take $g = h = f$. Assume $\exists n f(n) = 1$. Take g and h such that $x \in \mathscr{A}(g)$, $y \in \mathscr{A}(h)$. Let $g(m) = 0$, hence $x \notin A_m$ and for every k $\langle x, y\rangle \notin A_m \times A_k$. We infer that $f(J(m, k)) = 0$. Let $g(m) = 1$, hence $x \in A_m$. On the other hand we have $\langle x, y\rangle \in A_{K(n)} \times A_{L(n)}$, hence $y \in A_{L(n)}$ and we get $\langle x, y\rangle \in A_m \times A_{L(n)}$, so

$$f(J(m, L(n))) = 1 .$$

For h we argue similarly and thus one inclusion is proved. Now take $f \in \mathscr{F} \cap G$ and appropriate functions $g, h \in \mathrm{Rg}(\varphi_{\mathscr{A}})$. We claim that if $x \in \mathscr{A}(g)$, $y \in \mathscr{A}(h)$ then $\langle x, y\rangle \in \mathscr{A}^*(f)$ and hence $f \in \mathrm{Rg}(\varphi_{\mathscr{A}^*})$. Indeed for every $m \in \omega$ we have:

$$\begin{aligned}\langle x, y\rangle \in A^*_m &\equiv \langle x, y\rangle \in A_{K(m)} \times A_{L(m)} \equiv x \in A_{K(m)} \,\&\, y \in A_{L(m)}\\ &\equiv g(K(m)) = 1 \,\&\, h(L(m)) = 1\\ &\equiv \exists k f(J(K(m), k) = 1 \,\&\, \exists l f(J(l, L(m)) = 1\\ &\equiv f(J(K(m), L(m)) = 1 \equiv f(m) = 1.\end{aligned}$$

This proves that $\mathrm{Rg}(\varphi_{\mathscr{A}^*}) = \mathscr{F} \cap G$ and finishes the proof of our theorem. ■

It would be interesting to find a necessary and sufficient condition for the existence of a preserving injection for a sequence in terms of its components. We would like to state this problem as a natural remainder of Ulam's question.

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Monotone decompositions of hereditarily smooth continua

by

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Abstract. It is proved that if a Hausdorff compact continuum X is hereditarily smooth at a certain point (see below), then there is an upper semi-continuous decomposition $\mathfrak{D}$ of X into continua such that the quotient space $X/\mathfrak{D}$ is arcwise connected and hereditarily smooth and $\mathfrak{D}$ is minimal with respect to these properties. This result generalizes theorems obtained by Gordh [3] and by Maćkowiak [6].

1. Introduction. A *continuum* is a compact connected Hausdorff space. A continuum I is *irreducible between its points a and b* if no proper subcontinuum of I contains them. The symbol $I(a, b)$ always denotes a continuum irreducible between a and b. We use the following notation: $\mathrm{cl}\,A$ ($\mathrm{int}\,A$) denotes the closure (the interior) of A. A continuum X is *smooth at a point p* [4], [7] provided that for each subcontinuum K of X such that $p \in K$ and for each open set V which includes K, there is an open connected set U such that $K \subset U \subset V$. The following is well known [7].

PROPOSITION 1. *Let p be a point of a continuum X. Then the following conditions are equivalent*:

(i) *X is smooth at p*,

(ii) *for each convergent net $x_n \in X$ with* $\lim x_n = x$ *and for each continuum $I(p, x)$ irreducible between p and x there are continua $I(p, x_n)$ each one irreducible between p and x_n such that* $\mathrm{Lim}\, I(p, x_n) = I(p, x)$,

(iii) *for each subcontinuum K of X containing p and for each convergent net $\{x_n,\ n \in D\}$ with* $\lim x_n = x \in K$ *there is a net $\{K_i,\ i \in E\}$ of subcontinua of X such that each K_i contains a certain x_n and p and* $\mathrm{Lim}\, K_i = K$ (*if K is irreducible, then it is possible to have each K_i irreducible also*).

