On components of MANR-spaces

by

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Abstract. A metrizable space X is a MANR-space [5] if and only if all components of X are MANR-spaces open in X. For two MANR-spaces having the same shape in the sense of Fox [4] there exists one-to-one correspondence between components of these spaces such that corresponding components have the same shape.

The notion of MANR-space introduced by the author in [5] and studied in [6] and [7] is a generalization of the notion of FANR-space introduced and studied by K. Borsuk in [3]. In the case of compacta these notions coincide ([5], p. 62). In [3] K. Borsuk proved that components of a FANR-space X are FANR-spaces open in X. In this paper we obtain analogous result for components of MANR-spaces. In [2] K. Borsuk has proved that for compacta having the same shape there exists one-to-one correspondence between components of these compacta such that corresponding components have the same shape. It is not known if it is true for arbitrary metrizable spaces when we consider shape in the sense of Fox [4]. In this paper we obtain the analogous result for MANR-spaces.

§ 1. Shape in the sense of Fox and connectivity. First, let us recall the basic notions of Fox shape theory [4].

Let X be a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space P. The family $\boldsymbol{U}(X, P)$ of all open neighborhoods of X in P is called the *complete neighborhood system* of X in P.

Consider two arbitrary complete neighborhood systems $\boldsymbol{U}(X, P)$ and $\boldsymbol{V}(Y, Q)$. A *mutation* $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ is defined as a collection of maps $f\colon U \to V$, where $U \in \boldsymbol{U}(X, P)$, $V \in \boldsymbol{V}(Y, Q)$ such that

(1.1) If $f \in \boldsymbol{f}$, $f\colon U \to V$, $U' \in \boldsymbol{U}(X, P)$, $U' \subset U$, $V \subset V' \in \boldsymbol{V}(Y, Q)$, and $f'\colon U' \to V'$ is defined by $f'(x) = f(x)$ for $x \in U'$, then $f' \in \boldsymbol{f}$;

(1.2) Every neighborhood $V \in \boldsymbol{V}(Y, Q)$ is the range of a map $f \in \boldsymbol{f}$;

(1.3) If $f_1, f_2 \in \boldsymbol{f}$ and $f_1, f_2\colon U \to V$, then there exists $U' \in \boldsymbol{U}(X, P)$ such that $U' \subset U$ and $f_1|U' \simeq f_2|U'$.

It is easy to see that the condition (1.3) may be replaced by the following condition

(1.3′) If $f_1, f_2 \in \boldsymbol{f}$ and $f_i\colon U_i \to V_i$ for $i = 1, 2$ then there exists $U' \in \boldsymbol{U}(X, P)$ such that $U' \subset U_1 \cap U_2$ and $f_1|U' \simeq f_2|U'$ in $V_1 \cup V_2$.

Consider two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{W}(Z, R)$. The *composition* $\boldsymbol{gf}\colon \boldsymbol{U}(X, P) \to \boldsymbol{W}(Z, R)$ of the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ is the collection of all compositions gf such that $f \in \boldsymbol{f}$, $g \in \boldsymbol{g}$, and gf is defined.

For any complete neighborhood system $\boldsymbol{U}(X, P)$ the collection $\boldsymbol{u}$ of all inclusions $u\colon U' \to U$, where $U, U' \in \boldsymbol{U}(X, P)$ and $U' \subset U$, is a mutation from $\boldsymbol{U}(X, P)$ to itself. It is called the *identity mutation* for the system $\boldsymbol{U}(X, P)$, because $\boldsymbol{uf} = \boldsymbol{f}$ and $\boldsymbol{gu} = \boldsymbol{g}$ whenever the compositions $\boldsymbol{uf}$ and $\boldsymbol{gu}$ are defined.

Two mutations $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ are *homotopic* (notation $\boldsymbol{f} \simeq \boldsymbol{g}$) if

(1.4) For every $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$ such that $f, g\colon U \to V$ there exists $U' \in \boldsymbol{U}(X, P)$ such that $U' \subset U$ and $f|U' \simeq g|U'$.

