Indicators, recursive saturation and expandability

by

L. Kirby (Paris, Princeton), **K. McAloon** (Paris)
and **R. Murawski** (Warszawa, Poznań)

Abstract. We investigate the recursive saturation and the expandability of initial segments of countable nonstandard models of Peano arithmetic. The main tools which are used are nonstandard satisfaction classes and indicators.

§ 1. Introduction. In this paper we apply the techniques of indicator theory to recursively saturated models of arithmetic. The connection between recursive saturation and expandability is well-known; we go on to look at the "natural" expansion of an initial segment I of a model of (first order) arithmetic to a structure I^* for the language of second order arithmetic by adding in the coded subsets of I, and prove (something very slightly more general than) the:

THEOREM. *Let T be a recursively axiomatizable theory in the language of second order arithmetic extending arithmetic comprehension. The following classes of initial segments are symbiotic*:

(α) *I such that $I^* \models T$.*

(β) *I such that I is a model of the Π_2^0 consequences of T.*

(The notion of symbiotic means roughly speaking that members of (α) are about equally as common as members of (β). See [K], [P$_2$], and Definition 7 below.)

Thus in particular, *any* model of (part of) arithmetic abounds in initial segments I such that I^* is a model of second order arithmetic or even of the second-order-arithmetical consequences of any set theory you care to believe consistent. (See the corollary at the end of the paper.)

The theorem is proved by the intermediary of another class:

(γ) recursively saturated models of the first order consequences of T.

We lean heavily on results by Barwise, Schlipf and Wilmers on recursive saturation; Robinson, Mostowski and Krajewski (see [Kr]) on satisfaction classes; and Kirby and Paris on indicators.

In § 2 we look at recursive saturation and characterize it in terms of satisfaction classes. In § 3 we use this to produce indicators for recursively saturated models, showing that (β) and (γ) are symbiotic. § 4 is another application: an indicator for recursively saturated elementary initial segments. § 5 (independent of § 4) looks at the expansions I^* and shows that (α) and (γ) are symbiotic.

Historical remark. This paper represents a union of results obtained independently and through somewhat different approaches by Murawski and by Kirby–McAloon. Murawski introduced in his thesis (¹) the notion of substitutable satisfaction class with ω-overspill (cf. Defs. 4 and 5) and obtained Theorems 3, 7, Lemma 9, Corollaries 11 (i)-(iii) and (v)-(vii), Theorem 12 and Cor. 13(i)(ii). Kirby-McAloon developed indicators for 2nd order theories and obtained Thm. 7, 8, Cor. 11(i)-(iv) and the results of Section 5.

Notation. Let $\mathscr{L}^{\mathrm{I}}$ be the language of first order arithmetic and let P be Peano's Axioms in this language. We denote the standard model of P by $\mathscr{N}_0$. Any other model $\mathscr{M} \models P$ is said to be nonstandard and has a "standard part" ω which is isomorphic to $\mathscr{N}_0$. Elements in $M-\omega$ are called nonstandard. (We use M for the universe of $\mathscr{M}$.)

Let $\mathscr{M} \models P$. An initial segment of $\mathscr{M}$ is a subset I of M such that $\forall xy \in M$ $(x \in I \wedge y<x \rightarrow y \in I)$. I is a proper initial segment of M ($I \subset_e M$) if further I is closed under successor and $\emptyset \neq I \neq M$.

If $I \subset_e M$ is closed under multiplication then I becomes a structure for $\mathscr{L}^{\mathrm{I}}$ with operations inherited from $\mathscr{M}$; we sometimes use the latter I ambiguously to mean this structure on the initial segment.

Write $\mathscr{M}' \succ_e \mathscr{M}$ iff, $\mathscr{M}' \succ \mathscr{M}$ and $\mathscr{M} \subset_e \mathscr{M}'$. Robinson's overspill principle says that if $I \subset_e \mathscr{M}$ and for unboundedly many $i \in I$,

$$\mathscr{M} \models \varphi(\vec{a}, i)$$

where $\varphi(\vec{x}, y)$ is a formula of $\mathscr{L}^{\mathrm{I}}$ and $\vec{a} \in M$, then for some $b \in M-I$,

$$\mathscr{M} \models \varphi(\vec{a}, b).$$

If T is a theory in some language extending $\mathscr{L}^{\mathrm{I}}$, P^T is the theory consisting of all consequences in $\mathscr{L}^{\mathrm{I}}$ of T.

We denote by $\mathscr{L}^{\mathrm{II}}$ the language of second order arithmetic — the two-sorted language containing set variables and the membership symbol ε. The theory A_2^- in $\mathscr{L}^{\mathrm{II}}$ consists of the axioms of P with induction replaced by a single axiom, extensionality, and the comprehension schema. The theory arith-CA is the same with comprehension restricted to arithmetic formulae (i.e. no set quantifiers). For details see e.g. [AM].

§ 2. First we survey "classical" results about recursive saturation, due largely to Barwise and Schlipf and (independently) to Wilmers. For a detailed exposition see [M]. This will be followed by a characterization of recursive saturation in terms of satisfaction classes.

(¹) Written under the supervision of W. Marek and submitted to the University of Warsaw.

Definition 1. Let $\mathscr{M}$ be a structure for $\mathscr{L}^{\mathrm{I}}$, T a theory in $\mathscr{L}^{\mathrm{II}}$. M is *expandable* to a model of T (T-expandable) iff there exists $\mathscr{X} \subseteq \mathscr{P}(M)$ such that $\langle \mathscr{M}, \mathscr{X} \rangle \models T$.

Definition 2 [BS]. Let $\mathscr{M}$ be a structure for a first order language $\mathscr{L}$. $\mathscr{M}$ is *recursively saturated* iff any recursive set of finitary formulae of $\mathscr{L}$ (with parameters from M) finitely satisfiable in $\mathscr{M}$ is realized in $\mathscr{M}$.

