Collapsing algebras and Suslin trees

by

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Abstract. In this paper we construct an Easton-like notion of forcing T which preserves cofinalities but such that $T\times T$ collapses all uncountable cardinals. T is the product of the Devlin–Johnsbräten homogeneous $\aleph_1$-Suslin tree $T_{\aleph_1}$ together with homogeneous $\varkappa^{++}$ Suslin trees $T_{\varkappa^{++}}$ ($\varkappa$ an infinite cardinal) which are neatly $\varkappa^+$-closed and such that $T_{\varkappa^{++}}\times T_{\varkappa^{++}}$ collapses $\varkappa^{++}$.

Introduction. This paper provides a positive answer to the following question: Does there exist an Easton-like notion of forcing P such that $P\times P$ is not an Easton-like notion of forcing? Of course, if P is Easton-like, then P preserves a proper class of cardinals.

The goal of the present paper is to construct an extreme example P of an Easton-like notion of forcing such that $P\times P$ is not Easton-like.

Assuming that inaccessible cardinals do not exist, GCH and $\diamondsuit_{\mu^{++}}$ (μ an infinite cardinal) holds, we prove that there exists a P which does not collapse any cardinal but $P\times P$ collapses all cardinals onto $\aleph_0$. The construction is inspired by a result of Devlin and Johnsbråten [1].

They have constructed an $\aleph_1$-Suslin tree $T_{\aleph_1}$ such that $T_{\aleph_1}\times T_{\aleph_1}$ collapses $\aleph_1$ onto $\aleph_0$. We generalize their method to successors of regular cardinals. Namely, we prove that if $\diamondsuit_{\mu^+}$ holds, μ is regular and $\mu^{\underline{\mu}}=\mu$, then there exists a μ^+-Suslin tree T_{μ^+} such that $T_{\mu^+}\times T_{\mu^+}$ collapses μ^+ onto μ, T_{μ^+} is homogeneous and neatly μ-closed. Now $T=T_{\aleph_1}\times\prod_{\alpha\in On}T_{\aleph_{\alpha+2}}$ is the required notion of forcing. To show that $T\times T$ collapses all cardinals onto $\aleph_0$ we prove a generalization of McAloon's theorem on collapsing algebras (see [3]), from which it follows that if $(\mu^+)^{\underline{\mu}}=\mu^+$ then $C(\mu,\mu^+)$ is isomorphic to a dense subset of $T_{\mu^+}\times T_{\mu^+}$($C(\mu,\mu^+)$ is the usual notion of forcing collapsing μ^+ onto μ). We then prove that if GCH holds and inaccessible cardinals do not exist then

$$\prod_{\gamma\leqslant\xi<\eta}C(\aleph_{\xi+1},\aleph_{\xi+2})\cong_d C(\aleph_{\gamma+1},\aleph_{\eta+1})\,.$$

By $P_1\cong_d P_2$, where P_1 and P_2 are notions of forcing, we mean that P_1 is isomorphic to P_2 up to density. Consequently

$$T\times T\cong_d\prod_{\alpha\in On}C(\aleph_{\alpha+1},\aleph_{\alpha+2})\cong_d\prod_{\alpha\in On}C(\aleph_0,\aleph_{\alpha+1})\,.$$

So $\Vdash_{T\times T}$ "V = HC" (HC is the class of hereditarily countable sets). By a result of Zarach [4], $\Vdash_{T\times T} A$ for each axiom A of ZFC$^-$ since, for each $\varkappa$, $T_\varkappa \times T_\varkappa$ is homogeneous.

We work in ZFC. Notation, definitions and terminology are standard.

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I. $\varkappa$-Suslin trees. Let M be a countable transitive model for ZFC + there are no inaccessible cardinals + $\diamondsuit_{\varkappa^+}(E^\varkappa_{\varkappa^+})$ ($\varkappa$ a regular cardinal of M). M is fixed for the rest of the paper. Here, for $\varkappa$ regular, $\diamondsuit_{\varkappa^+}(E^\varkappa_{\varkappa^+})$ is Jensen's principle:

There is a sequence $\langle S_\alpha\colon \alpha<\varkappa^+\rangle$ such that $S_\alpha\subseteq\alpha$ for $\alpha<\varkappa^+$ and if $X\subseteq\varkappa^+$ then the set $\{\alpha<\varkappa^+\colon X\cap\alpha = S_\alpha \ \&\ \mathrm{cf}(\alpha)=\varkappa\}$ is stationary in $\varkappa^+$.

