Homogeneously wild curves and infinite knot products

by

H. G. Bothe (Berlin)

Abstract: An oriented simple closed curve C in euclidian 3-space E^3 is called *homogeneously embedded* if for each pair p, q of points in C there is an orientation preserving homeomorphism $h\colon E^3 \to E^3$ such that $h(C) = C$ respecting the orientation of C and $h(p) = q$. Since all tame simple closed curves are homogeneously embedded we are mainly interested in the case where C is wildly embedded, i.e. where there is no homeomorphism of E^3 onto itself mapping C onto a closed polygon. The main result is a classification of all possible positions of homogeneously embedded simple closed curves in E^3 which pierce a disk by associating to each of these curves a possibly infinite product of tame knots. As a corollary we see that for each homogeneously embedded simple closed curve C in E^3 which pierces a disk either each orientation preserving homeomorphism $h_0\colon C \to C$ can be extended to an orientation preserving homeomorphism $h\colon E^3 \to E^3$ or for each pair $p, q \in C$ there is exactly one orientation preserving homeomorphism $h_0\colon C \to C$ with $h_0(p) = q$ which can be extended to E^3. Moreover, it turns out that being homogeneously embedded is a local property of C.

1. Introduction. In this section we introduce some concepts which are used in this paper and state the main results. The following conventions will be convenient: E^3 and S^3 denote the euclidian 3-space and the 3-sphere respectively which are assumed to have a fixed orientation and to carry the usual PL (i.e. piecewise linear) structure. All simple closed curves which appear are tacitly assumed to be oriented. Moreover, we say that a homeomorphism h maps a simple closed curve C onto a simple closed curve C' only if h respects the orientations on C and C'. By a *polyhedron* we mean the underlying space of a locally finite simplicial complex in E^3 or in S^3. A *disk* is a closed 2-cell and a *ball* a closed 3-cell. By $\operatorname{Int} M$ and $\operatorname{Bd} M$ we denote the interior and the boundary of a manifold M, and $\operatorname{Cl} X$ is the point set closure of X.

We shall consider *knots*, i.e. equivalence classes of simple closed curves (oriented by our conventions) in E^3 or S^3 under orientation preserving homeomorphisms of E^3 or S^3 onto itself. The knot which is represented by a simple closed curve C will be denoted by $\varkappa(C)$. Since E^3 is S^3 minus a point, our theorems will hold in E^3 as well as in S^3, and it is sufficient to formulate and to prove them in the case which is most convenient.

There are two main classes of knots: the *tame knots* which can be represented by finite polygons or, equivalently, by smooth simple closed curves and the *wild* *knots* which have no such representation. In general the theory of tame knots (the classical knot theory) and the theory of wild knots are two separate parts of geometric topology. It is the aim of this paper to show how for a special class of wild knots a complete description and classification by formal infinite products of tame knots (as defined below) is possible. This class consists of all homogeneous knots which pierce a disk. Here a knot is called *homogeneous*, if its representants C are homogeneously embedded in S^3, i.e., if for $p, q \in C$ there is an orientation preserving homeomorphism $h\colon S^3 \to S^3$ such that $h(C) = C$ and $h(p) = q$ (respecting the orientation of C by our assumptions). A knot $\varkappa$ *pierces a disk* if for a representant C of $\varkappa$ (and therefore for all representants) we can find a disk D such that $C \cap D$ consists of an interior point of D and the boundary $\mathrm{Bd}\,D$ of D is linked with C in the sense that the natural map $H_1(\mathrm{Bd}\,D) \to H_1(S^3 \setminus C)$ is an isomorphism.

By the way we shall prove that a locally homogeneous knot which pierces a disk is homogeneous. Here a knot is called *locally homogeneous* if for each representing curve C and each pair $p, q \in C$ there are subarcs L_1, L_2 of C and an orientation preserving homeomorphism $h\colon S^3 \to S^3$ such that $p \in \mathrm{Int}L_1$, $q \in \mathrm{Int}L_2$, $h(p) = q$, and $h(L_1) = L_2$ respecting the orientations on these arcs which are induced by the orientation of C. Another result (Theorem 4) determines for a homogeneously embedded simple closed curve C in S^3 which pierces a disk the group $\mathscr{E}(C)$ of all those orientation preserving homeomorphisms $g\colon C \to C$ which can be extended to orientation preserving homeomorphisms $h\colon S^3 \to S^3$.