Definition 3 [BS]. A structure $\mathscr{M}$ is *resplendent* iff for any $\mathscr{N} \succ \mathscr{M}$, and Σ_1^1 formula φ and any $\vec{a} \in |\mathscr{M}|$ we have

$$\langle \mathscr{N}, \mathscr{P}(N) \rangle \models \varphi[\vec{a}] \Rightarrow \langle \mathscr{M}, \mathscr{P}(M) \rangle \models \varphi[\vec{a}].$$

Result 1. (a) (Barwise). *Every resplendent structure is recursively saturated.*

(b) (Barwise, Ressayre). *Every countable recursively saturated structure is resplendent.*

Result 2 (essentially due to Schlipf for $\mathscr{L}^{\mathrm{II}}$ and independently Wilmers for a more general case). *Let T be a recursively axiomatizable theory in $\mathscr{L}^{\mathrm{II}}$ such that $T \vdash P$, and let $\mathscr{M}$ be a countable model of P^T. Then $\mathscr{M}$ is T-expandable iff $\mathscr{M}$ is recursively saturated.*

For a precise definition of satisfaction class cf. Krajewski's paper [Kr]; roughly speaking, the notion of a satisfaction class extends the Tarski's definition of satisfaction to the case of nonstandard formulas. So in particular if S is a satisfaction class for $\mathscr{M}$ then for all standard formulas φ of $\mathscr{L}^{\mathrm{I}}$ we have

$$\mathscr{M} \models \varphi[\vec{a}] \Leftrightarrow \langle \mathscr{M}, S \rangle \models \underline{S}(\overline{\varphi}, \vec{a}).$$

Definition 4. A satisfaction class S over a model $\mathscr{M} \models T$, T being a theory in a language $\mathscr{L} \subseteq \mathscr{L}^{\mathrm{I}}$, has *$\omega$-overspill* iff for any formula φ of the language $\mathscr{L} \cup \{\underline{S}\}$, if $\{n \in \omega : \langle \mathscr{M}, S \rangle \models \varphi(\underline{n})\}$ is unbounded in ω then there is a nonstandard $a \in M$ such that $\langle \mathscr{M}, S \rangle \models \varphi[a]$.

Before introducing the next notion let us fix the following notation. If $\mathscr{M}$ is a structure and $X \subseteq M^k$ then $\mathrm{Def}(\mathscr{M}, X)$ is the family of all subsets of M definable with parameters over $\langle \mathscr{M}, X \rangle$. T_{pr} denotes the predicative extension (in $\mathscr{L}^{\mathrm{II}}$) of the theory T (cf. [Kr]).

Definition 5. A satisfaction class S over a model $\mathscr{M} \models T$ (T being a theory in $\mathscr{L} \subseteq \mathscr{L}^{\mathrm{I}}$ such that $T \vdash P$) is said to be *substitutable* iff $\langle \mathscr{M}, \mathrm{Def}(\mathscr{M}, S) \rangle \models T_{\mathrm{pr}}$.

Observe that if T is axiomatized by some sentences and some schemas then S is substitutable over $\mathscr{M} \models T$ iff all substitutions of the schemas of T by formulas of the language $\mathscr{L} \cup \{\underline{S}\}$ are true in $\langle \mathscr{M}, S \rangle$.

Theorem 3. *Let T be a recursively axiomatizable theory in $\mathscr{L}^{\mathrm{I}}$ such that $T \vdash P$, and let $\mathscr{M} \models T$ be nonstandard and countable. The following are equivalent:*

(i) *$\mathscr{M}$ is recursively saturated;*

(ii) *$\mathscr{M}$ has a satisfaction class with ω-overspill;*

(iii) *$\mathscr{M}$ has a substitutable satisfaction class.*

Proof. The implication (iii) → (ii) is not hard. For (ii) → (i), let us take a satisfaction class S over $\mathscr{M}$ with ω-overspill. We show $\mathscr{M}$ is recursively saturated. Let

Φ be a recursive type in the parameters $\vec{a} \in M$ and let $\underset{\cdot}{\Phi}$ be a formula of $\mathscr{L}^{\mathrm{I}}$ strongly representing in P the sequence of formulae $\psi(x, \vec{x})$ such that $\psi(x, \vec{a})$ is in Φ. If Φ is finitely satisfiable in $\mathscr{M}$ then for every $K \in \omega$

$$\langle \mathscr{M}, S\rangle \models \exists x \forall y [y < \underline{K} \wedge \underset{\cdot}{\Phi}(y) \rightarrow \underline{S}(y, \langle x, \vec{a}\rangle)].$$

By ω-overspill there is $c > \omega$ such that

$$\langle \mathscr{M}, S\rangle \models \exists x \forall y [y < c \wedge \underset{\cdot}{\Phi}(y) \rightarrow \underline{S}(y, \langle x, \vec{a}\rangle)].$$

Hence Φ is realized in $\mathscr{M}$.

To prove (i) $\rightarrow$ (iii) notice that for any φ in $\mathscr{L}^{\mathrm{I}}$ the theory

$$\mathrm{Th}(\mathscr{M}, a)_{a \in M} + \text{schemas of } T \text{ for } \underline{S} + \text{“}\forall \vec{x}[\varphi(\vec{x}) \longleftrightarrow \underline{S}(\ulcorner \varphi \urcorner, \langle \vec{x}\rangle)]\text{”}$$

is finitely consistent. So there is a model of it, say $\langle \mathscr{N}, S\rangle$. Of course $\mathscr{M} \prec \mathscr{N}$. Since $\mathscr{M}$ is recursively saturated and countable, it is resplendent. Using this and the fact, shown by Kleene in 1952, that every recursive conjunction of formulae can be replaced by a Σ_1^1 formula, we obtain that there is a substitutable satisfaction class over $\mathscr{M}$.