A continuum X is *hereditarily unicoherent at a point p* [3] if the intersection of any two subcontinua of X, each of which contains p, is connected. Any Hausdorff compactification αJ of the set J consisting of the interval $[0, 1)$ of reals and of a circle S such that $[0, 1) \cap S = \{0\}$ is a continuum which is smooth at each point of J but not hereditarily unicoherent at any point of $\alpha J \setminus J$. A continuum X is *hereditarily smooth at a point p* if each subcontinuum of X containing p is smooth at p.

If X is either hereditarily unicoherent at p or irreducible, then X is hereditarily smooth at p [3], p. 55, but the notion of smoothness is not hereditary in general, even for a metrizable X (cf. [4], p. 94).

THEOREM 1. *Let a continuum X be hereditarily smooth at p and let Q be a subcontinuum of X. If there is a continuum N in X irreducible between p and q and $N\cap Q=\{q\}$, then Q is hereditarily smooth at q.*

Proof. Take a subcontinuum M of Q containing q and a net $\{x_n,\ n\in D\}$ of points of M converging to x. Take a subcontinuum of M containing q and x, say K. Since X is hereditarily smooth at p, the continuum $N\cup M$ is smooth at p. Therefore there is a net $\{R_i,\ i\in E\}$ of subcontinua of $N\cup M$ such that each R_i contains p and a certain x_n and $\operatorname{Lim} R_i=N\cup K$. Put $K_i=M\cap R_i$. Obviously the net $\{K_i,\ i\in E\}$ satisfies the required conditions and hence M is smooth at q.

Remark. If a continuum X is arcwise connected and hereditarily smooth at a certain point, then each subcontinuum of X is hereditarily smooth, and therefore X is hereditarily decomposable (because each subcontinuum of X is locally connected at a certain point).

An *arc* is a continuum (not necessarily metrizable) ordered by a linear order $\leqslant$. Symbol a_0a_1 always denotes the arc with endpoints a_0 and a_1 with $a_0\leqslant a_1$. We use also the notation $[a_0,a_1]$ or a_0bca_1 or $[a_0,b,c,a_1]$ for a_0a_1 provided $b,c\in a_0a_1$ and $a_0\leqslant b\leqslant c\leqslant a_1$. Also $[a_0,a_1)=a_0a_1\setminus\{a_1\}$, $(a_0,a_1)=a_0a_1\setminus\{a_0,a_1\}$.

A subcontinuum K of a continuum X is said to be a *continuum of convergence in X* if K is a limit of a net K_n of mutually disjoint subcontinua of X such that $K_n\cap K\neq\emptyset$.

2. Irreducible subcontinua of hereditarily smooth continua. The main purpose of this section is Theorem 2. The very technical proof is, perhaps, necessary.

An arbitrary irreducible continuum $I(x,y)$ which is smooth at a certain point admits a monotone mapping f onto an arc a_0a_1 each point-inverse of which is a layer of $I(x,y)$ (cf. [2], Theorem 2.7, p. 650 and [3], Corollary 3.3, p. 55). A layer $f^{-1}(t)$ is said to be of *left* (*right*) *cohesion* if either $t=a_0$ or $f^{-1}(t)\subset\operatorname{cl}f^{-1}([a_0,t))$ (either $t=a_1$ or $f^{-1}(t)\subset\operatorname{cl}f^{-1}((t,a_1])$) ($f(x)=a_0$ is assumed). If $I(x,y)$ is smooth at a point p, then each layer $f^{-1}(t)$ satisfying $a_0\leqslant t<f(p)$ is of right cohesion, each layer $f^{-1}(t)$ satisfying $f(p)<t\leqslant a_1$ is of left cohesion and $f^{-1}(f(p))=\{p\}$ (the proof is obtained by generalizing the argument of [1]).

