Let $\sigma$ be an automorphism of a compact totally disconnected metric abelian group $X$. Then there exists in $X$ a $\sigma$-invariant subgroup $K$ such that $(X/K, \sigma)$ has zero entropy and $(K, \sigma)$ is Bernoulli (with respect to the normalized Haar measure).

Let $X$ be a compact metric abelian group and $X$ split into a direct sum $X = \bigoplus_{i=0}^{m} H_i$ where $H_i = H_i, i = 0, \pm 1, \ldots$ The automorphism $\sigma$ of $X$ defined by $\sigma[H_i] = [H_{i+1}]$ will be called the Bernoulli group automorphism. A Bernoulli group automorphism having a group of states which is different from the identity and having no proper non-trivial subgroups will be called a simple Bernoulli group automorphism.

Theorem C (11.5 of [16]). Let $X$ be a compact totally disconnected metric abelian group and $\sigma$ be an automorphism of $X$. Let $H$ be an open subgroup of $X$ such that $\bigcap \sigma^i H = \{0\}$. If $X/H$ is simple, then $X$ is finite or $\sigma$ is a simple Bernoulli group automorphism of $X$.

The following is a reform of Theorem 11.7 in [16]. In compact totally disconnected metric abelian groups, the statement of the theorem is not certain (see [2]).

Lemma 7. Let $X$ be a compact totally disconnected metric abelian group. If $\sigma$ is an ergodic automorphism of $X$, then $X$ contains a sequence
\[ X = F_0 \supset F_1 \supset \ldots \]
of $\sigma$-invariant subgroups such that $\bigcap F_n = \{0\}$ and for any $n > 0$, there is a sequence
\[ F_n = F_n \supset F_{n+1} \supset \ldots \]
of $\sigma$-invariant subgroups such that for any $i \geq 1$, $\sigma|_{F_n \cap F_{n+i}}$ is a simple Bernoulli group automorphism.

Proof. Since $X$ is totally disconnected, $X$ contains a sequence $X = A_0 \supset A_1 \supset \ldots$ of open subgroups such that $\bigcap A_n = \{0\}$. Writing $H_n = \bigcap A_n$ for any $n > 0$, by Theorem B there is a subgroup $H_n$ of $H_n$ such that $H_n/H_n, \sigma$ has zero entropy and $H_n, \sigma$ has completely positive entropy. The sequence $\{H_n\}$ decreases and $\bigcap H_n = \{0\}$.
Without loss of generality we may assume that the sequence contains no repetitions.

We fix the integer $n$ and carry out a recursive construction of the sequence
\[ H_n = F_n \supset F_{n+1} \supset \ldots \]
of $\sigma$-invariant subgroups, imposing the following conditions: for any $i \geq 0$,
\[ F_n \supset F_{n+i} \supset \ldots \]
such that for any $j \geq 1$, $\sigma|_{F_n \cap F_{n+j}}$ is a simple Bernoulli group automorphism.

Assume that the subgroups $F_n \supset F_{n+1} \supset \ldots$ satisfy all the conditions that we have imposed and $F_n H_n \supset H_{n+1}$. Then $F_n$ contains an open proper subgroup $B$ such that $B \supset H_{n+1}$ and $F_n/B$ is simple. Put $H = \bigcap B$, then $\sigma|_H$ is a simple Bernoulli group automorphism (Theorem C), and $h(\sigma|_H) = \log p$ where $p$ is some prime. By Theorem B there is a subgroup $F_{n+i+1}$ of $H$ such that $H/F_{n+i+1}, \sigma$ has zero entropy and $F_{n+i+1}$, $\sigma$ has completely positive entropy.

Let $(G, \gamma)$ be the dual of $(F_n \cap F_{n+i+1}, \sigma)$. Since $h(\sigma|_{F_n \cap F_{n+i+1}}) = \log p$ (by Theorem A), $G$ is a $p$-group. For, if $G$ is not a $p$-group then $G$ splits into a direct sum $G = \bigoplus_{i=0}^{m} G_i$ of $\gamma$-invariant prime groups $G_i$ (p. 137 of [11]).