FACT 1.1. *$\diamondsuit_{\varkappa^+}(E^\varkappa_{\varkappa^+})$ implies $2^\varkappa=\varkappa^+$.*

This is a direct generalization of the fact that $\diamondsuit$ implies CH.

Let $H_{\varkappa^+}$ denote the set of all sets of hereditary power less than or equal to $\varkappa$. It follows from (1.1) that $H_{\varkappa^+}$ has cardinality $\varkappa^+$. Since $H_{\varkappa^+}$ is a transitive set, $H_{\varkappa^+}$ can be coded by some $A\subseteq\varkappa^+$. Hence $L_{\varkappa^+}[A] = H_{\varkappa^+}$ and $(\varkappa^+)^{L[A]}=\varkappa^+$. By induction on $\alpha<\varkappa^+$ we define δ_α to be the least ordinal $\delta>\alpha$ such that:

(i) $L_\delta[A]\prec H_{\varkappa^+}$,

(ii) $S_\alpha, \langle\delta_\nu\colon \nu<\alpha\rangle, \varkappa\in L_\delta[A]$.

Let $M_\alpha = L_{\delta_\alpha}[A] = L_{\delta_\alpha}[A\cap\delta_\alpha]$. Then $|M_\alpha|=\varkappa$.

Now we can formulate a property which implies that a tree is Suslin.

LEMMA 1.2. *Let T be a normal tree of height $\varkappa^+$ such that if $x\in T$ then $x\in H_{\varkappa^+}$. Let $C\subseteq\varkappa^+$ be a closed unbounded set of limit ordinals such that for all $\alpha\in C$ if $T\restriction\alpha\in M_\alpha$, $\mathrm{cf}(\alpha)=\varkappa$ and $x\in T_\alpha$, then $\{y\colon y<_T x\}$ is M_α-generic for $T\restriction\alpha$. Then T is a Suslin tree.*

Proof. A slight modification of the proof given in [1] for the case of $\aleph_1$-trees works. The basic idea is that for $\mathrm{cf}(\alpha)=\varkappa$, S_α is often a maximal antichain of $T\restriction\alpha$. If so, and if $X\supseteq S_\alpha$ is an antichain of T, then $X=S_\alpha$ since points of T_α are tops of M_α-generic branches through $T\restriction\alpha$, and $S_\alpha\in M_\alpha$ is a pre-dense subset of $T\restriction\alpha$ (viewing $T\restriction\alpha$ with the reverse order as a forcing notion).

If $\varkappa^{\underset{\smile}{\varkappa}}=\varkappa$ then M_α's have an additional property.

LEMMA 1.3. *Let $\varkappa^{\underset{\smile}{\varkappa}}=\varkappa$ and let $\{A_\xi\colon \xi<\alpha\}\in M_\alpha$ be a family of sets of functions such that, for each $\eta<\xi<\alpha$, $A_\eta=\{f\restriction\eta\colon f\in A_\xi\}$. Let $A_\alpha \overset{\mathrm{df}}{=} \{f\colon \mathrm{Func}(f)\ \&\ \mathrm{dom}(f)=\alpha\ \&\ (\xi)_\alpha(f\restriction\xi\in A_\xi)\}$. If α is a limit ordinal and $\mathrm{cf}(\alpha)<\varkappa$ then $A_\alpha\in M_\alpha$.*

Proof. Let $\gamma=\mathrm{cf}(\alpha)$ and let $h\colon\gamma\to\alpha$ be a cofinality function. Then $A_\alpha=\{f\colon \mathrm{Func}(f)\ \&\ \mathrm{dom}(f)=\alpha\ \&\ (\eta)_\gamma(f\restriction h(\eta)\in A_{h(\eta)})\}$, and so

$$|A_\alpha|\leqslant\Big|\prod_{\eta<\gamma}A_{h(\eta)}\Big|\leqslant\varkappa^\gamma\leqslant\varkappa^{\underset{\smile}{\varkappa}}=\varkappa.$$

Therefore $A_\alpha\in H_{\varkappa^+}$ and A_α is definable with parameters from M_α. Since $M_\alpha\prec H_{\varkappa^+}$ we have $A_\alpha\in M_\alpha$.