LEMMA 1. *A continuum $I(x,y)$ is supposed to be smooth at a point b. Take $I(a,b)\subset I(x,y)$. Then either each layer of $I(a,b)$ is a layer of $I(x,y)$, or $I(a,b)$ is contained in a layer of $I(x,y)$.*

Proof. Let $\{f^{-1}(t):\ t\in a_0a_1\}$ be the decomposition of $I(x,y)$ into layers. It follows by [3], Corollary 3.2, p. 55 that $I(x,y)$ is hereditarily unicoherent at b. Suppose that $f(a)<f(b)$. Thus $I(a,b)\subset f^{-1}([f(a),f(b)])$. It is clear that $f^{-1}([a_0,f(a)])\cup I(a,b)\cup f^{-1}([f(b),a_1])=I(x,y)$. Therefore $f^{-1}((f(a),f(b)))\subset I(a,b)$, and hence $f^{-1}([f(a),f(b)])\subset\operatorname{cl}f^{-1}([f(a),f(b)])\subset I(a,b)$ because $f^{-1}(f(b))=\{b\}$ and $f^{-1}(f(a))$ is of right cohesion. Consequently, $I(a,b)=f^{-1}([f(a),f(b)])$ which implies that each layer of $I(a,b)$ is a layer of $I(x,y)$. If $f(a)=f(b)$, then $I(a,b)$ is contained in a layer of $I(x,y)$.

LEMMA 2. *A continuum $I(x,y)$ is supposed to be smooth at x. Let*

$$\{f^{-1}(t):\ t\in a_0a_1\}$$

be the decomposition of $I(x,y)$ into layers. Take $I(a,b)\subset I(x,y)$ such that $f(a)<f(b)$. Then $f^{-1}(b)$ is a layer of $I(a,b)$.

Proof. It is clear that $f^{-1}([a_0,f(a)])\cup I(a,b)\cup f^{-1}([f(b),a_1])=I(x,y)$. Thus $f^{-1}((f(a),f(b))\subset I(a,b)$ and hence $f^{-1}(f(b))\subset I(a,b)$ and $f^{-1}(f(b))$ is nowhere dense in $I(a,b)$ because $f^{-1}(f(b))$ is of left cohesion in $I(x,y)$. Since each $f^{-1}(f(t))$ with $f(a)<t<f(b)$ separates $I(a,b)$ (cf. [2], Theorem 2.3, p. 649) we infer that $f^{-1}(f(b))=\{b'\in I(a,b):\ I(a,b)=I(a,b')\}$. Thus $f^{-1}(f(b))$ is a layer of $I(a,b)$ (cf. [2], Corollary 2.2, p. 649).

LEMMA 3. *Let a continuum X be hereditarily smooth at p, let L be a subcontinuum of X containing p and let K be a continuum of convergence in X. Then the set $K\cap L$ is connected.*

The proof is obtained by generalizing the argument of [6], Theorem 3, p. 26. The details are omitted.

LEMMA 4. *Let $I(x,y)$ be smooth at a certain point and let T be a layer of $I(x,y)$. If $x_1,x_2\in T$, then there is a continuum of convergence K such that $x_1,x_2\in K\subset T$.*

Proof. Let $\{f^{-1}(t):\ t\in a_0a_1\}$ be the decomposition of $I(x,y)$ into layers. We can assume that $T=f^{-1}(t)$ is of left cohesion. Let $\mathscr{U}_i$ denote the family of all neighbourhoods of x_i and put $D_i=\bigcup\{f(U):\ U\in\mathscr{U}_i\}\setminus[t,a_1]$, $i=1,2$. There is a family, say D, consisting of mutually disjoint subarcs m_1m_2 of a_0a_1, $m_i\in D_i$, such that $\operatorname{Lim}\{m_1m_2,\ m_1m_2\in D\}=\{t\}$. Observe that the net $\{f^{-1}(m_1m_2),\ m_1m_2\in D\}$ has a convergent subnet, say $\{K_n,\ n\in D\}$. By construction,

$$x_1,x_2\in\operatorname{Lim\,inf}\{f^{-1}(m_1m_2),\ m_1m_2\in D\}\subset\operatorname{Lim\,inf}\{K_n,\ n\in D\}=\operatorname{Lim}\{K_n,\ n\in D\}\subset T$$

and hence $K=\operatorname{Lim}K_n$ is a continuum of convergence as required.