Hence $F_n \supset F_{n+i+1}$ splits into a direct sum $F_n \supset F_{n+i+1} = \bigoplus_{i=0}^{m} F_i$ where $F_i$ is a $\gamma$-invariant subgroup with character group $G_i$. Since each $F_i$ is ergodic (Bernoullian), if $q$ is a prime and $G_i$ is a $q$-group then $h(\sigma|_{F_i}) = \log q$ (by Theorem B). On the other hand, let $G_n$ be the annihilator of $H_n/F_{n+i+1}$, then $G_n$ is the character group of $F_n/H_n$. Since $F_n/H_n = \bigoplus_{i=0}^{m} F_i$ where $F_i$ is a $p$-cyclic group, $G_n$ is annihilated by multiplication by $p$ and hence $G_n = G_i$ for some $b$. This is inconsistent with $h(\sigma|_{F_n \cap F_{n+i+1}}) = \log p$.

It is easy to see that $G$ is annihilated by multiplication by the prime $p$. Indeed,
A group automorphism is a factor of a direct product.

Lemma 9. Let $X$ and $\sigma$ be as in Lemma 8. If $(X, \sigma)$ is expansive and $h(\sigma) = 0$, then $X$ is finite.

Proof. Assume that $X$ is infinite (whence it is not discrete). Let $U$ be an expansive neighborhood of the identity for $(X, \sigma)$. Then $U$ contains a non-trivial open subgroup $N$. Since $h(\sigma) = 0$, it is easy to see that $\sigma^N \cap N' \neq e$ holds for all $k > 0$. Indeed, if $\sigma^N \subseteq N$ for all $k > 0$, then it follows that $|X|^{1/k} \geq 2k$ for all $k > 0$. Let $g(N)$ be the partition of $X$ consisting of the cosets of $N$, then $g(N)$ is a finite measurable partition and $\sigma^N \subseteq N'$ holds, by definition we have $h(\sigma) = \lim (1/(k+1)) \log 2k = \log 2$. But, since $h(\sigma) = 0$, we get the required result.

Hence $N = \bigcap k \sigma^N$ for some $k$. If $N' = \bigcap 1 \sigma^N$ then $\sigma^N = N'$. But, $(0) \neq N' \subseteq N \subseteq U$, which is a contradiction.

Lemma 10. Let $X$ and $\sigma$ be as in Lemma 8. If $\sigma$ is a simple Bernoulli group automorphism of $X$, then every $\sigma$-invariant proper subgroup is finite.

Proof. Let $K$ be a $\sigma$-invariant proper subgroup. Since $h(\sigma) = \log(p)$ for some prime $p$, $h(\sigma^{i|k|}) = \log(p)$ by Lemma 8. Hence $h(\sigma) = 0$ (by Theorem A). Since $(X, \sigma)$ is expansive, $K$ is finite by Lemma 9.

Lemma 11. Let $X$ and $\sigma$ be as in Lemma 8. Assume that $Y_2$ is a $\sigma$-invariant subgroup such that $\sigma_0$ is a simple Bernoulli group automorphism and $(X, \sigma_0, h)$ has zero entropy. If $Y_2$ is open in $X$, then $X$ contains a $\sigma$-invariant subgroup $Y_1$ such that $(Y_1, h)$ has zero entropy and $X = Y_1 + Y_2$.

Proof. Since $\sigma_0$ is a simple Bernoulli automorphism, $Y_2 = \bigoplus_{x \in Y_2} \sigma^x W$ where $W$ is a $p$-cyclic group ($p$ is some prime). Hence $x = 0$ for any $x \in Y_2$. Since $Y_2$ is open, $X$ contains a finite subgroup $F$ such that $X = Y_2 + F$. Since $F$ is finite, there is the smallest non-negative integer $n$ such that $\sigma^n x \neq x$ and $X = \bigoplus_{x \in X} \sigma^x W + F$. Thus $X/K$ is simple (indeed, by the choice of $n$ we have $X/K = (\sigma^n W + K)/K \cong (\sigma^n W)$).