By the Kuratowski–Wojdysławski theorem ([1], p. 78) any metrizable space X may be considered as a closed subset of an ANR($\mathfrak{M}$)-space P.

Two metrizable spaces X and Y are said to be of the same *shape* in the sense of Fox (notation $\operatorname{Sh} X = \operatorname{Sh} Y$) if there exist two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that

(1.5) $$\boldsymbol{fg} \simeq \boldsymbol{v} \quad \text{and} \quad \boldsymbol{gf} \simeq \boldsymbol{u},$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are identity mutations for the systems $\boldsymbol{U}(X, P)$ and $\boldsymbol{V}(Y, Q)$, respectively. By Theorem (3.2) of [4] the choice of ANR($\mathfrak{M}$)-spaces P, Q and the manner of imbedding of X and Y in P and Q, respectively, is immaterial. If the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the first of conditions (1.5) then we say that the shape of X (in the sense of Fox) dominates the shape of Y (notation $\operatorname{Sh} X \geqslant \operatorname{Sh} Y$).

Let us prove the following

(1.6) THEOREM. *If X is a connected metrizable space and $\operatorname{Sh} X \geqslant \operatorname{Sh} Y$, then Y is connected, i.e. connectivity is a hereditary shape invariant.*

Proof. By the hypothesis there exist two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that $\boldsymbol{fg} \simeq \boldsymbol{v}$. Let us observe that

(1.7) For every neighborhood $U \in \boldsymbol{U}(X, P)$ there exists a connected neighborhood $U' \in \boldsymbol{U}(X, P)$ contained in U.

Indeed, by the theorem of Hanner ([1], p. 96) $U \in \mathrm{ANR}(\mathfrak{M})$ and hence it is locally connected. Therefore for every $x \in X$ there exists a connected neighborhood U_x of x contained in U. Setting $U' = \bigcup_{x \in X} U_x$ we get the required neighborhood.

Suppose, to the contrary, that Y is not connected. Then $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are non-empty, disjoint sets, closed in Y, and hence closed in Q. By the normality of Q there exist disjoint neighborhoods V_1 and V_2 of Y_1 and Y_2, respectively. Let us set $V = V_1 \cup V_2$. Obviously $V \in \boldsymbol{V}(Y, Q)$. By (1.2) there exists $f \in \boldsymbol{f}$ such that $f\colon U \to V$. By (1.1) and (1.7) we can assume that the neighborhood U is connected. By (1.2) there exists $g \in \boldsymbol{g}$ such that $g\colon V' \to U$, and by (1.1) we can assume that $V' \subset V$. Consider the inclusion $v\colon V' \to V$, $v \in \boldsymbol{v}$. Since $\boldsymbol{fg} \simeq \boldsymbol{v}$, then by (1.4) for sufficiently small neighborhood V' we have $fg \simeq v\colon V' \to V$. So, we have

$$Y_1 \cup Y_2 = Y \subset V' \subset V = V_1 \cup V_2 \quad \text{and} \quad Y_i \subset V_i \quad \text{for} \quad i = 1, 2.$$

Let us set

$$V_i' = V' \cap V_i \quad \text{for} \quad i = 1, 2.$$

Consider the maps $f\colon U \to V_1 \cup V_2$ and $g\colon V_1' \cup V_2' \to U$. Since U is connected and V_1 and V_2 are disjoint neighborhoods, then $f(U) \subset V_1$ or $f(U) \subset V_2$. Suppose $f(U) \subset V_1$. Then $fg(V_1' \cup V_2') \subset V_1$. Since $v \simeq fg$, then $v(V_1' \cup V_2') \subset V_1$, but it is not possible because $v(V_2') \subset V_2$.

Thus, the proof is finished.

(1.8) COROLLARY. *If X is a connected metrizable space and $\operatorname{Sh} X = \operatorname{Sh} Y$, then Y is connected, i.e. connectivity is shape invariant.*

§ 2. MAR and MANR-spaces. Let us recall some notions introduced in [5] and studied in [6] and [7].