For the definition of infinite knot products we repeat some facts from classical knot theory (see [4]). Let C_1, C_2 be simple closed polygons in S^3 which represent the knots $\varkappa_1$, $\varkappa_2$ respectively. We assume that there is a polyhedral ball B in S^3 such that, if S^2 denotes its boundary, we have $C_1 \subset B$, $C_2 \subset (S^3 \setminus B) \cup S^2$, and $L = C_1 \cap S^2 = C_2 \cap S^2$ is an arc on which C_1, C_2 define opposite orientations. Then $C = (C_1 \cup C_2) \setminus \mathrm{Int}L$ represents a knot $\varkappa = \varkappa(C)$ which depends only on $\varkappa_1$ and $\varkappa_2$. This knot $\varkappa$ is called the *product* $\varkappa_1 \cdot \varkappa_2$ of $\varkappa_1$ and $\varkappa_2$. A fundamental fact concerning this product is that the set of all tame knots with this multiplication is a semigroup isomorphic to the semigroup of all positive integers with the ordinary multiplication. This implies the existence of prime knots (corresponding to prime numbers) and the fact that each tame knot is a unique product of prime knots: $\varkappa = \prod \pi^{e(\pi)}$ where π ranges over all prime knots and only a finite number of exponents $e(\pi)$ are different from zero. A possibly infinite knot product (p.i. knot product) is a formal product $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ where the exponents are arbitrary non negative integers or the symbol ∞.

The product of tame knots can be represented in various ways. It is clear how a polyhedral solid torus T in S^3 in whose fundamental group a generator is distinguished (we shall speak about a *directed solid torus* in this case) represents a tame knot $\varkappa(T)$. Now let T_0, T_1 be directed solid tori in S^3 where T_1 lies *normally* in $\mathrm{Int}T_0$ in the sense that there is a latitudinal disk of T_0 (i.e. a disk D in T_0 for which $D \cap \mathrm{Bd}T_0 = \mathrm{Bd}\,D$ is not contractible on $\mathrm{Bd}T_0$) whose intersection with T_1 is a latitudinal disk of T_1 and that T_0, T_1 are coherently directed. Then, if $h\colon T_0 \to S^3$

is an orientation preserving polyhedral reimbedding for which $h(T_0)$ is unknotted, we have $\varkappa(T_1) = \varkappa(T_0)\cdot\varkappa(h(T_1))$. Therefore, for a directed solid torus T' which lies normally in a directed solid torus T the knot $\varkappa(T)$ is a divisor of $\varkappa(T')$ (i.e. in $\varkappa(T) = \prod \pi^{e(\pi)}$, $\varkappa(T') = \prod \pi^{e'(\pi)}$ the inequalities $e(\pi) \leqslant e'(\pi)$ hold).

Another way to represent tame knots uses tunnels in balls. A *tunnel* in an oriented ball B is a ball B' in B such that for a suitable PL structure of B the ball B' is a subpolyhedron of B for which $B' \cap \mathrm{Bd}B = \mathrm{Bd}B' \cap \mathrm{Bd}B$ is the union of two separate disks one of which is called the *entrance* and the other the *exit* of B'. Each such tunnel represents a tame knot: Embed B by an orientation preserving embedding in S^3 such that $\mathrm{Cl}(B \setminus B')$ is a polyhedron. Then $(S^3 \setminus \mathrm{Int}B) \cup B'$ is a polyhedral solid torus T in whose fundamental group a generator is extinguished by a curve which runs in B' from entrance to exit and then outside B back to the entrance. The knot $\varkappa(T)$ does not depend on the embedding of B in S^3 but only on the pair (B, B') and will be denoted by $\varkappa(B, B')$. In our applications of this representation of tame knots B will be already embedded in S^3 (or E^3) in such a way that $\mathrm{Cl}(B \setminus B')$ is a subpolyhedron of S^3. Let a tunnel B' in B and a tunnel B'' in B' be given such that $D_1' \supset D_1''$ and $D_2' \supset D_2''$ where D_1', D_2' are entrance and exit of B', and D_1'', D_2'' are entrance and exit of B''. Then $\varkappa(B, B'') = \varkappa(B, B')\cdot\varkappa(B', B'')$ holds.