II. Cardinals with products of Suslin trees. Let $\varkappa$ be a regular cardinal of M.

DEFINITION 2.1. A $\varkappa$-Suslin tree T is called a *collapsing $\varkappa$-tree* if $T\times T$ collapses $\varkappa$ onto some $\mu<\varkappa$.

DEFINITION 2.2. A notion of forcing C is *neatly $\varkappa$-closed* if, whenever $\langle p_\alpha\colon \alpha<\beta\rangle$, $\beta<\varkappa$ is a descreasing sequence of elements of C, then $p\in C$, where $p=\bigwedge_{\alpha<\beta}p_\alpha$ in $\mathrm{RO}(C)$.

THEOREM 2.3. *In M, if $\varkappa$ is regular and $\varkappa^{\underset{\smile}{\varkappa}}=\varkappa$, then there exists a homogeneous collapsing $\varkappa^+$-Suslin tree $T_{\varkappa^+}$ which is neatly $\varkappa$-closed.*

Proof. The tree $T_{\varkappa^+}$ will be defined by induction on the levels $T^\alpha \overset{\mathrm{df}}{=} T^\alpha_{\varkappa^+}$. The elements of $T_{\varkappa^+}$ are $0,1$-sequences s such that $ht(s)=\mathrm{dom}(s)$ and $|\mathrm{dom}(s)|<\varkappa^+$. The ordering on $T_{\varkappa^+}$ will be ordinary inclusion. By induction it will follow that $T\restriction\alpha \overset{\mathrm{df}}{=} T_{\varkappa^+}\restriction\alpha\in M_\alpha$ for every $\alpha<\varkappa^+$ and $T\restriction\alpha$ is neatly $\varkappa$-closed if $\mathrm{cf}(\alpha)=\varkappa$. $T^0=\{\emptyset\}$. If $\alpha=\beta+1$ then $T^\alpha=\{s^\frown\langle i\rangle\colon s\in T^\beta\ \&\ i\in 2\}$. If $\lim(\alpha)$ and $\mathrm{cf}(\alpha)<\varkappa$, we extend all branches. By Lemma (1.3), $T^\alpha\in M_\alpha\subseteq M_{\alpha+1}$ in this case. Now we consider the case $\lim(\alpha)$ and $\mathrm{cf}(\alpha)=\varkappa$. Let $\langle D_\beta\colon \beta<\varkappa\rangle$ be an enumeration of the set of strongly dense subsets of $T\restriction\alpha$ which lie in M_α. Define by induction a sequence $\langle x_\beta\colon \beta<\varkappa\rangle$ of elements of $T\restriction\alpha$. Let $x_0\in T\restriction\alpha$ be such that $x_0\in D_0$. If $\beta=\gamma+1$ then x_β is such that $x_\beta\leqslant x_\gamma$ and $x_\beta\in D_\beta$ ($T\restriction\alpha$ is considered as a notion of forcing with reversed ordering). Now let β be a limit ordinal and suppose that $\langle x_\gamma\colon \gamma<\beta\rangle$ is defined. Since $T\restriction\alpha$ is $\varkappa$-closed, there exists an x_β such that $x_\beta\leqslant x_\gamma$ for $\gamma<\beta$ and $x_\beta\in D_\beta$. The sequence $\langle x_\beta\colon \beta<\varkappa\rangle$ fixes a certain M_α-generic branch for $T\restriction\alpha$. Let b be the first M_α-generic branch for $T\restriction\alpha$ in the sense of $<_{L[A]}$. Now define $T^\alpha=\{\bigcup d\colon d$ is an α-branch through $T\restriction\alpha$ and $\bigcup d$ differs from $\bigcup b$ on an initial segment only$\}$.

Let $T_{\varkappa^+}=\bigcup_{\alpha<\varkappa^+}T^\alpha$. It is obvious that $T_{\varkappa^+}$ is a normal tree of height $\varkappa^+$.