Let X be a closed subset of a metrizable space X' considered as a closed subset of an ANR($\mathfrak{M}$)-space P. A mutation $\boldsymbol{r}\colon \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ is called a *mutational retraction* ([5], p. 52) if $r(x) = x$ for every $r \in \boldsymbol{r}$ and every $x \in X$. A closed subset X of a metrizable space X' is called a *mutational retract* ([5], p. 53) of X' if there exists a mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$. A metrizable space X is called a *mutational absolute retract* (shortly MAR, [5], p. 57) if for every metrizable space X' containing X as a closed subset, the set X is a mutational retract of X'. A closed subset X of a metrizable space X' is called a *mutational neighborhood retract* ([5], p. 56) of X' if there exists a closed neighborhood W of X in X' such that X is a mutational retract of W. A metrizable space X is said to be a *mutational absolute neighborhood retract* (shortly MANR, [5], p. 57) if for every metrizable space X' containing X as a closed subset, the set X is a mutational neighborhood retract of X'. By the *trivial shape* we mean the shape of a space consisting of one point.

In [6] (Theorem (3.5), p. 90) we have proved the following

(2.1) THEOREM. *A metrizable space X is a* MAR-*space if and only if the shape* $\operatorname{Sh} X$ *is trivial.*

From Corollary (1.8) and Theorem (2.1) we obtain the following

(2.2) COROLLARY. MAR-*spaces are connected.*

§ 3. Some properties of components of MANR-**spaces.** The notions of MAR and MANR-space are generalizations of the notions of FAR and FANR-space, respectively, introduced by K. Borsuk [3]. In the case of compacta these notions coincide ([5], (4.2), (4.4), (5.8), pp. 57, 62). K. Borsuk has proved ([2], p. 193) that components of a FANR-space X are FANR-spaces open in X. So, it is natural to consider the following

(3.1) QUESTION. Is it true that every component of a MANR-space X is a MANR-space open in X?

We are going to get a positive answer for Question (3.1). First let us prove the following

(3.2) THEOREM. *If X is a mutational retract of an* ANR$(\mathfrak{M})$*-space P, then every component of X is a mutational retract of a component of P.*

Proof. Take an arbitrary component X_0 of X. Let P_0 be a component of P containing the component X_0. By the hypothesis there exists a mutational retraction $\boldsymbol{r}\colon \boldsymbol{W}(P,P)\to \boldsymbol{U}(X,P)$. Since $P\in \mathrm{ANR}(\mathfrak{M})$, then P_0 is open in P and hence $P_0\in \mathrm{ANR}(\mathfrak{M})$. Since the set X is closed in P, then $X\cap P_0$ is closed in P_0. So, we can consider complete neighborhood systems $\boldsymbol{W}_0(P_0,P_0)$ and $\boldsymbol{U}_0(X\cap P_0,P_0)$. Let us construct a mutational retraction $\boldsymbol{r}_0\colon \boldsymbol{W}_0(P_0,P_0)\to \boldsymbol{U}_0(X\cap P_0,P_0)$.

Take an arbitrary $r\in\boldsymbol{r}$, $r\colon P\to U$, $U\in \boldsymbol{U}(X,P)$. Then $U\cap P_0\in \boldsymbol{U}_0(X\cap P_0,P_0)$. Let us show that $r(P_0)\subset U\cap P_0$. Since $r(P_0)\subset r(P)\subset U$, it suffices to show that $r(P_0)\subset P_0$, and since P_0 is a component of P and $r(P_0)$ is connected, if suffices to show that $r(P_0)\cap P_0\neq\emptyset$. Take an arbitrary point $x\in X_0$. Then $x\in P_0$, and $x=r(x)\in r(P_0)$. Hence $x\in P_0\cap r(P_0)$. Thus, we have $r(P_0)\subset U\cap P_0$. Let us define a map $r_0\colon P_0\to U\cap P_0$ by the formula $r_0(x)=r(x)$ for every $x\in P_0$.