Now we show how a simple closed curve C in S^3 which pierces a disk and which is definable by solid tori determines a p.i. knot product. (C is called definable by solid tori, if there is a sequence $T_1, T_2, \ldots$ of polyhedral solid tori in S^3 such that $T_{i+1} \subset \mathrm{Int}T_i$ and $\bigcap_{i=1}^{\infty} T_i = C$). Since C pierces a disk D (which by [1] can be chosen so that $D \setminus C$ is a polyhedron) it is not hard to find a defining sequence $T_1, T_2, \ldots$ for C in which T_{i+1} lies normally in T_i where each T_i is directed by the orientation of C. Then the knots $\varkappa(T_i) = \prod \pi^{e_i(\pi)}$ satisfy $e_1(\pi) \leqslant e_2(\pi) \leqslant \ldots$, and for each prime knot π the sequence $(e_i(\pi))$ converges to a limit $e(\pi)$ which is an integer or ∞. It is clear that $e(\pi)$ depends only on C (indeed on $\varkappa(C)$), and we define the p.i. knot product $\prod \pi^{e(\pi)}$ which we shall denote by $\tilde{\varkappa}(C)$. If C is tame, then $\tilde{\varkappa}(C) = \varkappa(C)$, but simple examples show that in general $\tilde{\varkappa}(C)$ does not determine the knot $\varkappa(C)$ (see Fig. 1).

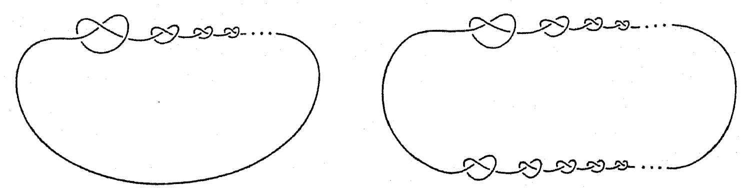

Fig. 1

Now we can state the main results:

THEOREM 1. *Each locally homogeneous knot $\varkappa$ which pierces a disk is definable by solid tori and determines therefore a p.i. knot product $\tilde{\varkappa}$.*

THEOREM 2. *A locally homogeneous knot $\varkappa$ which pierces a disk is uniquely determined by the corresponding p.i. knot product $\tilde{\varkappa}$, and $\varkappa$ is tame if and only if $\tilde{\varkappa}$ is finite.*

THEOREM 3. *A p.i. knot product $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ corresponds to a locally homogeneous knot which pierces a disk if and only if for each integer e there are only finitely many prime knots π for which $0<e(\pi)<e$ holds.*

THEOREM 4. *Let C be an oriented simple closed curve in S^3 which represents a locally homogeneous knot $\varkappa$ which pierces a disk, and let $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ be the corresponding p.i. knot product. If $\mathscr{E}(C)$ denotes the group of all orientation preserving homeomorphisms $g\colon C \to C$ which can be extended to orientation preserving homeomorphisms $h\colon S^3 \to S^3$, then we can say: if $0<e(\pi)<\infty$ holds only for a finite number of prime knots π, then $\mathscr{E}(C)$ is the ful group of all orientation preserving homeomorphisms of C; if $0<e(\pi)<\infty$ holds for infinitely many prime knots π, then $\mathscr{E}(C)$ is a full rotation group of C, i.e., there is a homeomorphism $\varphi\colon C \to S^1$ onto the unit circle S^1 such that g belongs to $\mathscr{E}(C)$ if and only if $\varphi g \varphi^{-1}\colon S^1 \to S^1$ is a (euclidian) rotation of the circle S^1.*

Remark. Theorem 4 implies that each locally homogeneous knot which pierces a disk is homogeneous.

2. Proof of Theorem 1. In the first part of this section we give a proof of Theorem 1. The second part is devoted to a strengthened version of this theorem (see Proposition (2.1) below) which will be needed later. For its formulation some definitions are necessary. (In this section the ambient space will be E^3.)