LEMMA 2.4. *For each pair $s, s'\in T_{\varkappa^+}$ such that $ht(s)=ht(s')$, $|\{\nu\colon s_\nu\neq s'_\nu\}|<\varkappa$.*

Proof. We proceed by induction on $\alpha=ht(s)=ht(s')$. The assertion is obvious for $\alpha=0$, successor α and $\lim(\alpha)$ if $\mathrm{cf}(\alpha)<\varkappa$. If $\lim(\alpha)$ and $\mathrm{cf}(\alpha)=\varkappa$ then s and s' are equal except for an initial segment for which the induction hypothesis holds.

LEMMA 2.5. *For $a\subseteq\varkappa^+$ and $|a|<\varkappa$ let $\sigma_a\colon 2^{<\varkappa^+}\to 2^{<\varkappa^+}$ be such that $ht(\sigma_a(s))=ht(s)$ and*

$$(\sigma_a(s))_\nu=\begin{cases} s_\nu & \text{if } \nu\notin a,\\ 1-s_\nu & \text{if } \nu\in a.\end{cases}$$

Then $T_{\varkappa^+}$ is closed under the maps σ_a.

Proof. By construction.

(2.4) and (2.5) imply that $T_{\varkappa^+}$ is homogeneous. Note that for $\lim(\alpha)$, if $\mathrm{cf}(\alpha)=\varkappa$, each α-branch which is extended in $T_{\varkappa^+}$ is M_α-generic for $T\restriction\alpha$. So (1.2) implies that $T_{\varkappa^+}$ is a Suslin tree.

LEMMA 2.6. *$T_{\varkappa^+}\times T_{\varkappa^+}$ collapses $\varkappa^+$ onto $\varkappa$.*

Proof. Let b be M-generic for $T_{\varkappa^+}$ and let d be $M[b]$-generic for $T_{\varkappa^+}$. Then $|\{\nu\colon (\bigcup b)_\nu \neq (\bigcup d)_\nu\}| = \varkappa^+$ but for $\alpha<\varkappa^+$, $|\{\nu\colon ((\bigcup b)\restriction\alpha)_\nu \neq ((\bigcup d)\restriction\alpha)_\nu\}|<\varkappa$. Hence $\varkappa^+$ is $\varkappa$-cofinal in $M[b,d]$, and so $\varkappa^+$ is collapsed onto $\varkappa$.

This concludes the proof of Theorem (2.3).

III. The Easton conditions. Let $\langle\varkappa_\alpha\colon \alpha\in On\rangle$ be, in M, the increasing enumeration of successors of regular cardinals. Note that for $n<\omega$, $\varkappa_n = \aleph_{1+n}$ and if $\lambda>0$, $\lim(\lambda)$, $\varkappa_{\lambda+n} = \aleph_{\lambda+2+n}$. Add to our assumptions on M that $M\models$ GCH. Working in M, we define

$$T = \{f\colon \mathrm{Func}(f)\ \&\ \mathrm{dom}(f)\subseteq On\ \&\ (\xi)_{\mathrm{dom}(f)}(f(\xi)\in T_{\varkappa_\xi}\ \&\ f(\xi)\neq\emptyset)\}$$

where $T_{\varkappa_\alpha}$ is the $\varkappa_\alpha$-Suslin tree constructed above. T is ordered by componentwise ordering.

Let G_T be T-generic over M. Then the following holds:

FACT 3.1. *For all $\alpha\in On\cap M$, $M[G_T]\models \mathrm{ZFC}+\varkappa_\alpha$ is regular.*

The last statement is an immediate consequence of the following standard Easton forcing lemma, based on the $\varkappa_\alpha$-closure of $T_{\varkappa_\beta}$ $(\alpha<\beta)$, the $\varkappa_\alpha$-chain-condition and the $\varkappa_\alpha$-closure of $T_{\varkappa_\alpha}$ and the fact that $|\prod_{\beta<\alpha} T_{\varkappa_\beta}|<\varkappa_\alpha$.