So, to every $r\in\boldsymbol{r}$ we have assigned a map r_0. Let us denote by $\boldsymbol{r}_0$ the collection of all maps r_0 which can be obtained in such a way. Let us show that $\boldsymbol{r}_0\colon \boldsymbol{W}_0(P_0,P_0)\to \boldsymbol{U}_0(X\cap P_0,P_0)$ is a mutational retraction.

For an arbitrary point $x\in X\cap P_0$ we have $r_0(x)=r(x)=x$, because $r\in\boldsymbol{r}$. It remains to show that $\boldsymbol{r}_0$ is a mutation. We are going to werify the conditions (1.1)-(1.3).

Take an arbitrary $r_0\in\boldsymbol{r}_0$, $r_0\colon P_0\to U_0$. Take $U_0'\in \boldsymbol{U}_0(X\cap P_0,P_0)$ such that $U_0\subset U_0'$ and consider the map $r_0'\colon P_0\to U_0'$ defined by $r_0'(x)=r_0(x)$ for $x\in P_0$. We must show that $r_0'\in\boldsymbol{r}_0$.

By the definition of $\boldsymbol{r}_0$ there exists $r\in\boldsymbol{r}$ such that $r\colon P\to U$, $U\cap P_0=U_0$, and $r_0(x)=r(x)$ for $x\in P_0$. Then $U\cap P_0\subset U_0'$. Let $U'=U_0'\cup(P\setminus P_0)$. The set U' is open in P and it contains the set X. Hence $U'\in \boldsymbol{U}(X,P)$. Moreover $U\subset U'$. Let us define a map $r'\colon P\to U'$ by the formula $r'(x)=r(x)$ for $x\in P$. Since $\boldsymbol{r}$ is a mutation, $r'\in\boldsymbol{r}$. Let us observe that $U'\cap P_0=U_0$ and for arbitrary $x\in P_0$ we have $r_0'(x)=r_0(x)=r(x)=r'(x)$. Therefore $r_0'\in\boldsymbol{r}_0$. Thus, the condition (1.1) is verified.

Let us verify the condition (1.2). Take an arbitrary neighborhood $U_0\in \boldsymbol{U}_0(X\cap P_0,P_0)$. Let us set $U=U_0\cup(P\setminus P_0)$. Obviously $U\in \boldsymbol{U}(X,P)$. Since $\boldsymbol{r}$ is a mutation, there exists $r\in\boldsymbol{r}$ such that $r\colon P\to U$. By the definition of $\boldsymbol{r}_0$ there exists $r_0\in\boldsymbol{r}_0$ such that $r_0\colon P_0\to U\cap P_0$. But $U\cap P_0=U_0$, hence $r_0\colon P_0\to U_0$. Thus, the condition (1.2) is satisfied.

Let us verify the condition (1.3). Take two arbitrary maps $r_0,r_0'\in\boldsymbol{r}_0$ such that $r_0,r_0\colon P_0\to U_0$. We must show that $r_0\simeq r_0'$. By the definition of $\boldsymbol{r}_0$ there exist maps $r,r'\in\boldsymbol{r}$ such that $r\colon P\to U$, $r'\colon P\to U'$, $U,U'\in \boldsymbol{U}(X,P)$, $U\cap P_0=U'\cap P_0=U_0$, and $r_0(x)=r(x)$, $r_0'(x)=r'(x)$ for $x\in P_0$. Since $\boldsymbol{r}$ is a mutation, then $r\simeq r'$ in $U\cup U'$.

Therefore, there exists a homotopy $H\colon P\times\langle 0,1\rangle\to U\cup U'$ such that

$$H(x,0)=r(x)\quad\text{and}\quad H(x,1)=r'(x)\quad\text{for}\quad x\in P.$$

We are going to show that $H(P_0\times\langle 0,1\rangle)\subset U_0$. First, let us show that $H(P_0\times\langle 0,1\rangle)\subset P_0$. Since $H(P_0\times\langle 0,1\rangle)$ is a connected set lying in P and P_0 is a component of P, it suffices to show that $H(P_0\times\langle 0,1\rangle)\cap P_0\neq\emptyset$. Take an arbitrary point $x\in X_0$. Then $H(x,0)=r(x)=x\in X_0\subset P_0$ and

$$H(x,0)\in H(X_0\times\langle 0,1\rangle)\subset H(P_0\times\langle 0,1\rangle).$$