Let C be a simple closed curve in E^3. By a *normal disk* D of C we mean a disk D in E^3 intersecting C in exactly one point p at which C pierces D. Moreover we assume that $D \setminus \{p\}$ is a polyhedron. By a *normal neighborhood* of a point $p \in C$ we mean a ball B in E^3 containing p in its interior whose boundary S intersects C in exactly two points q_1, q_2 at which C pierces S. Moreover we assume that $S \setminus \{q_1, q_2\}$ is a polyhedron. By a *tubular neighborhood* of C we mean a polyhedral solid torus T in E^3 containing C in its interior whose fundamental group is generated by C. It is clear that each polyhedral solid torus in E^3 containing C in its interior which is sufficiently close to C must be a tubular neighborhood of C. By a *normal cell decomposition* of a tubular neighborhood T of C we mean a cyclically ordered collection $\mathfrak{C} = \{Z_1, \ldots, Z_r\}$ of balls (indices are counted modulo r) such that 1) each Z_j is a normal neighborhood of a point $p_j \in C$, 2) $Z_j \cap Z_k = \emptyset$ if $k \neq j-1$, j, $j+1$, 3) $Z_j \cap Z_{j+1}$ is a normal disk of C, 4) $r \geqslant 4$, and 5) the cyclic order of $\mathfrak{C}$ corresponds to the orientation of C.

Let $\mathfrak{C} = \{Z_1, \ldots, Z_r\}$ and $\mathfrak{C}' = \{Z'_1, \ldots, Z'_{r'}\}$ be normal subdivisions of tubular neighborhoods T, T' respectively of C. We write $\mathfrak{C}' < \mathfrak{C}$ if for each Z_i the intersection $Z_i \cap T'$ is a tunnel in Z_i and if each ball Z'_j is contained in a ball Z_i such that $Z'_j \cap \mathrm{Bd} Z_i$ is empty or one of the disks $Z'_j \cap Z'_{j-1}$ or $Z'_j \cap Z'_{j+1}$. If we have a defining sequence $T_1, T_2, \ldots$ of tubular neighborhoods for C and normal subdivisions $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ of these tubular neighborhoods, we call $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ a *defining sequence of normally subdivided tubular* (abbreviated: n.s.t.) *neighborhoods* of C provided we have $\mathfrak{C}_1 > \mathfrak{C}_2 > \ldots$, and the maximal diameter of balls in $\mathfrak{C}_i$ tends to 0 if $i \to \infty$. If we write $\mathfrak{C}_i = \{Z_{i,1}, \ldots, Z_{i,r_i}\}$, we assume moreover that

$$Z_{i+1,r_{i+1}} \cap Z_{i+1,1} \subset Z_{i,r_i} \cap Z_{i,1}\,.$$

We shall prove in the second part of this section the following stronger version of Theorem 1:

PROPOSITION (2.1). *Each locally homogeneously embedded simple closed curve in E^3 which pierces a disk has a defining sequence of n.s.t. neighborhoods.*

We begin the proofs with some remarks about simplicial complexes in E^3.

By an ordered complex we mean a finite simplicial complex whose vertices are given in a fixed order $v_1, \ldots, v_n$. For two ordered complexes $\mathfrak{K}$, $\mathfrak{K}'$ with the same number of vertices there is a natural one-to-one mapping of the set a vertices of $\mathfrak{K}$ onto the set of vertices of $\mathfrak{K}'$. If this mapping and its inverse map simplexes onto simplexes, then $\mathfrak{K}$ and $\mathfrak{K}'$ are called isomorphic. If $\mathfrak{K}$ is an ordered complex in E^3 with vertices $v_1, \ldots, v_n$, we can find neighborhoods $U_1, \ldots, U_n$ of $v_1, \ldots, v_n$ respectively such that for each sequence $v'_1, \ldots, v'_n$ of points where $v'_i \in U_i$ $(i = 1, \ldots, n)$ there is a unique ordered complex $\mathfrak{K}'$ in E^3 with vertices $v'_1, \ldots, v'_n$ which is isomorphic to $\mathfrak{K}$. If $\varepsilon > 0$ is given, then we can choose these neighborhoods so small that for any two complexes $\mathfrak{K}'$, $\mathfrak{K}''$ which are obtained in this manner there is a PL ε-homeomorphism of E^3 onto E^3 which is the identity outside the ε-neighborhood of $\mathfrak{K}'$ and which maps each simplex of $\mathfrak{K}'$ linearly onto the corresponding simplex of $\mathfrak{K}''$. To each ordered complex $\mathfrak{K}$ in E^3 with vertices $v_1, \ldots, v_n$ there corresponds the point $p(\mathfrak{K}) = (v_1, \ldots, v_n)$ in $E^{3n} = E^3 \times \ldots \times E^3$, and to choose points $v'_1, \ldots, v'_n$ in neighborhoods $U_1, \ldots, U_n$ of $v_1, \ldots, v_n$ respectively is the same as to choose a point in a certain neighborhood U of $p(\mathfrak{K})$. Therefore we have:

LEMMA (2.2). *If $\mathfrak{K}$ is an ordered complex in E^3 and ε is a positive number, then there is a neighborhood U of $p(\mathfrak{K})$ in E^{3n} for which the following assertions are true:*

(1) *For each $p' \in U$ there is exactly one ordered complex $\mathfrak{K}'$ in E^3 which is isomorphic to $\mathfrak{K}$ and for which $p(\mathfrak{K}') = p'$.*

(2) *If $\mathfrak{K}'$, $\mathfrak{K}''$ are ordered complexes which are isomorphic to $\mathfrak{K}$ and for which $p(\mathfrak{K}')$ and $p(\mathfrak{K}'')$ are in U, then there is a PL ε-homeomorphism of E^3 onto E^3 which is the identity outside the ε-neighborhood of $\mathfrak{K}'$ and which maps each simplex of $\mathfrak{K}'$ linearly onto the corresponding simplex of $\mathfrak{K}''$.*

We assume now that C is a fixed simple closed curve in E^3 which pierces in each of its points a disk. If C is a locally homogeneously embedded simple closed curve which pierces a disk (in at least one of its points), then this condition is satisfied.

For each $p \in C$ let D_p be a disk which is pierced by C in p. By Bing's approximation theorem (see [1]) we may assume that $D \setminus \{p\}$ is a polyhedron, and we choose a fixed triangulation $\mathscr{D}_p$ of $D_p \setminus \{p\}$. Now we select subdisks $D_{p,n}$ of D_p $(n = 1, 2, \ldots)$

such that

(1) $D_{p,0} = D_p$,

(2) $p \in \operatorname{Int} D_{p,n}$,

(3) $D_{p,n+1} \subset \operatorname{Int} D_{p,n}$,

(4) $\operatorname{diam}(D_{p,n}) < n^{-1}$,

(5) each annulus $A_{p,n} = D_p \setminus \operatorname{Int} D_{p,n}$ is covered by a subcomplex $\mathscr{A}_{p,n}$ of $\mathscr{D}_p$ $(n = 1, 2, \ldots)$.

For the vertices of $\mathscr{D}_p$ we choose an arbitrary order so that each $\mathscr{A}_{p,n}$ becomes an ordered complex.

LEMMA (2.3). *If L is a subarc of C we can find a sequence $\delta_0 > \delta_1 > \delta_2 > \ldots$ of positive numbers and a sequence $P_0 \supset P_1 \supset P_2 \supset \ldots$ of uncountable subsets of L such that the following conditions are satisfied:*

(1) *If $p \in P_0$, then $\operatorname{dist}(p, \operatorname{Bd} D_p) > \delta_0$.*

(2) *If $n \geq 1$, $p \in P_n$, and D is a subdisk of D_p whose boundary lies in the δ_n-neighborhood of p, then D is contained in the n^{-1}-neighborhood of p.*

(3) *If $n \geq 1$ and $p \in P_n$, then the intersection of D_p with the δ_n-neighborhood of C is contained in a subdisk of D_p with diameter less than n^{-1}.*

(4) *If $n \geq 1$ and $p, q \in P_n$, then there is a PL n^{-1}-homeomorphism of E^3 onto E^3 which is the identity on C and which maps $A_{p,n}$ onto $A_{q,n}$.*

Proof. We shall use the following simple fact:

(2.4) If M is a set with more than countably many elements and if $f\colon M \to E^n$ is any mapping, then there is an element $x_0 \in M$ such that for each neighborhood U of $f(x_0)$ there are more than countably many elements $x \in M$ for which $f(x)$ lies in U.