Let $Y_1 = \sigma^n K$, then $X/K$ is finite. For, if $X(Y_1)$ is finite, then $(Y_1 + Y_2)/Y_1 \cong Y_2/(Y_1 \cap Y_2)$ is also finite. Since $Y_1 \cap Y_2 \neq Y_2$, $Y_1 \cap Y_2$ is finite by Lemma 10, whence $Y_2$ is finite. This is impossible.

Since $X/K$ is simple and $X(Y_1)$ is finite, $\sigma_0(K)$ is a simple Bernoulli group automorphism (by Theorem C). Hence the entropy of $(X(Y_1), \sigma_0, h)$ is zero and positive if $X(Y_1 + Y_2)$ is non-trivial. Therefore, we have $Y_1 + Y_2$ where $h(\sigma_0(K)) = 0$. 

Let $G_a$ be the subgroup of $G$ being annihilated by multiplication by $p^n$ if $m = 1, 2$.

Then $a[G_a \subset G]$ and $\gamma_G[0, \infty)$ has no finite orbits except the identity. For $m = 1, 2$ if $[G_a]$ is the annihilator of $G_a$ in $F_{a+}\mathbb{Z}$, then $F_{a+}G_a = \mathbb{Z}$ for $a$ is the annihilator of $G_a$ in $F_{a+}\mathbb{Z}$ and $F_{a+}G_a = \mathbb{Z}$ for $m = 1, 2$.

Therefore we can consider $G$ to be a $\mathbb{Z}[p]\mathbb{Z}[x, x^{-1}]$-module $(Z[p]x, x^{-1})$ denotes the ring of polynomials in $x$ and $x^{-1}$ with coefficients in the field $Z(p)$. We claim that $\gamma$ has no finite orbits except the identity. Since $Z[p]x, x^{-1}$ is a principal ideal domain, there is a sequence $G_1 \subset G_2 \subset \ldots$ of free $Z[p]x, x^{-1}$-modules such that any $G_1 \subset G_2 \subset \mathbb{Z}$ and any $x \in Z$. 

for some $a \in G_1$. If $F_{a+}G_a = \mathbb{Z}$, then $F_{a+}G_a = \mathbb{Z}$ and $F_{a+}G_a = \mathbb{Z}$ for $m = 1, 2$.

It remains to show that there is an integer $i$ such that $F_{a+}G_a = \mathbb{Z}$. Put $a = X[H_{a+}]$ and $A = H_{a+}$, then $A$ is open in $X$ and $\sigma^A = \sigma$.

Therefore our requirement was obtained.

For the subgroups $F_{a+}$, $(i = 0, 1, 2, \ldots, \sigma^A = \mathbb{Z})$, write $F_{a+} = F_{a+} + i$ for $a = 0, 1, 2, \ldots$. Obviously, the sequence $F_{a+}$ satisfies all the required conditions.

Lemma 8. Let $X$ be a compact totally disconnected metric abelian group and $\sigma$ be an automorphism of $X$. If $h(\sigma) > 0$, then $h(\sigma) = \log(n)$ where $n$ is some integer.

Proof. Let $\{A_n\}$ be a decreasing sequence of open subgroups such that $\bigcap A_n = \{0\}$ and $\bigcap A_n$ is the finite group partition of $X$ consisting of the cosets of $A_n$.

Then, $h(\sigma, A_n) = \lim n \log(\sigma) = \log(\bigcap A_n)$ (the notation $|E|$ means the cardinality of a set $E$).