Hence $H(x,0)\in H(P_0\times\langle 0,1\rangle)\cap P_0$. Therefore $H(P_0\times\langle 0,1\rangle)\subset P_0$. Thus

$$H(P_0\times\langle 0,1\rangle)\subset P_0\cap(U\cup U')=(P_0\cap U)\cup(P_0\cap U')=U_0.$$

Let us define a map $H_0\colon P_0\times\langle 0,1\rangle\to U_0$ by the formula

$$H_0(x,t)=H(x,t)\quad\text{for}\quad x\in P_0,\ 0\leqslant t\leqslant 1.$$

For every $x\in P_0$ we have

$$H_0(x,0)=H(x,0)=r(x)=r_0(x)\quad\text{and}\quad H_0(x,1)=H(x,1)=r'(x)=r_0'(x).$$

Therefore H_0 is a homotopy joining r_0 and r_0', thus $r_0\simeq r_0'$, and the condition (1.3) is verified.

Thus, $\boldsymbol{r}_0$ is a mutational retraction, and the set $X\cap P_0$ is a mutational retract of P_0. Hence, by Corollary (3.13) of [5] (p. 55) $\mathrm{Sh}P_0\geqslant\mathrm{Sh}X\cap P_0$. Hence, by Theorem (1.6) the set $X\cap P_0$ is connected. Let us observe that $X_0\subset X\cap P_0\subset X$. Therefore, the set $X\cap P_0$ is a connected set lying in X and containing the component X_0 of X. Hence $X\cap P_0=X_0$.

Thus, X_0 is a mutational retract of P_0 and the proof is finished.

In [5] (p. 59, Theorem (4.11)) we have proved the following

(3.3) THEOREM. MANR*-spaces are the same as mutational retracts of* ANR$(\mathfrak{M})$*-spaces.*

From Theorems (3.2) and (3.3) we obtain the following

(3.4) COROLLARY. *Every component of a* MANR*-spaces is a* MANR*-space.*

Proof. Take an arbitrary MANR-space X and let X_0 be a component of X. By (3.3) X is a mutational retract of an ANR$(\mathfrak{M})$-space P. By (3.2) X_0 is a mutational retract of a component P_0 of P, and since $P_0\in\mathrm{ANR}(\mathfrak{M})$, then by (3.3) $X_0\in\mathrm{MANR}$.

(3.5) THEOREM. *Components of* MANR*-space X are open in X.*

Proof. Let X_0 be a component of a MANR-space X. By Theorem (3.3) there exists an ANR$(\mathfrak{M})$-space P such that X is a mutational retract of P. By Theorem (3.2) the component X_0 is a mutational retract of a component P_0 of P. Then $P_0\cap X=X_0$. Since P_0 is open in P, then X_0 is open in X.

(3.6) COROLLARY. *Every component of a* MANR*-space X is a retract of X.*

Dr J. Olędzki and Mr W. Matuszewski independently obtained the following

(3.7) THEOREM. *If X is a mutational retract of X' and $X \in$ MANR then every component of X is a mutational retract of a component of X'.*

Proof. Take an arbitrary component X_0 of X. By (3.6) X_0 is a retract of X and since X is a mutational retract of X', then X_0 is a mutational retract of X', and hence X_0 is a mutational retract of X_0', where X_0' is a component of X containing X_0.

(3.8) PROBLEM. Does Theorem (3.7) remain true without the hypothesis $X \in$ MANR?