Observe that the implication (ii) $\rightarrow$ (i) is true for nonstandard models of any cardinality. In fact the countability of $\mathscr{M}$ was used only in the proof of (i) $\rightarrow$ (iii). Note also that this last implication does not hold for full (i.e. deciding all formulas on all valuations) satisfaction classes. (This fact was communicated by H. Kotlarski. It can be found also in [Kr].) For let $\mathscr{M}$ be a recursively saturated model of $P + \neg\mathrm{Con}P$. We claim that $\mathscr{M}$ has no full substitutable satisfaction class. For let T be the theory: $P + \neg\mathrm{Con}P +$ “$\underline{S}$ is a full satisfaction class” + induction schema for formulas of $\mathscr{L}^{\mathrm{I}} \cup \{\underline{S}\}$. It suffices to prove that T is inconsistent. One can easily show, by induction in T, that

$$(*) \qquad T \vdash \forall \varphi [\mathrm{Fm}(\varphi) \rightarrow \underline{S}(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x')) \rightarrow \forall x \varphi(x))].$$

(To be precise we ought to write $\mathrm{Sub}(\varphi, x, \ulcorner 0 \urcorner)$ for $\varphi(0)$, etc. (cf. [F], [Sh]).)

Assume now that T has a model, say $\langle \mathscr{N}, S\rangle$. Using (*) we obtain that

$$\langle \mathscr{N}, S\rangle \models \forall x (\mathrm{Fm}(x) \wedge \mathrm{Pr}_P(x) \rightarrow \underline{S}(x, \emptyset)]$$

where Pr_P is a formula of $\mathscr{L}^{\mathrm{I}}$ strongly representing in P the set of Gödel numbers of theorems of P. But $\langle \mathscr{N}, S\rangle \models \neg\underline{S}(\ulcorner 0 \neq 0 \urcorner, \emptyset)$. By the transposition law $\langle \mathscr{N}, S\rangle \models \neg\mathrm{Pr}_P(\ulcorner 0 \neq 0 \urcorner)$. Consequently $\langle \mathscr{N}, S\rangle \models \mathrm{Con}P$, contradicting the fact that $T \vdash \neg\mathrm{Con}P$. So T is inconsistent.

§ **3.** We review the theory of indicators, and study indicators for recursively saturated models.

DEFINITION 6 (Kirby, Paris [KP], [K], [P]). Let $\mathscr{M} \models P$ and let Q be a family of proper initial segments of $\mathscr{M}$. The function $Y\colon M^2 \rightarrow M$ definable (possibly with parameters) in $\mathscr{M}$ is an *indicator for* Q iff for any $a, b \in M$ with $a \leqslant b$:

there exists $I \in Q$ such that $a \in I < b$ iff for some $c > \omega$ in M, $\mathscr{M} \models Y(a, b) = c$.

If Q is a property of initial segments of models of P and $Y(x, y)$ a term of $\mathscr{L}^{\mathrm{I}}$, Y is a *well-behaved indicator* for the property Q iff

(1) $P \vdash \forall xy \exists ! z\, Y(x, y) = z$,

(2) “$Y(x, y) = z$” is $\Sigma_1^0(P)$,

(3) for any countable non-standard model $\mathscr{M} \models P$,

$$Y^{(\mathscr{M})} = \{(a, b, c) \in M^3\colon \mathscr{M} \models Y(a, b) = c\}$$

is an indicator for the family of initial segments of $\mathscr{M}$ having property Q,

(4) $P \vdash \forall xy\, Y(x, y) \leqslant y$,

(5) $P \vdash \forall xy x_1 y_1 (x_1 \leqslant x < y \leqslant y_1 \rightarrow Y(x, y) \leqslant Y(x_1, y_1))$.

Nearly all the indicators that we consider will be well-behaved. Hence, we shall simply use the word “indicator” and assume good behavior unless otherwise stated.

In fact, we can develop the theory of indicators for initial segments of models of $I\Sigma_2^0$ (Σ_2^0-induction) and results such as the following work equally well:

RESULT 4 (Kirby, Paris [KP], [K]). *Let Q be a class of initial segments with indicator Y.*

(a) *If $\exists I \in Q$, $a \in I < b$ then we can find e with $a < e < b$ such that $\exists I_1, I_2 \in Q$,*

$$a \in I_1 < e \quad \text{and} \quad e \in I_2 < b.$$

(b) *Hence if $I \in Q$ then I is either the union or intersection of other initial segments in Q.*

(c) *If $I \in Q$ and $I \models I\Sigma_2^0$ then I is both the union and the intersection of other initial segments in Q.*

We sketch the proof (c): Suppose I were not the union of members of Q. Then for some $a \in I$, $\forall b \in I\, Y(a, b) \in \omega$. Now $I\Sigma_2^0$ is equivalent to the least number principle for Π_2^0 sets: see [PK]. Let c be the least element of the class $\{x\colon I \models \forall y\, Y(a, y) < x\}$. c must be in ω; by overspill there must be $d > I$ such that $Y(a, d) < c$; but the value of $Y(a, d)$ must be $> \omega$ since $a \in I < d$. Contradiction.

DEFINITION 7 ([K]). Two classes Q_1, Q_2 of initial segments are *symbiotic* iff whenever $a < b$:

$$\exists I \in Q_1\, a \in I < b \quad \text{iff} \quad \exists I \in Q_2\, a \in I < b.$$

So any indicator for one is an indicator for the other.

RESULT 5 ([K]). *Let T be a recursively axiomatizable theory in $\mathscr{L}^{\mathrm{I}}$. Then there is an indicator for the class of models of T.*

COROLLARY. *Let T be a recursively axiomatizable theory in some language extending $\mathscr{L}^{\mathrm{I}}$. Then there is an indicator for the class of models of P^T.*

Proof. P^T is a recursively axiomatizable theory in $\mathscr{L}^{\mathrm{I}}$.