As an immediate consequence of (2.4) we have:

(2.5) If M is as in (2.4) and if $f\colon M \to R$ is a mapping with positive values, then there are positive numbers δ, η and more than countably many elements $x \in M$ for which $\delta < f(x) < \eta$ holds.

(2.6) If M is an uncountable subset of an arc L, then there is a point p in $\operatorname{Int} L$ such that for each subarc L' of L which contains p in its interior both components of $L' \setminus \{p\}$ contain uncountably many points of M.

We define (δ_0, P_0), (δ_1, P_1), ... successively. For the definition of δ_0, P_0 we use (2.5) with $M = L$ and $f(p) = \operatorname{dist}(p, \operatorname{Bd} D_p)$. Now assume that δ_n, P_n are already fixed. We define δ_{n+1}, P' satisfying (2) and (3) where $P' \subset P_n$ is uncountable by (2.5) with $M = P_n$ and $f(p) > 0$ such that each subdisk of D_p whose boundary lies in the $f(p)$-neighborhood of p is contained in the n^{-1}-neighborhood of p and that the intersection of D_p with the $f(p)$-neighborhood of C is contained in a subdisk of D_p with diameter less than n^{-1}. To get P_{n+1} we consider for each $p \in P'$ the complex $\mathscr{A}_{p,n+1}$. Once more using (2.5) we can find an uncountable subset P'' of P' and a number m such that for each $p \in P''$ the complex $\mathscr{A}_{p,n+1}$ has exactly m vertices.

Since there are only finitely many types of ordered complexes with m vertices, we may assume that for all $p \in P''$ the complexes $\mathscr{A}_{p,n+1}$ are isomorphic. Now we use (2.4) and Lemma (2.2) to find an uncountable subset P_{n+1} of P'' for which (4) is satisfied.

The following lemma is an easy consequence of (2.6) and the proof of Lemma (2.3).

LEMMA (2.7). *We can find two sequences (p_n), (q_n) of points in L both converging to a point p in $\operatorname{Int} L$ such that $p_n, q_n \in P_n$, and p lies between p_m and q_n $(m, n = 1, 2, \ldots)$.*

Let (p_n), (q_n) be chosen according to this lemma.

LEMMA (2.8). *The point p has arbitrarily small normal neighborhoods.*

Proof. For a given positive ε let n be so large that $D_{p_n,n}$ and $D_{q_n,n}$ lie in the $\frac{1}{2}\varepsilon$-neighborhood of p and that there is a $\frac{1}{2}\varepsilon$-homeomorphism $h\colon E^3 \to E^3$ which is the identity on C and which maps $A_{p_n,n}$ onto $A_{q_n,n}$. Then the set $S' = h(D_{p_n,n}) \cup \cup D_{q_n,n}$ is a singular sphere in the ε-neighborhood of p for which the following holds: $S' \cap C = \{p_n, q_n\}$, and C pierces S' in p_n and in q_n. Moreover, $S' \setminus \{p_n, q_n\}$ is a polyhedron. On the boundary curve $K = S' \cap A_{q_n,n}$ of $A_{q_n,n}$ there are no singularities of S'. After a simple modification of S' we may assume that there is an annular neighborhood N of K in S' which contains no singularities of S', and that the two disks D', D'' of $S' \setminus \operatorname{Int} N$ are in general position. Then the singularities of S' are simple closed curves in which D' intersects D'', and we can remove these singularities by standard techniques. So we get the boundary sphere S of an ε-small normal neighborhood of p.

PROPOSITION (2.9). *If a simple closed curve C in E^3 pierces in each of its points p a disk D_p where $D_p \setminus \{p\}$ is a polyhedron, then the set Q of all $q \in C$ for which the disk D_q is tame must be dense in C.*

COROLLARY (2.10). *If a simple closed curve C in E^3 pierces in each of its points a disk, then there is a dense subset Q of C in whose points C pierces a tame disk.*

COROLLARY (2.11). *If C is a locally homogeneously embedded simple closed curve in E^3 which pierces a disk, then C pierces in each of its points a tame disk.*

Proof of Proposition (2.9). Let L be a subarc of C. We show $L \cap Q \neq \emptyset$. By Lemma (2.8) we can find a point $p \in \operatorname{Int} L$ and an arbitrarily small 2-spere S containing p in its interior for which $S \cap C$ consists of two points in which C pierces S. Moreover, S can be chosen to be a polyhedron outside $S \cap C$. By easy modifications of the spheres S we may assume that each S intersects D_p in a simple closed polygon. Then by a result of O. G. Harrold, Jr. [3] D_p is tame.