LEMMA 5. *With X and L as in Lemma 3, take $I(p,x)$ included in X and a layer T of $I(p,x)$. Then the set $T\cap L$ is connected.*

Proof. Fix a point $x_1\in T$. Taking a point $x_2\in T$ we have a continuum of convergence, say $K(x_1,x_2)$, which contains x_1 and x_2 and is included in T (Lemma 4). Thus $T=\bigcup\{K(x_1,x_2):\ x_2\in T\}$ and hence $T\cap L=\bigcup\{K(x_1,x_2)\cap L:\ x_2\in T\}$ is connected by Lemma 3.

LEMMA 6. *Let a continuum $X=I(x,y)\cup pxy$ be hereditarily smooth at p and $py\cap I(x,y)=\{x\}$ (the case $p=x$ is acceptable). Let $\{f^{-1}(t),\ t\in a_0a_1\}$ be the decompositions of $I(x,y)$ into layers. Let $(t_0,t_1)\subset[a_0,a_1]$ be a component of the set $\{t\in a_0a_1,\ f^{-1}(t)\cap py=\emptyset\}$. Put $K=\operatorname{cl}f^{-1}((t_0,t_1))$, $K_0=K\cap f^{-1}(t_0)$, $K_1=K\cap f^{-1}(t_1)$. Then $K_0\cup K_1\subset py$.*

Proof. We can assume without loss of generality that there are points $b_0, b_1 \in py$ such that $py = [p, b_0, b_1, y]$ and $pb_0 \cap f^{-1}(t_0) = \{b_0\}$ and $pb_1 \cap f^{-1}(t_1) = \{b_1\}$. It is clear that $I(x, y)$ is smooth at x and hence each layer of $I(x, y)$ is of left cohesion. Take $t \in (t_0, t_1)$ and consider a continuum $N \subset I(x, y)$ irreducible from $f^{-1}(t)$ to $f^{-1}(t_1)$. By Lemma 2 the set $f^{-1}(t_1)$ is a layer of N. We infer that $py \cap f^{-1}(t_1) = py \cap N$ and hence N is smooth at a certain point of $f^{-1}(t_1)$ (Theorem 1). Thus $f^{-1}(t_1) = \{b_1\}$, i.e., $K_1 = \{b_1\}$. If $b_0 \in K_0$ then take $b_0' \in pb_0$ such that $pb_0' \cap K_0 = \{b_0'\}$. Thus K is hereditarily smooth at b_0 and hence $K_0 = \{b_0'\}$. Therefore we can assume that

$$(*) \qquad pb_1 \cap K_0 = \emptyset .$$

Take a subarc $db_1 \subset b_0 b_1$ such that $db_1 \cap f^{-1}([a_0, t_0]) = \{d\}$.

Case 1. $f(d) < t_0$. Take $s \in a_0 a_1$ with $f(d) < s < t_0$. Put $J = db_1 \cup \operatorname{cl} f^{-1}((s, t_1])$. Then J is a continuum irreducible between b_1 and each point of $f^{-1}(s) \cap J$. Consequently, J is smooth at b_1 (Theorem 1) and hence $f^{-1}(t_0)$ is a layer of J (Lemma 2) such that $f^{-1}(t_0) \subset K$. Thus $b_0 \in K_0$ which contradicts to $(*)$.

Case 2. $f(d) = t_0$. Take a point $c \in K_0$. By Lemma 4 there is a convergent net $\{R_n, n \in D\}$ of mutually disjoint subcontinua of $f^{-1}([a_0, t_0))$ such that $c, d \in \operatorname{Lim} R_n \subset f^{-1}(t_0)$. The continuum $M = pd \cup f^{-1}([a_0, t_0])$ is smooth at p and $pd \cap K = \emptyset$ (here pd is a subarc of pb_1), thus there is a continuum Q such that $pd \subset \operatorname{int} Q \subset Q \subset M \setminus K$ (here int denotes the interior in M). Since each $R_n \subset M$ and $c, d \in \operatorname{Lim} R_n$ we can assume that $R_n \cap Q \neq \emptyset$ for each n and that there is a net $\{c_n, n \in D\}$, $c_n \in R_n$, such that $\lim c_n = c$. Take continua $I(c_n, q_n) \subset R_n$ irreducible between c_n and Q. We can assume that $\lim q_n = q \in Q$. Put

$$P = Q \cup db_1 \cup f^{-1}([t_0, t_1]) \cup \operatorname{cl} \bigcup \{I(c_n, q_n)\colon n \in D\} \quad \text{and} \quad R = pb_1 \cup K .$$