The following result is due to Paris and (independently) Lessan.

LEMMA 6. *Let T be a recursively axiomatizable theory in $\mathscr{L}^{\mathrm{I}}$. Then the class of models of T and the class of models of $\Pi_2^0(T)$, the Π_2^0 consequences of T, are symbiotic.*

Proof. Fix $\mathcal{M} \models P$. For any $n \in \omega$, $T \vdash \forall x \exists y\, Y(x, y) > n$, where Y is an indicator for the class of models of T. Since this is a Π_2^0 statement, if $I \subset_e \mathcal{M}$ and $I \models \Pi_2^0(T)$, then $I \models \forall x \exists y\, Y(x, y) > n$. Now if $a \in I < b$ it suffices to show that $\mathcal{M} \models Y(a, b) > \omega$. But if not, then for some $n \in \omega$, $\mathcal{M} \models Y(a, b) = n$.

Hence $I \models \forall y\, Y(a, y) \leqslant n$ (by the absoluteness of Σ_1^0 formulae), a contradiction.

THEOREM 7. *Let T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{I}}$ such that $T \vdash P$. There is an indicator for the family of initial segments which are recursively saturated models of T.*

Proof. Fix a recursive axiomatization of T. Let $T' = T +$ "$\underline{S}$ is a satisfaction class" + substitutions of all schemas of T by formulae of the language $\mathcal{L} = \mathcal{L}^{\mathrm{I}} \cup \{\underline{S}\} \cup \{\underline{K}_n\}_{n \in \omega}$ where $\underline{S}$ is a new binary predicate and the $\underline{K}_n$ are new constant symbols. T' is Σ_1^0 definable over $\mathcal{N}_0$; let θ be a recursive function enumerating T', $\boldsymbol{\theta}$ a formula of $\mathcal{L}^{\mathrm{I}}$ strongly representing θ in P.

Let $\mathcal{M}$ be any countable nonstandard model of P. As in [K] we define a game $G_c(a, b)$ for $a, b, c \in M$, $a \leqslant b$. The game has two players, each with c moves. The idea of the game is that player II claims that there is an initial segment $N \subseteq_e \mathcal{M}$ such that $a \in N < b$, $\mathcal{N} \models T$ and $\mathcal{N}$ has a substitutable satisfaction class. Player I verifies this claim by asking questions. Set $a_0 = a$. The nth step is as follows:

Assume we already have $a_n < b$ and numbers $K_1, \ldots, K_m < a_n$ for some $m \leqslant 2n$, and a set T_n of sentences of $\mathcal{L}$ involving only $\underline{K}_1, \ldots, \underline{K}_m$ among the new constant symbols. Player I produces an element $K_{m+1} < a_n$ and asks:

Does $\mathcal{N} \models \exists x \varphi(x)$ (φ formula of $\mathcal{L}$ with Gödel number $\leqslant n$ and involving only $\underline{K}_1, \ldots, \underline{K}_{m+1}$ among the new constant symbols), where $\underline{K}_1, \ldots, \underline{K}_{m+1}$ are interpreted by $K_1, \ldots, K_{m+1}$?

If the answer is Yes then II chooses $K_{m+2} < b$ and they put

$$a_{n+1} = [\max(a_n, K_{m+2})]^2 \quad \text{and} \quad T_{n+1} = T_n \cup \{\varphi(\underline{K}_{m+2})\}.$$

If No then they put $a_{n+1} = a_n^2$, $T_{n+1} = T_n \cup \{\neg \exists x \varphi(x)\}$.

Let R_{m+1} be the set of atomic formulae and negations of atomic formulae satisfied by $K_1, \ldots, K_{m+1}$ ($,K_{m+2}$).

Player I wins $G_c(a, b)$ iff there is a proof with Gödel number $\leqslant c$ of a contradiction from $T_c \cup \{\boldsymbol{\theta}(i) \colon i < c\} \cup R_c$.

Player II wins otherwise. This game is definable in $\mathcal{M}$ and finite in $\mathcal{M}$, hence it is determined. We claim that the function $Y(a, b) = \max c$: player II has a winning strategy in $G_c(a, b)$ is an indicator for

$$Q = \{\mathcal{N} \subset_e \mathcal{M} \colon \mathcal{N} \models T \text{ and } \mathcal{N} \text{ has a substitutable satisfaction class}\},$$

which suffices by Theorem 3. (The definition can be made Σ_1^0 in P and the indicator is well-behaved.)

Indeed, assume that there is $\mathcal{N} \subset_e \mathcal{M}$ such that $a \in N < b$ and $\mathcal{N} \in Q^{\mathcal{M}}$. Suppose $n \in \omega$ and I has a winning strategy for $G_n(a, b)$. Then II could play the strategy given by answering the truth about $\mathcal{N}$. This strategy is definable in $\mathcal{M}$ since II gives only a (truly) finite number of answers; hence II wins. Thus I could not have had a winning strategy, so II has a winning strategy for $G_n(a, b)$ for any $n \in \omega$ and hence for $G_c(a, b)$ for some $c > \omega$.

Conversely, suppose $Y(a, b) = c > \omega$. Take (outside $\mathcal{M}$) an enumeration of all formulae of $\mathcal{L}$ of form $\exists x \varphi(x)$ with parameters from $< b$. At the nth stage let I play the first suitable formula from this list not so far considered (i.e. the formula must have Gödel number $\leqslant n$ and involve only one new parameter $< a_n$, to be christened K_{m+1}) and let II play according to a winning strategy for $G_c(a, b)$.

For any $n \in \omega$, the game up to the nth move is (truly) finite and hence can be played in $\mathcal{M}$.