In [5] (Theorem (4.13), p. 59) we have proved the following

(3.9) THEOREM. *If X_t are MANR-spaces for every $t \in T$, then $\bigoplus_{t \in T} X_t$ is a MANR-space.*

The converse is also true, because X_t is a retract of $\bigoplus_{t \in T} X_t$ if and hence $\bigoplus_{t \in T} X_t \in$ MANR then by Theorem (4.12) of [5] (p. 59) we have $X_t \in$ MANR. Therefore, we get the following

(3.10) COROLLARY. *$\bigoplus_{t \in T} X_t$ is a MANR-space if and only if X_t is a MANR-space for every $t \in T$.*

From Theorem (3.5) and Corollary (3.10) we obtain the following

(3.11) COROLLARY. *A metrizable space X is a MANR-space if and only if all components of X are MANR-spaces open in X.*

§ 4. Components of MANR-spaces having comparable shapes.

Following K. Borsuk ([2], p. 17) let us denote by $\square(X)$ the set of all components of a space X.

K. Borsuk has proved ([2], pp. 215, 216) that

(4.1) *If X and Y are metric compacta of the same shape then there exists one-to-one corespondence $\Lambda\colon \square(X) \to \square(Y)$ such that for every component $X_0 \in \square(X)$ we have $\operatorname{Sh} \Lambda(X_0) = \operatorname{Sh} X_0$.*

(4.2) *If X and Y are metric compacta such that $\operatorname{Sh} X \geqslant \operatorname{Sh} Y$ then there exist functions $\Lambda\colon \square(X) \to \square(Y)$ and $\Lambda'\colon \square(Y) \to \square(X)$ such that the composition $\Lambda\Lambda'$ is the identity function on $\square(Y)$ and then for every component $Y_0 \in \square(Y)$ we have $\operatorname{Sh} \Lambda'(Y_0) \geqslant \operatorname{Sh} Y_0$.*

(4.3) PROBLEM. Do the results (4.1) and (4.2) remain true for arbitrary metrizable spaces X and Y, where Sh denote the shape in the sense of Fox?

We are going to show that for MANR-spaces the answer for Problem (4.3) is "yes".

Let us observe that

(4.4) For every MANR-space X there exists an ANR($\mathfrak{M}$)-space P such that X is a mutational retract of P and every component of P contains exactly one component of X.

Indeed, by Theorem (4.11) of [5] (p. 59) there exists an ANR($\mathfrak{M}$)-space Q such that X is a mutational retract of Q. The set P being the union of all components of Q intersecting X satisfies the required conditions.

Suppose $X_0 \subset X \subset P$; $X_0 \subset P_0 \subset P$; $Y_0 \subset Y \subset Q$; $Y_0 \subset Q_0 \subset Q$; $P, Q, P_0, Q_0 \in$ ANR($\mathfrak{M}$); X and X_0 are closed in P; Y and Y_0 are closed in Q. Consider two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{f}_0\colon \boldsymbol{U}_0(X_0, P_0) \to \boldsymbol{V}_0(Y_0, Q_0)$. We say that $\boldsymbol{f}_0$ is a *submutation* of $\boldsymbol{f}$ if for every $f_0 \in \boldsymbol{f}_0$ there exists $f \in \boldsymbol{f}$ such that the domain of f_0 is a subset of the domain of f, the range of f_0 is an intersection of the range of f with Q_0, and $f_0(x) = f(x)$ for every x belonging to the domain of f_0.

(4.5) LEMMA. *Suppose $P, Q \in$ ANR($\mathfrak{M}$), X and Y are closed subsets of P and Q, respectively, and every component of P (of Q) contains exactly one component of X (of Y). Let $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ be a mutation. Then there exists exactly one function $\Lambda_{\boldsymbol{f}}\colon \square(X) \to \square(Y)$ such that for every component $X_0 \in \square X$ there exists a submutation $\boldsymbol{f}_0\colon \boldsymbol{U}_0(X_0, P_0) \to \boldsymbol{V}_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0)$, where P_0 and Q_0 are components of P and Q containing X_0 and $\Lambda_{\boldsymbol{f}}(X_0)$, respectively.*

Proof. Take an arbitrary component $X_0 \in \square(X)$ and arbitrary $f \in \boldsymbol{f}$. Then there exists exactly one component $Q_0 \in \square(Q)$ such that $f(X_0) \subset Q_0$. Since $\boldsymbol{f}$ is a mutation, then by (1.1) and (1.3) the choice of Q_0 does not depend of the choice of f. By the hypothesis the component Q_0 contains exactly one component $Y_0 \in \square(Y)$. Let us set $\Lambda_{\boldsymbol{f}}(X_0) = Y_0$. So, we have defined the function $\Lambda_{\boldsymbol{f}}\colon \square(X) \to \square(Y)$.