The continuum R is irreducible between p and c. Thus by the smoothness of P at p there are continua $I(p, c_n) \subset P$ such that $\operatorname{Lim} I(p, c_n) = R$. By the definition of P we can assume that $I(c_n, q_n) \subset I(p, c_n)$ and hence $L = \operatorname{Lim\,sup} I(c_n, q_n)$ is a continuum included in $R \cap f^{-1}(t_0)$. Furthermore, $L = (pb_1 \cap L) \cup (K \cap L)$ and $c \in K \cap L$. We have either $pb_1 \cap L = \emptyset$ or $pb_1 \cap L \cap K \neq \emptyset$. If $pb_1 \cap L = \emptyset$, then $q \in L \subset K$. But $q \in Q$ and hence $q \in Q \cap K$. On the other hand $Q \cap K = \emptyset$, a contradiction. If $pb_1 \cap L \cap K \neq \emptyset$ then $pb_1 \cap f^{-1}(t_0) \cap K \neq \emptyset$, a contradiction. This completes the proof.

LEMMA 7. *Let a continuum X be hereditarily smooth at a point p. Let $I(a, b) \subset X$ and let T be a layer of $I(a, b)$. If there is an arc py such that $py \cap T = \{y\}$, then $T = \{y\}$.*

Proof. If $py \cap T = py \cap I(a, b)$, then $I(a, b)$ is smooth at y and $T = \{y\}$. Assume that T is non-degenerate and $py \cap I(a, b) \neq \{y\}$. Take an arc $px \subset py$ such that $px \cap I(a, b) = \{x\}$ and a continuum $I(x, y) \subset I(a, b)$. Let $\{f^{-1}(t), t \in a_0 a_1\}$ be the decomposition of $I(x, y)$ into layers. Then $I(x, y)$ is smooth at x and $f^{-1}(a_0) = \{x\}, f^{-1}(a_1) = T$. Each component of the set $\{t \in a_0 a_1\colon f^{-1}(t) \cap px = \emptyset\}$

is an interval of the form (t_0, t_1). Denote all of them by $\{(t_0^\alpha, t_1^\alpha)\colon \alpha \in \mathscr{A}\}$ and put

$$R = \big(I(x, y) \cap py\big) \cup \bigcup \{f^{-1}((t_0^\alpha, t_1^\alpha))\colon \alpha \in \mathscr{A}\} .$$

Take $s \in a_0 a_1$. If $f^{-1}(s) \cap py = \emptyset$, then $s \in (t_0^\alpha, t_1^\alpha)$ for a certain α, and hence $f^{-1}(s) \subset R$. If $f^{-1}(s) \cap py \neq \emptyset$, then $f^{-1}(s) \cap R = f^{-1}(s) \cap py$, and hence the set $f^{-1}(s) \cap R$ is connected. If R were closed, then the partial mapping $f|R$ would be monotone and R would be a proper subcontinuum of $I(x, y)$ containing x and y, a contradiction. Therefore R is not closed. Since $I(x, y) \cap py$ is closed there is a convergent net $\{q_n, n \in D\}$ of points of $\bigcup \{f^{-1}((t_0^\alpha, t_1^\alpha))\colon \alpha \in \mathscr{A}\}$ such that $\lim q_n = q \in I(x, y) \setminus R$. Thus

$$(*) \qquad f^{-1}(f(q)) \cap py \neq \emptyset \quad \text{and} \quad q \notin py .$$