Put $N = \sup\{a_n \colon n \in \omega\}$. N is closed under multiplication and is made into a structure for $\mathcal{L}$ in obvious way with $\underline{S}$ interpreted by

$$S = \{(e, f) \colon \ulcorner \underline{S}(\underline{e}, \underline{f}) \urcorner \in \bigcup_{n \in \omega} T_n\}.$$

The construction ensures that $\bigcup_{n \in \omega} T_n \vdash \mathrm{Th}(\mathcal{N}, S)$, and hence $\langle \mathcal{N}, S \rangle \models T'$ for otherwise there would be a sentence $\theta(i)$ ($i \in \omega$) such that $\langle \mathcal{N}, S \rangle \models \neg \theta(i)$. So $\bigcup_{n \in \omega} T_n \vdash \neg \theta(i)$. By compactness for some $n \in \omega$ $\bigcup_{j<n} T_j \vdash \neg \theta(i)$. Hence there is a (finitely derived) contradiction from $\bigcup_{j<n} T_j$ and $\theta(i)$ and player I would win at some finite stage, whereas we know that II wins at any finite stage.

THEOREM 8. *Let T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{I}}$, $T \vdash P$. The class of models of T and the class of recursively saturated models of T are symbiotic.*

Proof. This will follow immediately from:

LEMMA 9. *Let T be as in Theorem 8, and $\mathcal{M} \models T$ countable, nonstandard. Then $\mathcal{M}$ has arbitrarily large initial segments which are recursively saturated models of T.*

Proof. We can find $\mathcal{M} \subseteq_e \mathcal{M}' \models T$ such that $\mathcal{M}' \succ_{\Sigma_1} \mathcal{M}$ and $\mathcal{M}'$ is recursively saturated (see e.g. [Sm], [L]). Also take $\mathcal{M}'' \supseteq_e \mathcal{M}'$ such that $\mathcal{M}'' \succ_{\Sigma_1} \mathcal{M}'$. Let Y be an indicator for recursively saturated models of T. Take any $a \in M$. For $b \in M'' - M'$,

$$\mathcal{M}'' \models Y(a, b) > c$$

for some nonstandard c which we may assume is small enough to be in M.

$$\text{Hence } \mathcal{M}'' \models \exists y\, Y(a, y) > c.$$

$$\text{Hence } \mathcal{M} \models \exists y\, Y(a, y) > c.$$

By the definition of indicators, this means that there is an initial segment of $\mathcal{M}$ lying above a which is a recursively saturated model of T.

Remarks. Lemma 9 was first proved by Lessan by other methods [L, Chapter 5]. In fact Theorem 8 (and hence Lemma 9, which follows from it by Result 4 (c)) is also implicit in Chapter 9 of [K].

COROLLARY 10. *Let T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{II}}$, $T \vdash P$. The following classes are symbiotic*:

(i) *Models of P^T.*
(ii) *Models of $\Pi_2^0(T)$.*
(iii) *Initial segments which are expandable to models of T.*

Proof. The corollary follows from Result 2, Lemma 6 and Theorem 8.

We can apply indicator-theoretic arguments to get corollaries such as:

COROLLARY 11. *If T is as in Theorem* 8, *then* (i) *Any countable nonstandard $\mathcal{M} \models T$ has arbitrarily large initial segments which are recursively saturated models of T.* (ii) *$\mathcal{M}$ has $2^{\aleph_0}$ such initial segments.* (*We see it by repeating the "splitting" of Result* 4(a). *For take e as given in Result* 4(a): *then for some nonstandard c_1, the games $G_{c_1}(a, e)$, $G_{c_1}(e, b)$ both have winning strategies for* II. *We keep splitting to obtain a binary tree the branches of which converge to distinct initial segments.*). (iii) *If $\mathcal{N}_0 \models T$ then any countable nonstandard $\mathcal{M} \models P$ has arbitrary small nonstandard initial segments which are recursively saturated models of T.* (iv) *Assuming that* ZF *has an ω-model, any countable nonstandard model $\mathcal{M} \models P$ has arbitrarily small nonstandard initial segments which are expandable to models of* ZF. (*This follows from a generalization due to G. Wilmers of Result* 2.) (v) *Every model of $P^{\Sigma_n^1\text{-CA}}$ $(n \in \omega)$ has arbitrarily large Σ_n^1-CA-expandable initial segments.* (vi) *Every model $\mathcal{M} \models P$ has $2^{\aleph_0}$ initial segments that are Δ_2^--expandable.* (vii) *Every $\mathcal{M} \models P$ has arbitrarily small Δ_2^--expandable initial segments.*

Remarks. Corollaries 11(iii) and (iv) can be strengthened somewhat: the condition $\mathcal{N}_0 \models T$ can be weakened to $\mathcal{N}_0 \models \Sigma_1^0(T)$ and the condition $\mathcal{M} \models P$ to $\mathcal{M} \models I\Sigma_0^0$ by results of McAloon, cf. [Mc2].

§ 4. We wish to spend a moment producing another indicator whose existence follows nicely from the characterization of Theorem 3. This indicator is *not* well-behaved as it reigns only inside the model in which it is defined: it has the properties of Definition 6 if we replace provability from P by truth in $\mathcal{M}$.

THEOREM 12. *Let $\mathcal{M} \models P$ be countable and nonstandard. Then there is an indicator (defined in $\mathcal{M}$) for the class of recursively saturated elementary initial segments of $\mathcal{M}$.*

Proof. Let $T' = P +$"$\underline{S}$ is a satisfaction class" + substitutions of the induction schema by formulae of the language $\mathcal{L} = \mathcal{L}^{\mathrm{I}} \cup \{\underline{S}\} \cup \{\underline{K}_n\}_{n\in\omega}$ where $\underline{S}$ is a new binary predicate and the $\underline{K}_n$ are new constant symbols.