Since $\bigcap A_n$ is the partition of $X$ into single points, we get $h(\sigma) = \log(\bigcap A_n)$. But since $\sigma^A = \sigma$, there is $m > 0$ such that $h(\sigma) = h(\sigma, A_m)$ and $h(\sigma) = \log(\bigcap A_m)$.

Let $\sigma$ be an automorphism of a compact metric abelian group $X$. We shall call $\sigma$ expansive if there exists an open neighborhood $U$ of the identity in $X$ such that $\sigma^n U \neq U$ for some integer $n$. Some obvious examples of expansive automorphisms are the Bernoulli group automorphism and an automorphism of a finite-dimensional torus such that all the eigenvalues of the matrix corresponding to the automorphism are off the unit circle.
Lemma 12. Let $\sigma$ be an automorphism of a compact metric abelian group $X$ and $H$ be a $\sigma$-invariant subgroup ($\sigma H = H$) of $X$. If $(X, \sigma)$ and $(X/H, \sigma)$ are expansive, then $(X, \sigma)$ is expansive.

Proof. Let $U$ and $U'$ be expansive neighborhoods for $(X/H, \sigma)$ and $(H, \sigma)$, respectively. Obviously, $U = \{x+H: x \in V\}$ and $U' = H \cap V$ where $U$ and $V$ are suitable neighborhoods of $X$. Letting $W = U \cap V$, we have $\cup_{n=\infty} \sigma^n W = \cup_{n=\infty} \sigma^n (W \cap H) = (0)$. Therefore, $W$ is an expansive neighborhood for $(X, \sigma)$.

Lemma 13. Let $X$ be a compact totally disconnected metric abelian group. If $\sigma$ is a Bernoulli group automorphism of $X$ and if $|\sigma| < \infty$, then for any $\sigma$-invariant subgroup $H$, $(X/H, \sigma)$ is expansive.

Proof. As before let $(G, \gamma)$ be the dual of $(X/H, \sigma)$. By assumption, $G$ is expressed as $G = \bigoplus_{\infty} G_j$ where $G_j$ is a finite group. Since $G$ is a torsion group, $G_1$ splits into a finite direct sum $G_1 = \bigoplus_{j=1}^k G_{1,j}$ of prime groups $G_{1,j}$. Hence,

$$G = \bigoplus_{\infty} G_j = \bigoplus_{j=1}^k G_{1,j} = \bigoplus_{j=1}^k G^{(j)}$$

where $G^{(j)} = \sum_{i=0}^\infty G_{1,i}$ for $1 \leq j \leq k$. Then $X/H$ splits into a direct sum

$$X/H = \bigoplus_{j=1}^k X_j$$

where each $X_j$ is a subgroup with character group $G^{(j)}$.

Let $G^{(j)}$ be a $p$-group and $G_j$ be a subgroup of $G^{(j)}$ being annihilated by multiplication by $p^m$ for $m \geq 1$. Then there is an integer $a > 0$ such that

$$G_1 = G_2 = \ldots = G_a = G^{(j)}$$

since $G_{1,j}$ is a finite group. It is easy to see that for $1 \leq m < a$, $G^{(j)}_{m+1}/G^{(j)}_m$ has no finite orbits except the identity. Since $G^{(j)} = \sum_{i=0}^\infty G_{1,i}$ and $G_{1,i}$ is finite, each $G^{(j)}_{m+1}/G^{(j)}_m$ is finitely generated under $Z(pZ[y, x^{-1}])$, and so $G^{(j)}_{m+1}/G^{(j)}_m$ is expressed as $G^{(j)}_{m+1}/G^{(j)}_m = Z(pZ[y, x^{-1}]), G^{(j)}_{m+1}/G^{(j)}_m$, where $G^{(j)}_{m+1}/G^{(j)}_m$ is expressed as

$$G^{(j)}_{m+1}/G^{(j)}_m = Z[pZ[y, x^{-1}]],$$

for some $\delta_1, \ldots, \delta_a \in G^{(j)}_{m+1}/G^{(j)}_m$.