Suppose that there were a subnet $\{q_i, i \in E\}$ of $\{q_n, n \in D\}$ and a certain $\alpha \in \mathscr{A}$ such that each q_i is contained in (t_0^α, t_1^α). Then $\lim f(q_i) = f(q)$ and by $(*)$ either $f(q) = t_0^\alpha$ or $f(q) = t_1^\alpha$. But then $q \in \operatorname{cl} f^{-1}((t_0^\alpha, t_1^\alpha)) \setminus f^{-1}((t_0^\alpha, t_1^\alpha)) \subset py$ according to Lemma 6, contrary to $(*)$. Therefore there is no such a subnet. Define $\alpha_n \in \mathscr{A}$ by $q_n \in f^{-1}((t_0^{\alpha_n}, t_1^{\alpha_n}))$ and choose $t_n \in f^{-1}((t_0^{\alpha_n}, t_1^{\alpha_n}))$. By Lemma 6, each $f^{-1}(f(t_1^{\alpha_n}))$ is degenerate and we can assume that the net $\{f^{-1}(f(t_1^{\alpha_n})), n \in D\}$ is convergent to a point $z \in py$. For $n \in D$ choose an arbitrary $z_n \in I(x, y)$ such that $t_0^{\alpha_n} < f(q_n) < f(z_n) < t_1^{\alpha_n}$. We can assume that the net $\{z_n, n \in D\}$ is convergent. Thus $\lim z_n = z$ and $z, q \in \operatorname{Lim\,sup} f^{-1}((t_0^{\alpha_n}), f(z_n)) = S$ and S is contained in a layer of $I(x, y)$. Put

$$P = py \cup S \cup \operatorname{cl} \bigcup \{f^{-1}((t_0^{\alpha_n}, f(z_n)))\colon n \in D\} .$$

The continuum P is smooth at p and hence there are continua $I(p, z_n) \subset P$ such that $\operatorname{Lim} I(p, z_n) = pz$ (here pz is a subarc of py). Each q_n is contained in $I(p, x_n)$, and hence $q \in pz$. But this contradicts to $(*)$. The proof is complete.

PROPOSITION 2. *Let a continuum X be hereditarily smooth at a point p. Let $f\colon X \to Y$ be a monotone mapping of X onto a Hausdorff space Y. Then Y is a continuum hereditarily smooth at $f(p)$.*

Proof. Suppose that Q is a subcontinuum of Y containing $f(p)$. Then $f^{-1}(Q)$ is a subcontinuum of X containing p, and hence $f^{-1}(Q)$ is smooth at p. Applying Proposition 1 to the proof of [4], Theorem 6.2, p. 90, we obtain the conclusion of the proposition.

THEOREM 2. *Let a continuum X be hereditarily smooth at a point p. If T_i are layers of the point x in irreducible continua $I_i(p, z)$, $i = 1, 2$, then $T_1 = T_2$.*

Proof. The continuum $K = I_1(p, x) \cup I_2(p, x)$ is hereditarily smooth at p. Define a relation ϱ on K by: $y \varrho z$ if and only if either $y = z$ or y and z lie in the same layer of $I_2(p, x)$. Let φ denote the quotient mapping of K onto K/ϱ. It is clear that $\varphi | I_2(p, x)$ is monotone and $\varphi(I_2(p, x))$ is an arc joining $\varphi(p)$ and $\varphi(x)$ in K/ϱ. By Lemma 5 the set $\varphi^{-1}(t) \cap I_1(p, x)$ is connected and by Lemma 7, $T_2 = \{x\}$. Then $\varphi(I_1(p, x))$ is a continuum irreducible between $\varphi(p)$ and $\varphi(x)$. Also K/ϱ is hereditarily smooth at $\varphi(p)$ by Proposition 2. Thus $T_1 \subset \varphi^{-1}\varphi(T_1) = \varphi^{-1}\varphi(x) = T_2$. The conclusion of the theorem follows because the role of T_1 and T_2 is symmetric.

COROLLARY. *Let a continuum X be hereditarily smooth at a point p. If $z \in I(p,x) \cap I(p,y)$ and T and S are layers of z in these continua, then $T = S$.*

Proof. It is an immediate consequence of Theorem 2.

3. Monotone decompositions of hereditarily smooth continua. Throughout this section let X denote a continuum which is hereditarily smooth at a point p. Define a relation ϱ on X by the condition:

$x\varrho y$ if and only if there are continua $I(p,x)$ and $I(p,y)$ such that $I(p,x) = I(p,y)$.