Let us define a mutation $\boldsymbol{f}_0\colon \boldsymbol{U}_0(X_0, P_0) \to \boldsymbol{V}_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0)$. Take an arbitrary $f \in \boldsymbol{f}$, $f\colon U \to V$. Then

$$f(U \cap P_0) \subset V \cap Q_0, \quad U \cap P_0 \in \boldsymbol{U}_0(X_0, P_0), \quad V \cap Q_0 \in \boldsymbol{V}_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0).$$

Let us define a map $f_0\colon U \cap P_0 \to V \cap Q_0$ by the formula

$$f_0(x) = f(x) \quad \text{for every } x \in U \cap P_Q.$$

Let us denote by $\boldsymbol{f}_0$ the collection of all maps f_0 which can be obtained in such a way. Let us show that $\boldsymbol{f}_0\colon \boldsymbol{U}_0(X_0, P_0), \boldsymbol{V}_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0)$ is a mutation.

Let us verify the condition (1.1). Take an arbitrary $f_0 \in \boldsymbol{f}_0$, $f_0\colon U_0 \to V_0$. Let $U_0' \in \boldsymbol{U}_0(X_0, P_0)$, $U_0' \subset U_0$, $V_0' \in \boldsymbol{V}_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0)$, $V_0 \subset V_0'$. Let us define $f_0'\colon U_0' \to V_0'$ by $f_0'(x) = f_0(x)$ for $x \in U_0'$. By the definition of $\boldsymbol{f}_0$ there exists $f \in \boldsymbol{f}$ such that $f\colon U \to V$, where $U \cap P_0 = U_0$, $V \cap Q_0 = V_0$. Let us set $U' = U_0' \cup (U \cap (P \setminus P_0))$ and $V' = V_0' \cup V$. Let us define a map $f'\colon U' \to V'$ by the formula $f'(x) = f(x)$ for $x \in U'$. Since $\boldsymbol{f}$ is a mutation, then $f' \in \boldsymbol{f}$. Let us observe that $U' \cap P_0 = U_0$ and $V' \cap Q_0 = V_0$. Applying the definition of $\boldsymbol{f}_0$ to the map f' we get the map f_0'. Therefore, $f_0' \in \boldsymbol{f}_0$ and the condition (1.1) is satisfied.

Let us verify the condition (1.2). Take an arbitrary $V_0 \in \boldsymbol{V}_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0)$. Let us set $V = V_0 \cup (Q \setminus Q_0)$. Then $V \in \boldsymbol{V}(Y, Q)$. Since $\boldsymbol{f}$ is a mutation, there exists $f \in \boldsymbol{f}$ such that $f\colon U \to V$. Let us set $U_0 = U \cap P_0$, and let us observe that $V \cap Q_0 = V_0$. Consider the map $f_0\colon U_0 \to V_0$ defined by $f_0(x) = f(x)$ for $x \in U_0$. By the definition of $\boldsymbol{f}_0$ we have $f_0 \in \boldsymbol{f}_0$. Therefore, the condition (1.2) is satisfied.