Let $X_{d,0}$ be the annihilator of $G_{d,0}$ in $X_j$. Obviously, $X_{d,0} + X_d = \ldots = X_{d,0} = (0)$. For $m \geq 1$, $X_{d,m}$ is $\sigma$-invariant and $\sigma|_{X_{d,m}} = \sigma|_{X_{d,m+1}}$ is a simple Bernoulli automorphism.

From this together with Lemma 12, we get that $(X_j, \sigma)$ is expansive. Therefore $(X/H, \sigma)$ is expansive and the proof is completed.

Lemma 14. Under the notations and the assumption of Lemma 11, if $Y_2$ is not open in $X$, then $X$ contains a $\sigma$-invariant subgroup $F$ such that $(F, \sigma)$ has zero entropy and $Y_2 + F$ is open in $X$.

Proof. Since $\sigma|_{Y_2}$ is a simple Bernoulli automorphism, obviously $h(\sigma|_{Y_2}) \geq \log 2$.

If $X$ contains an open subgroup $L$ such that $F_L = \bigcup_{n=\infty} \sigma^n L$ then $(Y_2 + F_L) \cap F_L$ is non-trivial, for, assume that $F_L \cap F_L = \{0\}$, which is a contradiction. Obviously $h(\sigma|_{X(L+F_L)}) = 0$ since $h(\sigma|_{X(L+F_L)}) = 0$.

By Theorem A,

$$h(\sigma|_{F_L}) = h(\sigma) = h(\sigma|_{X(L+F_L)}) + h(\sigma|_{X(L+F_L+F_L)}) + h(\sigma|_{F_L}).$$

Hence, $h(\sigma|_{F_L}) = 0$. We write $\tilde{X} = \bigoplus X_i$, where $X_i = Y_i$ for $i = 0, \pm 1, \ldots$. Let $P$ be the canonical projection from $X_i F_L$ onto $X/L$ and $\tilde{p}$ be a Bernoulli group automorphism of $X$. Define the map $\psi: X/L \rightarrow \tilde{X}$ by

$$\psi(\tilde{x}) = \{P(x)\}_{n=\infty}^{n=\infty}, \quad \tilde{x} \in X/L.$$

Then $\psi$ is an isomorphism from $X/L$ on some subgroup of $X$ and $\psi^{-1} \tilde{p} \psi = \sigma$ on $X_i F_L$ holds. $\tilde{p}$ is the Bernoulli group automorphism of $X$ and $h(\tilde{p}) < \infty$, $\psi(\{Y_2 + F_L\})$ is expansive by Lemma 13 and $\psi(\{X_i F_L\}) \psi(\{Y_2 + F_L\})$ is also expansive, whence so is $(\{X_i Y_2 + F_L\})$. Since $h(\sigma|_{X(L+F_L)}) = 0$, $X(L + F_L)$ is finite by Lemma 9 and $Y_2 + F_L$ is open in $X$.

Proposition 3 now is proved as follows. Let $X_2$ be a subgroup of the group $X$ such that $(X/X_2, \sigma)$ has zero entropy and $(X_2, \sigma)$ has completely positive entropy (see Theorem B). Using Lemma 7, we see that $X_2$ contains a sequence

$$X_2 \leftarrow X_{2,1} \leftarrow X_{2,2} \leftarrow \ldots$$

of $\sigma$-invariant subgroups such that $\bigcap X_2 = \{0\}$ and for any $n \geq 0$ there is a sequence $X_2 \leftarrow X_{2,n+1} \leftarrow \ldots$ of $\sigma$-invariant subgroups such that $\bigcap_{j=1}^\infty X_{2,2} = \{0\}$ and for any $j > 0$, $\sigma|_{X_{2,2} \cup X_{2,n+1}}$ is a simple Bernoulli group automorphism.