Let θ be a recursive function enumerating T' and $\underset{\cdot}{\theta}$ a formula of $\mathcal{L}^{\mathrm{I}}$ strongly representating θ in P. Let S be a substitutable satisfaction class over $\mathcal{M}$. We shall define in $\langle \mathcal{M}, S\rangle$ a game $G_c(a, b)$ for $a \leqslant b$. The definition of the indicator from the game and the proof that it is an indicator will follow just as in Theorem 7.

The idea of the game is that player II claims that there is an $\mathcal{N} \prec_e \mathcal{M}$ such that $a \in N < b$ and $\mathcal{N}$ has a substitutable satisfaction class and player I verifies this claim by asking questions.

The nth step is as in Theorem 7. But player I wins this game iff there is a proof with Gödel number $\leqslant c$ of a contradiction from

$$T_c \cup \{\underset{\cdot}{\theta}(i)\colon i < c\} \cup R_c \cup \{\varphi(\vec{a})\colon \vec{a} < a_c \wedge \underline{S}(\ulcorner\varphi\urcorner, \langle\vec{a}\rangle)\}.$$

We can now obtain corollaries of the same sort as before, such as:

COROLLARY 13. (i) *Every countable recursively saturated $\mathcal{M} \models P$ has $2^{\aleph_0}$ and arbitrarily large initial segments which are recursively saturated elementary submodels of $\mathcal{M}$.* (*This uses the Mac Dowell–Specker theorem and Theorem* 10.) (ii) *In fact if $\mathcal{M} \models P^{\Sigma_1^1\text{-CA}}$ then all of these initial segments are isomorphic to $\mathcal{M}$.* (*This follows from the generalized theorem of Wilkie* (cf. Theorem 18 in [M] part II, and [M$_2$]). *This says that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are Σ_1^1-CA-expandable, $\mathcal{M}_1 \subset_e \mathcal{M}_2$ and $\mathcal{M}_1 \equiv \mathcal{M}_2$ then $\mathcal{M}_1 \simeq \mathcal{M}_2$.*)

Similar results to (i) have been proved by other methods by Kotlarski [Ko] and Schlipf [S].

§ 5. We now seek to go beyond Corollary 10 by looking at a specific way to expand initial segments to the second order language.

DEFINITION 8. Let $I \subset_e M$. Write $\mathcal{R}_{\mathcal{M}}(I)$ for the set of subsets of I coded in $\mathcal{M}$, i.e. the intersections with I of sets definable with parameters in $\mathcal{M}$. We may assume that any set in $\mathcal{R}_{\mathcal{M}}(I)$ has the form

$$\{x \in I\colon \mathcal{M} \models p_x | a\}$$

for some $a \in M$, where p_x is the xth prime. If I is closed under multiplication it "is" a structure for $\mathcal{L}^{\mathrm{I}}$. Let I^* be

$$\langle I, \mathcal{R}_{\mathcal{M}}(I)\rangle$$

which is a structure for $\mathcal{L}^{\mathrm{II}}$.

THEOREM 14. *Let T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{II}}$. Then there is a (well-behaved) indicator for the class of initial segments I such that $I^* \models T$.*

Proof. Fix a recursive axiomatization of T. Let $T' = T +$ substitutions of all schemas of T by formulae of the language $\mathcal{L} = \mathcal{L}^{\mathrm{II}} \cup \{\underline{K}_n\}_{n\in\omega} \cup \{\underline{S}_n\}_{n\in\omega}$ where $\underline{K}_n$, $\underline{S}_n$ are new constant symbols, the $\underline{S}_n$ to be interpreted as sets. Let θ be a recursive function enumerating T', $\underset{\cdot}{\theta}$ a formula of $\mathcal{L}^{\mathrm{I}}$ strongly representing θ in P. We shall define a game $G_c(a, b)$; the rest of the proof will follow as usual. The nth step is as follows: Player II is claiming the existence of $\mathcal{N}$ such that $\mathcal{N}^* \models T$ and $a \in \mathcal{N} < b$.

Assume we already have $a_n < b$ and numbers $K_1, \ldots, K_m < a_n$ for some $m \leqslant 2n$ and $S_1, \ldots, S_{m'} < b$ for some $m' \leqslant 2n$ and a set T_n of sentence of $\mathcal{L}$ involving only $\underline{K}_1, \ldots, \underline{K}_m, \underline{S}_1, \ldots, \underline{S}_m$, among the new constant symbols. Player I produces $K_{m+1} < a_n$ and $S_{m'+1} < b$, and asks:

Does $\mathcal{N} \models \Phi$ (Φ formula of $\mathcal{L}$ with Gödel number $\leqslant n$ and involving only $\underline{K}_1, \ldots, \underline{K}_{n+1}, \underline{S}_1, \ldots, \underline{S}_{m'+1}$ among the new constant symbols), where $\underline{K}_1, \ldots, \underline{K}_{m+1}$, $\underline{S}_1, \ldots, \underline{S}_{m'+1}$ are to be interpreted by $K_1, \ldots, K_{m+1}$,

$$\{x \in I\colon p_x | S_1\}, \ldots, \{x \in I\colon p_x | S_{m'+1}\}?$$

If the answer is Yes and Φ is $\exists x\psi(x)$ then II chooses $K_{m+2}<b$ and they put $a_{n+1} = [\max(a_n, K_{m+2})]^2$ and $T_{n+1} = T_n \cup \{\psi(\underline{K}_{m+2})\}$.

If the answer is Yes and Φ is $\exists X\psi(X)$ then II chooses $S_{m'+2}<b$ and they put $a_{n+1} = a_n^2$, $T_{n+1} = T_n \cup \{\psi(\underline{S}_{m'+2})\}$.