LEMMA 3.2. *Let $P_\alpha = \{f_{(\alpha)}\colon f\in T\}$ and $P^\alpha = \{f^{(\alpha)}\colon f\in T\}$, where $f_{(\alpha)} = f\restriction(\alpha+1)$ and $f^{(\alpha)} = f-f_{(\alpha)}$. Then $\cup\colon P_\alpha\otimes P^\alpha\to T$ is an order isomorphism onto T. Moreover, P_α satisfies $\varkappa_\alpha$-c.c. and P^α is $\varkappa_\alpha$-closed for all ordinals α.*

Accordingly, for $\lim(\lambda)$, $\aleph_\lambda^M = \aleph_\lambda^{M[G_T]}$. In order to conclude that cofinalities, and hence cardinals, are preserved, it remains only to see that $\aleph_{\lambda+1}^M$ remains regular in $M[G_T]$. Suppose that it does not. Let $\nu = \aleph_{\lambda+1}^M$. Then, for some $\mu<\aleph_\lambda^M$, in $M[G_T]$, $\mathrm{cf}\,\nu = \mu$. Let $\mu\leqslant\varkappa_\alpha<\aleph_\lambda^M$. Then, letting $G_\alpha = G_T\cap P_\alpha$, in $M[G_\alpha]$, $\mathrm{cf}\,\nu = \mu$, since P^α is $\varkappa_\alpha$-closed. But this is impossible, since P_α is $\varkappa_\alpha$-*c.c.* This gives us:

FACT 3.3. $\mathrm{Card}^{M[G_T]} = \mathrm{Card}^M$: *in fact* $\mathrm{cf}^{M[G_T]} = \mathrm{cf}^M$.

Now let us consider $T\times T = (\prod_\alpha T_{\varkappa_\alpha})\times(\prod_\alpha T_{\varkappa_\alpha})$.

LEMMA 3.4. $T\times T\cong \prod_\alpha (T_{\varkappa_\alpha}\times T_{\varkappa_\alpha})$.

FACT 3.5. *If $G_{T\times T}$ is $T\times T$-generic over M then $M[G_{T\times T}]\models \mathrm{ZFC}^-$.*

Proof. This follows from the fact that $T\times T$ is a strongly homogeneous notion of forcing (see [4]).

Remark 3.6. $M[G_{T\times T}]\models(\alpha)(\exists f)\ (f\colon \mu_\alpha \xrightarrow{\text{onto}} \varkappa_\alpha$, where $\mu_\alpha^+ = \varkappa_\alpha)$.

Remark 3.7. Suppose $M[G_{T\times T}]\models \mathrm{V}\neq\mathrm{HC}$. Then, since $M[G_{T\times T}]\models$ AC, $\aleph_1^{M[G_{T\times T}]}$ is regular in $M[G_{T\times T}]$ and hence was regular in M.

IV. Products of collapsing algebras. In this section we prove that

$$M[G_{T\times T}]\models \mathrm{V} = \mathrm{HC}.$$

We start with the following standard technical lemma:

LEMMA 4.1. *Let $\varkappa$, μ be regular cardinals such that $\mu>\varkappa$ and $\mu^{\check\varkappa} = \mu$. By $C(\varkappa,\mu)$ we mean the usual notion of forcing which collapses μ onto $\varkappa$. Let C be a neatly $\varkappa$-closed notion of forcing such that $|C| = |C(\varkappa,\mu)| = \mu$ and $\|(\exists f)\,(f\colon \check\varkappa\xrightarrow{\text{onto}}\check\mu)\|^{\mathrm{RO}(C)} = \mathbf{1}$. Then C has a dense subset isomorphic to $C(\varkappa,\mu)$.*

Remark 4.2. From (2.3) and (4.1) it follows that $T_{\varkappa^+}\times T_{\varkappa^+}$ contains a dense subset isomorphic to $C(\varkappa,\varkappa^+)$. Thus $T\times T\cong_d \prod_\varkappa C(\varkappa,\varkappa^+)$.

We want to show that $T\times T$ collapses all cardinals onto $\aleph_0$. We know that the successors of regular cardinals will be collapsed and that the first cardinal which is not collapsed if it exists cannot be a singular cardinal. We first show that $\aleph_{\omega+1}$ will be collapsed.

Let us remark that:

FACT 4.3. $C(\aleph_n,\aleph_{n+1})\cong\prod_{k<\omega} C(\aleph_n,\aleph_{n+1})$.