A simple Bernoulli group automorphism $(X/X_{2,0}, \psi)$ and $\sigma|_{X_{2,0}}$ have zero entropy, by Lemmas 11 and 14 there is a $\sigma$-invariant subgroup $M_4$ of $X$ such that

$$X = M_4 + X_2$$

and $h(\sigma|_{X_{2,0} + X_2}) = 0$.

Since $X_{2,0} \times X_{2,1}$, by Theorem A we have that for $k \geq 1$,

$$h(\sigma|_{X_{2,0} + X_{2,1}}) = h(\sigma|_{X_{2,0} + X_{2,1}}) + \sum_{n=1}^k h(\sigma|_{X_{2,0} + X_{2,1}}).$$
Notice that $\sigma_{1,3,4,5,6,7,8}$ and $\sigma_{1,3,4,5,6,7,8,9}$ are simple Bernoulli group automorphisms. Then by Lemma 8, we get $\sum \sigma_{i,j,k} h(\sigma_{1,3,4,5,6,7,8}) = 0$ and so $h(\sigma_{1,3,4,5,6,7,8,9}) = 0$. Since $k$ is arbitrary, $h(\sigma_{1,3,4,5,6,7,8}) = 0$ and hence $h(\sigma_{i,j,k}) = 0$ (by Theorem A).

Since $\sigma_{1,3,4,5,6,7,8}$ is a simple Bernoulli group automorphism and

$$\left(\mathbb{M}_f(X_{1,2,3,4,5,6,7,8}, \sigma) \right)$$

has zero entropy, by the same argument we can find a $\sigma$-invariant subgroup $\mathbb{M}_2$ of $\mathbb{M}_1$ such that

$$\mathbb{M}_1 = \mathbb{M}_2 + X_{1,2} \quad \text{and} \quad h(\sigma_{1,3,4,5,6,7,8}) = 0.$$ 

Since $X_{1,2} \subset X$, we get $\mathbb{M}_1 + X_3 = \mathbb{M}_2 + X_3 = X$. It is easy to see that

$$h(\sigma_{1,3,4,5,6,7,8}) = 0 \quad \text{and} \quad h(\sigma_{1,3,4,5,6,7,8}) = 0.$$ 

Hence $\left(\mathbb{M}_f(X_{1,2,3,4,5,6,7,8}), \sigma \right)$ has zero entropy and $\sigma_{1,3,4,5,6,7,8}$ is a simple Bernoulli group automorphism, so that $\mathbb{M}_2$ contains a $\sigma$-invariant subgroup $\mathbb{M}_3$ such that

$$\mathbb{M}_2 = \mathbb{M}_3 + X_{1,2} \quad \text{and} \quad h(\sigma_{1,3,4,5,6,7,8}) = 0.$$ 

Obviously, $X = \mathbb{M}_3 + X_3$. Since $h(\sigma_{1,3,4,5,6,7,8}) = 0$, we get $h(\sigma_{1,3,4,5,6,7,8}) = 0$.

Continuing inductively this process, we have a sequence $\mathbb{M}_1 \supset \mathbb{M}_2 \supset \ldots$ of $\sigma$-invariant subgroups such that for any $n > 1$,

$$X = \mathbb{M}_n + X_3 \quad \text{and} \quad h(\sigma_{1,3,4,5,6,7,8}) = 0.$$ 

Put $X_1 = \cap \mathbb{M}_1$, then $X = X_1 + X_3$, and since $X_1/(X_1 \cap X_3) \cong (X_1 + X_3)/X_3$ is a subgroup of $\mathbb{M}_f(X_{1,2}), h(\sigma_{1,3,4,5,6,7,8}) = 0$ by Theorem A. The proof of Proposition 3 is completed.

Added in proof. An extension of the result of [9] (or [1]) to compact metric groups had already been developed by G. Miles and K. Thomas, Advances in Math. Supplementary Studies 2 (1978), pp. 207-249.

References


A group automorphism is a factor of a direct product