Proof of Theorem 1. We shall use the following simple facts which can be proved by elementary cutting and pasting techniques. Let $B(p)$, $B(q)$ be normal neighborhoods of $p, q \in C$. We denote by $S(p)$, $S(q)$ their boundary spheres and by $L(p)$, $L(q)$ the arcs of C in $B(p)$, $B(q)$ respectively.

(2.12) We assume $L(p) \cap L(q) = \emptyset$ and denote by A an annulus on $S(p)$ which separates on $S(p)$ the end points of $L(p)$ and which contains $B(q) \cap S(p)$. Then for each neighborhood N of A in E^3 we can find a normal neighborhood $B'(q)$ of q with boundary $S'(q)$ such that (1) $B'(q) \cap C = L(q)$, (2) $B'(q) \cap B(p) = \emptyset$, and (3) $S'(q) \subset S(q) \cup N$.

(2.13) We assume that $L(p) \cap L(q)$ is a proper subarc of $L(p)$ and of $L(q)$ and denote by D a subdisk of $S(p) \setminus (C \setminus L(q))$ which contains $S(p) \cap B(q)$. Then for each neighborhood N of D in E^3 we can find a normal neighborhood $B'(q)$ of q with boundary $S'(q)$ such that (1) $B'(q) \cap C = L(q)$, (2) $S'(q) \subset N \cup S(q)$, and (3) $S(p) \cap S'(q)$ is a simple closed curve. (By (3) $B(p) \cap B'(q)$ is a ball.)

To prove Theorem 1 it is sufficient to find in a given neighborhood U of C a solid torus which contains C in its interior. By Lemma (2.8) and the local homogeneity of C we can choose for each point p of C a normal neighborhood $B(p)$ in U. Let $S(p)$, $L(p)$ denote the boundary of $B(p)$ and the arc $B(p) \cap C$ respectively. We select a minimal set $p_1, \ldots, p_r$ of points in C for which $\mathrm{Int}L(p_1), \ldots, \mathrm{Int}L(p_r)$ cover C. Then we can find a cyclic order of these points such that

$$\mathrm{Int}L(p_i) \cap \mathrm{Int}L(p_j) \neq \emptyset$$

if and only if $j = i-1$, i, or $i+1$ (here indices are counted modulo r). To avoid technical complications we assume $r \geqslant 4$. It if happens that arcs $L(p_{i-1})$, $L(p_{i+1})$ have a common end point, we can easily replace $B(p_{i+1})$ by a normal neighborhood of p_{i+1} which is a little bit smaller, so that $L(p_{i-1}) \cap L(p_{i+1}) = \emptyset$ holds. This allows us to assume that $L(p_i) \cap L(p_j) \neq \emptyset$ if and only if $j = i-1$, i, $i+1$.

Now it is our aim to find new normal neighborhoods of $p_1, \ldots, p_r$ in U which will also be denoted by $B(p_1), \ldots, B(p_r)$ such that

(1) $B(p_i) \cap C$ is the old arc $L(p_i)$,

(2) $B(p_i) \cap B(p_j) \neq \emptyset$ if and only if $j = i-1$, i, $i+1$,

(3) $B(p_i) \cap B(p_{i+1})$ is a ball.

Then $T = \bigcup_{i=1}^{r} B(p_i)$ is a torus in U containing C in its interior.

To find the new normal neighborhoods $B(p_i)$ we make the following constructions.

(a) Applying (2.12) we may replace $B(p_3), \ldots, B(p_{r-1})$ by normal neighborhoods $B'(p_3), \ldots, B'(p_{r-1})$ of $p_3, \ldots, p_{r-1}$ respectively in U such that $B'(p_i) \cap C = L(p_i)$ and $B(p_1) \cap B'(p_i) = \emptyset$ for $i = 3, \ldots, r-1$. Denote $B'(p_i)$ by $B(p_i)$ and forget the old $B(p_i)$'s.