PROPOSITION 3. *The relation ϱ is reflexive, symmetric and transitive. The equivalence classes of ϱ are precisely layers of continua of the form $I(p,x)$.*

Proof. It is clear that ϱ is reflexive and symmetric. That ϱ is transitive follows from Theorem 2. Now if $T \subset I(p,x)$ is a layer of the point x, then T is contained in the equivalence class $[x]_\varrho$ of x with respect to ϱ. It follows by Corollary that $[x]_\varrho = T$.

For a continuum M, Vought [8] has obtained a unique monotone upper semi-continuous decomposition $\mathscr{D}$ of M such that layers of the irreducible subcontinua of M are contained in the elements of $\mathscr{D}$ and $\mathscr{D}$ is minimal with respect to these properties. We prove a similar theorem for hereditarily smooth continua. The following is a generalization of [3], Theorem 5.2, p. 58 and of [6], Theorem 12, p. 31.

THEOREM 3. *Let a continuum X be hereditarily smooth at a point p. Then the decomposition $\mathscr{D}$ of X induced by ϱ satisfies the following conditions:*

(i) *$\mathscr{D}$ is upper semi-continuous and monotone,*

(ii) *the quotient space X/ϱ is arcwise connected,*

(iii) *$\mathscr{D}$ is minimal among all decompositions of X satisfying* (i) *and* (ii) *(i.e., if $\mathscr{D}'$ satisfies* (i) *and* (ii) *then $\mathscr{D}$ refines $\mathscr{D}'$).*

Moreover, each element of $\mathscr{D}$ has empty interior, X/ϱ is hereditarily smooth at $[p]_\varrho$ and X/ϱ is hereditarily arcwise connected.

Proof. In order to prove (i) it suffices to show that ϱ is a closed subset of $X \times X$. Let $\{(x_n, y_n), n \in D\}$ be a net in ϱ which converges to $a(x,y)$. Let $I(p,x)$ be an irreducible subcontinuum of X. By the smoothness of X at p there are continua $I(p,x_n)$ such that $\operatorname{Lim} I(p,x_n) = I(p,x)$. Since $x_n \varrho y_n$ there are continua $I(p,x_n)$ and $I(p,y_n)$ such that $I(p,x_n) = I(p,y_n)$. Thus y_n belongs to the layer of x_n in $I(p,x_n)$. Therefore $y_n \in I(p,x_n)$, and hence $y \in I(p,x)$. Take an irreducible continuum $I(p,y) \subset I(p,x)$. As above we obtain $x \in I(p,y)$. Thus $I(p,x) = I(p,y)$, hence $x\varrho y$.

(ii) Let φ denote the quotient mapping of $\mathscr{D}$. If $\varphi(x)$ is an arbitrary point of $X/\varrho \setminus \{\varphi(p)\}$, take a continuum $I(p,x) \subset X$. By Proposition 3, $\varphi(I(p,x))$ is an arc in X/ϱ joining $\varphi(p)$ and $\varphi(x)$. Thus X/ϱ is arcwise connected.

(iii) Suppose that there is a mapping $\psi\colon X \to H$ such that the decomposition $\mathscr{E} = \{\psi^{-1}(h), h \in H\}$ satisfies (i) and (ii). If $\mathscr{D}$ does not refine $\mathscr{E}$ then there exists

an element $D \in \mathscr{D}$ and elements E_1 and E_2 of $\mathscr{E}$ such that $E_i \cap D \neq \emptyset$, $i = 1, 2$. Since H is arcwise connected, there exists an arc A in H containing the points $\psi(p)$ and $\psi(E_1)$ but missing $\psi(E_2)$. Now $\psi^{-1}(A)$ is a continuum which contains p and intersects D properly. This contradicts to the definition of $\mathscr{D}$; consequently, $\mathscr{D}$ refines $\mathscr{E}$.

It follows from Proposition 2 that X/ϱ is hereditarily smooth at $\varphi(p)$, thus X/ϱ being arcwise connected is hereditarily arcwise connected by Lemma 7. That the elements of $\mathscr{D}$ have empty interior follows by Proposition 3.

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