LEMMA 4.4. $\prod_{n>0} C(\aleph_n,\aleph_{n+1})\cong_d\prod_{n>1} C(\aleph_1,\aleph_n)$.

We now give the key technical lemma for the special case of $\aleph_{\omega+1}^M$.

LEMMA 4.5. $\prod_{n>1} C(\aleph_1,\aleph_n)$ *collapses* $\aleph_{\omega+1}$ *onto* $\aleph_1$.

Proof. If $G = \prod_{n>1} G_n$ is $\prod_{n>1} C(\aleph_1,\aleph_n)$-generic over M, we put $F_n = \bigcup G_n$. Let $\langle X_n\colon 2\leqslant n<\omega\rangle\in M$ be such that $X_n\subseteq\aleph_n$ and $|X_n|^M = \aleph_0$. Define

$$g(\langle X_n\colon 2\leqslant n<\omega\rangle) = \min_{\alpha<\aleph_1}\big((n)(F_n((\alpha+\omega)-\alpha) = X_n)\big).$$

Such an α exists since

$$D_{\langle X_n\colon 2\leqslant n<\omega\rangle} = \{p\colon (\exists\alpha)_{\aleph_1}(n)\big(p_n((\alpha+\omega)-\alpha) = X_n\big)\}$$

is dense in $\prod_{n>1} C(\aleph_1,\aleph_n)$. Clearly, g is 1-1. Working in M, we fined that

$$|\prod_{2\leqslant n<\omega}\wp_{\omega_1}(\aleph_n)| = \prod_{2\leqslant n<\omega}\aleph_n = \aleph_\omega^{\aleph_0} = \aleph_{\omega+1},$$

so $\aleph_{\omega+1}$ is collapsed.

From (4.1) it follows that:

LEMMA 4.6. $\prod_{n>1} C(\aleph_1,\aleph_n)\cong_d C(\aleph_1,\aleph_{\omega+1})$.

Now (4.5) can be generalized as follows. Recall that we are assuming that M has no inaccessible cardinals. Fix η, a limit ordinal. Let $f\colon \mathrm{cf}(\aleph_\eta)\to\aleph_\eta$ be an increasing cofinality function such that each $f(\xi)$ is regular and $f(o) = (\mathrm{cf}(\aleph_\eta))^+$. Since

$\mathrm{cf}(\aleph_\eta) = \aleph_{\eta_0+1}$ for some η_0 or $\mathrm{cf}(\aleph_\eta) = \aleph_0$, we have $f(0) = \aleph_{\eta_0+2}$ or $f(0) = \aleph_1$, and also $f(\xi) = \aleph_{\beta_\xi+1}$. Then, supposing $f(0) = \aleph_{\eta_0+2}$, we have

LEMMA 4.7 $\prod\limits_{\nu<\mathrm{cf}(\aleph_\eta)} C(\aleph_{\eta_0+2}, \aleph_{\beta_\nu+2})$ *collapses* $\aleph_{\eta+1}$ *onto* $\aleph_{\eta_0+2}$.

Proof. Let $G = \prod\limits_{\nu<\mathrm{cf}(\aleph_2)} G_\nu$ be $\prod\limits_{\nu<\mathrm{cf}(\aleph_\eta)} C(\aleph_{\eta_0+2}, \aleph_{\beta_\nu+2})$-generic over M. Then $F_\nu = \bigcup G_\nu\colon \aleph_{\eta_0+2} \xrightarrow{\text{onto}} \aleph_{\beta_\nu+2}$. Let $\langle X_\nu\colon \nu<\mathrm{cf}(\aleph_\eta)\rangle$ be such that $X_\nu \subseteq \aleph_{\beta_\nu+2}$ and $|X_\nu| = \mathrm{cf}(\aleph_\eta)$. Define

$$g(\langle X_\nu\colon \nu<\mathrm{cf}(\aleph_\eta)\rangle) = \min_{\alpha<\aleph_{\eta_0+2}} \big((\nu)_{\mathrm{cf}(\aleph_\eta)}(F_\nu((\alpha+\mathrm{cf}(\aleph_\eta))-\alpha) = X_\nu)\big).$$