Let us verify the condition (1.3). Take arbitrary two maps $f_{01}, f_{02} \in \boldsymbol{f}_0$ such that $f_{01}, f_{02}\colon U_0 \to V_0$. Then there exist $U_1, U_2 \in \boldsymbol{U}(X, P)$, $V_1, V_2 \in \boldsymbol{V}(Y, Q)$ such that $U_1 \cap P_0 = U_2 \cap P_0 = U_0$, $V_1 \cap Q_0 = V_2 \cap Q_0 = V_0$, and there exist maps $f_1, f_2 \in \boldsymbol{f}$, $f_i\colon U_i \to V_i$ for $i = 1, 2$, such that $f_{0i}(x) = f_i(x)$ for $x \in U_0$. Since $\boldsymbol{f}$ is a mutation, then by (1.3′) there exists $U' \in \boldsymbol{U}(X, P)$ such that $U' \subset U_1 \cap U_2$ and $f_1|U' \simeq f_2|U'$ in $V_1 \cup V_2$. Let us set $U_0' = U' \cap P_0$. Then $U_0' \in U_0(X_0, P_0)$ and $f_{01}|U_0' \simeq f_{02}|U_0'$ in V_0. Therefore, the condition (1.3) is satisfied.

Thus, $\boldsymbol{f}_0\colon U_0(X_0, P_0) \to V_0(\Lambda_{\boldsymbol{f}}(X_0), Q_0)$ is a mutation. It is evident that $\boldsymbol{f}_0$ is a submutation of $\boldsymbol{f}$.

It is obvious that the function $\Lambda_{\boldsymbol{f}}\colon \square(X) \to \square(Y)$ satisfying the required condition is unique. Thus, the proof is finished.

It is easy to see that under the hypotheses of Lemma (4.5) the following conditions are satisfied.

(4.6) *If* $\boldsymbol{f} \simeq \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$, *then* $\Lambda_{\boldsymbol{f}} = \Lambda_{\boldsymbol{g}}\colon \square(X) \to \square(Y)$ *and for every component* $X_0 \in \square X$ *the submutations* $\boldsymbol{f}_0$ *and* $\boldsymbol{g}_0$ *are homotopic.*

(4.7) *If* $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ *and* $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{W}(Z, R)$, *then* $\Lambda_{\boldsymbol{gf}} = \Lambda_{\boldsymbol{g}}\Lambda_{\boldsymbol{f}}$.

(4.8) *If* $\boldsymbol{u}$ *is the identity mutation for the system* $\boldsymbol{U}(X, P)$ *then* $\Lambda_{\boldsymbol{u}}\colon \square(X) \to \square(X)$ *is the identity function.*

(4.9) Theorem. *If* $X \in \mathrm{MANR}$ *and* $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$, *then there exist functions* $\Lambda\colon \square(X) \to \square(Y)$ *and* $\Lambda'\colon \square(Y) \to \square(X)$ *such that the composition* $\Lambda\Lambda'$ *is the identity function on* $\square(Y)$ *and* $\mathrm{Sh}\,\Lambda'(Y_0) \geqslant \mathrm{Sh}\,Y_0$ *for every component* $Y_0 \in \square(Y)$.

Proof. Since $X \in \mathrm{MANR}$ and $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$ then by Theorem (4.5) of [6] (p. 92) we have $Y \in \mathrm{MANR}$. By (4.4) there exist $\mathrm{ANR}(\mathfrak{M})$-spaces P and Q containing X and Y, respectively, and satisfying the hypotheses of Lemma (4.5). Since $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$, then there exist mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that $\boldsymbol{fg} \simeq \boldsymbol{v}$. By Lemma (4.5) there exist functions $\Lambda_{\boldsymbol{f}}\colon \square(X) \to \square(Y)$ and $\Lambda_{\boldsymbol{g}}\colon \square(Y) \to \square(X)$ sytisfying the thesis of Lemma (4.5). Let us set $\Lambda = \Lambda_{\boldsymbol{f}}$ and $\Lambda' = \Lambda_{\boldsymbol{g}}$. It follows by (4.6)-(4.8) that the functions Λ and Λ' satisfy the required conditions. Thus, the proof is finished.

Analogously we can prove the following

(4.10) Theorem. *If* $X \in \mathrm{MANR}$ *and* $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$, *then there exists one-to-one function* $\Lambda\colon \square(X) \to \square(Y)$ *such that* $\mathrm{Sh}\,X_0 = \mathrm{Sh}\,\Lambda(X_0)$ *for every component* $X_0 \in \square(X)$.

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