If the answer is No then they put $a_{n+1} = a_n^2$, $T_{n+1} = T_n \cup \{\neg\Phi\}$.

Let R_{n+1} be the set of atomic formulae satisfied by $K_1, \ldots, K_{m+1}$ ($,K_{m+2}$) together with all formulae of the form

$$\underline{K}_i \in \underline{S}_j \quad \text{when} \quad p_{K_i}|S_j,$$
$$\neg\underline{K}_i \in \underline{S}_j \quad \text{when} \quad p_{K_i}\nmid S_j \quad (i\leqslant m+2,\ j\leqslant n).$$

Player I wins iff there is a proof with Gödel number $\leqslant c$ of a contradiction from

$$T_c \cup \{\underline{\theta}(\mathrm{i})\colon i<c\} \cup R_c.$$

THEOREM 15. *Let $\langle\mathcal{M}, \mathcal{X}\rangle \models$ arith-CA be countable. Then there exists $\mathcal{M}'\succ_e\mathcal{M}$ such that*

$$\mathcal{R}_{\mathcal{M}'}(\mathcal{M}) = \mathcal{X}.$$

Proof. We give sketches of two proofs. First note that this theorem is a generalization of the generalization by Phillips [Ph] and Gaifman of the MacDowell–Specker theorem [MS]. They proved it with $\mathcal{X} = \mathrm{Def}(\mathcal{M})$, but the proof can be generalized.

Secondly remark that in any model $\langle\mathcal{M}, \mathcal{X}\rangle$ of arith-CA the following two statements are true:

(i) If $F\colon M \to \{x \in M\colon \mathcal{M} \models x<a\}$ is (coded) in $\mathcal{X}$ where $a \in M$, then for some $i<a$, in M, $F^{-1}\{i\}$ is unbounded in M.

(ii) If $G\colon [M]^3 \to \{0, 1\}$ is in $\mathcal{X}$ then there is a homogeneous set for G, in $\mathcal{X}$ and unbounded in M.

(Aside: indeed, these two, together with Δ_1^0-CA. are equivalent to arith-CA. This is intimately tied up with the fact that if $I\subset_e\mathcal{M}$ then I is a strong initial segment iff $I^* =$ arith-CA. See [K], Theorem 7.5).

Now produce an ultrafilter $\mathcal{U}$ on $\mathcal{X}$ such that

(1) $X \in \mathcal{U} \Rightarrow X$ unbounded in M;

(2) if $F\colon M \to \{x\colon x<a\}$ is in $\mathcal{X}$ then for some $i<a$, $F^{-1}\{i\} \in \mathcal{U}$;

(3) if $G\colon [M]^3 \to \{0, 1\}$ is in $\mathcal{X}$ then there is $H \in \mathcal{U}$ such that $|G''[H]^3| = 1$.

This is done by listing in sequences of order type ω all F as in (2), G as in (3). Form a decreasing sequence $\{X_n\}_{n\in\omega}$ of unbounded sets in $\mathcal{X}$:

Given X_n, let F, G be the nth functions on our lists. Choose $X_n' \subseteq X_n$ such that $X_n' \in \mathcal{X}$ and X_n' is unbounded and homogeneous for F (possible by (i)). Then choose $X_{n+1} \subseteq X_n'$ such that $X_{n+1} \in \mathcal{X}$ and X_{n+1} is unbounded and homogeneous for G.

Let $\mathcal{U}$ be the filter

$$\{X \in \mathcal{X}\colon \exists n \in \omega X \supseteq X_n\}.$$

Then $\mathcal{U}$ is an ultrafilter and the ultrapower $\mathcal{M}'$ of $\mathcal{M}$ with respect to $\mathcal{U}$ is (up to isomorphism) an elementary end extension of $\mathcal{M}$.

If $A \in \mathcal{X}$, to show $A \in \mathcal{R}_{\mathcal{M}'}(\mathcal{M})$ let $E\colon M \to M$ be defined by $E(x) = x$th element of A, in ascending order. (Put $E(x) = 1$ above A if A is bounded.) Then set

$$G(0) = 1,$$
$$G(i+1) = G(i)\cdot p_{E(i)}.$$

Then $G \in \mathcal{X}$ and for $i \in M$,

$$i \in A \Leftrightarrow \mathcal{M}' \models p_i|\bar{G}$$

where $\bar{G}$ is the equivalence class of G in the ultrapower. So A is coded in $\mathcal{M}'$.

Conversely if $A \in \mathcal{R}_{\mathcal{M}'}(\mathcal{M})$, say $\bar{G}$ codes A. In $\mathcal{M}$ define

$$F(a, b, c) = \begin{cases} 0 & \text{if } \{x<a\colon p_x|G(b)\} = \{x<a\colon p_x|G(c)\}, \\ 1 & \text{otherwise}. \end{cases}$$

Then $F \in \mathcal{X}$; take the homogeneous set H guaranteed by (3).

If $F''[H]^3 = \{1\}$ then let h_0 be the least element of H; then each new element of H introduces a new subset of $\{x\colon x<h_0\}$. But (in M) there are unboundedly many elements of H and only 2^{h_0} subsets of h_0, a contradiction.

So $F''[H]^3 = \{0\}$ and

$$i \in A \Leftrightarrow \{x \in M\colon \langle\mathcal{M}, \mathcal{X}\rangle \models p_i|G(x)\} \in \mathcal{U}$$
$$\Leftrightarrow \exists y\big[y \in H \wedge \forall z \in H(z>i \to z\geqslant y) \wedge \forall z \in H\big(z>y \to p_i|G(z)\big)\big],$$

so A is in $\mathcal{X}$.

THEOREM 16. *Let T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{II}}$, $T \vdash$ arith-CA. Then the class of initial segments which are T-expandable and the class of initial segments I such that $I^* \models T$ are symbiotic.*

Remarks. Since arith-CA $\vdash P$, this together with Corollary 10 gives us the theorem initially stated.