Then $g\colon \aleph_\eta^{\mathrm{cf}(\aleph_\eta)} = \aleph_{\eta+1} \xrightarrow{1\text{-}1} \aleph_{\eta_0+2}$ (note that

$$D_{\langle X_\nu\colon \nu<\mathrm{cf}(\aleph_\eta)\rangle} = \{p\colon (\exists\alpha)_{\aleph_{\eta_0+2}}(\nu)_{\mathrm{cf}(\aleph_\eta)}\big(p_\nu((\alpha+\mathrm{cf}(\aleph_\eta))-\alpha) = X_\nu\big)\}$$

is dense in $\prod\limits_{\nu<\mathrm{cf}(\aleph_\eta)} C(\aleph_{\eta_0+2}, \aleph_{\beta_\nu+2})$.

Now we can generalize (4.6) as follows.

LEMMA 4.8. *For all ordinals* η

$$(*) \qquad (\gamma)_{<\eta} \prod_{\gamma\leqslant\xi<\eta} C(\aleph_{\xi+1}, \aleph_{\xi+2}) \cong_d C(\aleph_{\gamma+1}, \aleph_{\eta+1}).$$

Proof. Let η be the least ordinal for which $(*)$ does not hold. Let $\gamma<\eta$ be a counterexample for η. We will consider two cases:

1) $\eta = \mu+1$. Then

$$\prod_{\gamma\leqslant\xi<\mu+1} C(\aleph_{\xi+1}, \aleph_{\xi+2}) = \prod_{\gamma\leqslant\xi<\mu} C(\aleph_{\xi+1}, \aleph_{\xi+2}) \times C(\aleph_{\mu+1}, \aleph_{\mu+2})$$
$$\cong_d C(\aleph_{\gamma+1}, \aleph_{\mu+1}) \times C(\aleph_{\mu+1}, \aleph_{\mu+2}) \cong_d C(\aleph_{\gamma+1}, \aleph_{\eta+1}).$$

2) $\eta \in \mathrm{Lim}$. Let $f\colon \mathrm{cf}(\aleph_\eta) \to \aleph_\eta$ be a cofinality function of $\aleph_\eta$ as in (4.7). We have

$$\prod_{\gamma\leqslant\xi<\eta} C(\aleph_{\xi+1}, \aleph_{\xi+2}) = \prod_{\gamma\leqslant\xi<\eta_0+1} C(\aleph_{\xi+1}, \aleph_{\xi+2}) \times \prod_{\eta_0+1\leqslant\xi<\eta} C(\aleph_{\xi+1}, \aleph_{\xi+2})$$
$$\cong_d C(\aleph_{\gamma+1}, \aleph_{\eta_0+2}) \times \prod_{\eta_0+1\leqslant\xi<\eta} \prod_{\nu<\mathrm{cf}(\aleph_\eta)} C(\aleph_{\xi+1}, \aleph_{\xi+2}).$$
$$\cong_d C(\aleph_{\gamma+1}, \aleph_{\eta_0+2}) \times \prod_{\nu<\mathrm{cf}(\aleph_\eta)} \prod_{\eta_0+1\leqslant\xi<\beta_\nu+1} C(\aleph_{\xi+1}, \aleph_{\xi+2})$$
$$\cong_d C(\aleph_{\gamma+1}, \aleph_{\eta_0+2}) \times \prod_{\nu<\mathrm{cf}(\aleph_\eta)} C(\aleph_{\eta_0+2}, \aleph_{\beta_\nu+2})$$
$$\cong_d C(\aleph_{\gamma+1}, \aleph_{\eta_0+2}) \times C(\aleph_{\eta_0+2}, \aleph_{\eta+1})$$
$$\cong_d C(\aleph_{\gamma+1}, \aleph_{\eta_0+1}).$$

This completes the proof of

THEOREM 4.9. $M[G_{T\times T}] \models V = HC$.

Remark 4.10. Let $X \subseteq On$, $T_X = \prod\limits_{\alpha\in X} (T_{\varkappa_\alpha} \times T_{\varkappa_\alpha})$. If $\alpha \in X$ implies that $\alpha+1 \notin X$ then T_X is Easton-like; so if G_{T_X} is T_X-generic over M then $M[G_{T_X}] \models$ ZFC.

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