Proof. It is enough to show that any countable T-expandable $\mathcal{M}$ has arbitrarily large initial segments I such that $I^* \models T$. But let Z be an indicator for such initial segments. Take $\mathcal{X} \subseteq \mathcal{P}(M)$ such that $\langle\mathcal{M}, \mathcal{X}\rangle \models T$. By Theorem 15 take $\mathcal{M}'\succ_e\mathcal{M}$ such that $\mathcal{R}_{\mathcal{M}'}(\mathcal{M}) = \mathcal{X}$. Let $a \in M$, $n \in \omega$. In $\mathcal{M}'$, $\mathcal{M}^* \models T$ so for any $b \in M'-M$, $Z(a, b)$ takes nonstandard value so in particular $\mathcal{M}' \models Z(a, b)>n$. By underspill there is some $b \in M$ for which this is true. Hence

$$\mathcal{M} \models \exists y Z(a, y)>n.$$

Now this is true for any $n \in \omega$ so by overspill for some $c>\omega$ $\mathcal{M} \models \exists y Z(a, y)>c$. Hence there is $I\subset_e M$, such that $a \in I$ and $I^* \models T$.

The usual sorts of corollaries follow, such as:

COROLLARY 17. *Let T be a recursively axiomatizable theory in $\mathscr{L}^{II}$ such that* $T \vdash$ arith-CA.

(i) *If $\mathscr{M}$ is a countable model of $\Pi_2^0(T)$ and $\mathscr{M}$ has an initial segment I such that $I^* \models T$ then $\mathscr{M}$ has $2^{\aleph_0}$ such initial segments.*

(ii) *If $\mathscr{M}$ is a countable model of $\Pi_2^0(T)+I\Sigma_2^0$ then $\mathscr{M}$ has arbitrarily large initial segments I such that $I^* \models T$.*

(iii) *If T has an ω-model then any countable $\mathscr{M} \models B$ has arbitrarily small nonstandard initial segments I such that $I^* \models T$.*

This is true for example if $T = A_2^-$, or $T =$ the second order consequences of ZF. Here B is basic number theory, the $\forall^2$ consequences of P, as introduced by Goldrei, Macintyre and Simmons [GMS]. Mc Allon [Mc] showed that $B = \Pi_2^0(P)$. A strengthening of the corollary can be obtained in the same way as in the remark after Corollary 11.

References

[AM] K. R. Apt and W. Marek, *Second order arithmetic and related topics*, Ann. Math. Logic 6 (1974), pp. 177–239.

[BS] J. Barwise and J. Schlipf, *An introduction to recursively saturated and resplendent models*, J. Symb. Logic 41 (1976), pp. 531–536.

[F] S. Feferman, *Arithmetization of metamathematics in a general setting*, Fund. Math. 59 (1960), pp. 35–92.

[GMS] D. Goldrei, A. Macintyre and R. Simmons, *The forcing companions of number theories*, Israel J. Math. 14 (1973) pp. 317–337.

[K] L. A. S. Kirby, *Initial Segments of Models of Arithmetic*, Ph. D. Thesis, Manchester University, 1977.

[KP] — J. Paris, *Initial segments of models of Peano's axioms*, Proc. Bierutowice Conf. 1976, Springer LNM 619, 1977, pp. 211–226.

[Ko] H. Kotlarski, *On elementary recursively saturated cuts in models of Peano arithmetic*, to appear.

[Kr] S. Krajewski, *Nonstandard satisfaction classes*, Set Theory and Hierarchy Theory, Bierutowice 1975, Springer LNM 537, pp. 121–144.

[L] H. Lessan, *Models of Arithmetic*, Ph. D. Thesis, Manchester, 1978.

[MC] K. McAloon, *Consistency statements and number theories*, Colloque International de Logique, Clermont-Ferrand, 1975, pp. 199–207.

[Mc$_2$] — *On the complexity of Models of Arithmetic*, to appear.

[MS] R. McDowell and E. Specker, *Modelle der Arithmetik*, Infinistic Methods, Pergamon Press & PWN, 1961, pp. 257–263.

[M] R. Murawski, *On expandability of models of Peano arithmetic*, Studia Logica 35 (1976), pp. 409–419, 421–431; 36 (1977), pp. 181–188.

[M$_2$] — *Models of Peano arithmetic expandable to models of fragments of second order arithmetic* (in Polish), Ph. D. Thesis, Warszawa 1978.

[P] J. B. Paris, *Some independence results for Peano arithmetic*, JSL 43 (1978), pp. 725–731.

[P$_2$] — *A combinatorial hierarchy in models of arithmetic I*, to appear.

[PK] J. B. Paris and L. A. S. Kirby, Σ_n-Collection Schemas in arithmetic, Proc. Wrocław Conf., North Holland, 1978, pp. 199–209.

[Ph] R. G. Phillips, *Omitting types in arithmetic and conservative extensions*, Springer LNM 369, 1972, pp. 195–202.

[S] J. Schlipf, *Toward model theory through recursive saturation*, JSL 43 (1978), pp. 183–206.

[Sm] C. Smorynski, *Recursively saturated nonstandard models of arithmetic* (mimeographed).

UNIVERSITÉ DE PARIS VII
U.E.R. DE MATHEMATIQUE
Paris

PRINCETON UNIVERSITY
DEPT OF MATHEMATICS
Princeton, New Jersey

UNIVERSITÉ DE PARIS VII
U.E.R. DE MATHEMATIQUE
Paris

UNIVERSITY OF WARSAW
INSTITUTE OF MATHEMATICS
Warszawa

ADAM MICKIEWICZ UNIVERSITY
INSTITUTE OF MATHEMATICS
Poznań

Accepté par la Rédaction le 6. 12. 1979