(b) Applying (2.12), (2.13), and some further simple surgery we may replace $B(p_2)$ and $B(p_r)$ by normal neighborhoods $B'(p_2)$, $B'(p_r)$ of p_2, p_r respectively in U such that $B'(p_i) \cap C = L(p_i)$, $B'(p_2) \cap B'(p_r) = \emptyset$, and $B'(p_i) \cap B(p_1)$ is a ball for $i = 1, r$. Denote $B'(p_2)$, $B'(p_r)$ by $B(p_2)$, $B(p_r)$ respectively and forget the old $B(p_2)$, $B(p_r)$.

If we repeat (a) and (b) in an obvious manner, we get new normal neighborhoods which satisfy (1), (2), and (3).

Proof of Proposition (2.1). It is sufficient to prove the following two assertions:

(2.14). C has a tubular neighborhood with a normal cell decomposition.

(2.15) If T is tubular neighborhood of C with a normal cell decomposition $\mathfrak{C}$ and if $\varepsilon > 0$, then there is a tubular neighborhood T' with a normal cell decomposition $\mathfrak{C}'$ such that $\mathfrak{C}' < \mathfrak{C}$ and the balls of $\mathfrak{C}'$ have diameters less then ε.

The proof of (2.14) is a simple consequence of the proof for Theorem 1. (Define $\mathfrak{C} = \{Z_1, \ldots, Z_r\}$ by $Z_1 = B(p_1)$, $Z_j = \mathrm{Cl}(B(p_j) \setminus B(p_{j-1}))$ if $2 \leqslant j \leqslant r-1$, and $Z_r = \mathrm{Cl}(B(p_r) \setminus (B(p_{r-1}) \cup B(p_1)))$.)

For the proof of (2.15) we use the following simple fact.

(2.16) Let L be a subarc of C with end points p_1, p_2 and let D_1, D_2 be normal disks of C which are pierced by C in p_1, p_2 respectively. Then there is a positive number δ such that for each tubular neighborhood T of C which is contained in the δ-neighborhood of C the component of $T \setminus (D_1 \cup D_2)$ whose closure contains L has diameter less than $2 \cdot \mathrm{diam} L$.

Now let $\mathfrak{C} = \{Z_1, \ldots, Z_r\}$ be the normal cell decomposition given in (2.15). Since C pierces in each of its points a normal disk, we can find a collection of normal disks $D_1, \ldots, D_{r'}$ such that the points $p_j = D_j \cap C$ are by their indices cyclically ordered in accordance with the orientation of C and such that the arcs L_j with end points p_j, p_{j+1} have diameters less than $\frac{1}{2}\varepsilon$. Using (2.16) we choose a tubular neighborhood T_1 of C with $\mathrm{Bd}T_1$ in general position to $D_1, \ldots, D_{r'}$ such that $T_1 \cap D_j$ is contained in a subdisk D_j' of $\mathrm{Int}D_j$, and each component of $T_1 \setminus (D_1 \cup \ldots \cup D_{r'})$ whose closure contains an arc L_j lies together with D_j', D_{j+1}' in a ball with diameter less than ε. If T_1 is sufficiently close to C, each component of $\mathrm{Bd}T_1 \cap D_j$ is a simple closed curve which bounds a disk on $\mathrm{Bd}T_1$ or bounds a disk in T_1. By simple surgery we can modify T_1 so that each intersection $T_1 \cap D_j'$ is a disk containing p_j in its interior. Then for each L_j the closure of the component of $T_1 \setminus (D_1' \cup \ldots \cup D_{r'}')$ which contains L_j is a ball Z_j'. If we choose the disks $D_1, \ldots, D_{r'}$ so that each disk $Z_i \cap Z_{i+1}$ appears among them, the cell decomposition $\mathfrak{C}' = \{Z_1', \ldots, Z_{r'}'\}$ of $T' = T_1$ has all properties required in (2.15).

3. First part of the proof of Theorem 3. Here we prove:

(3.1) *Let C be a locally homogeneous simple closed curve in S^3 which pierces a disk, and let $\tilde{\varkappa}(C) = \prod \pi^{e(\pi)}$ be the corresponding p.i. knot product. Then the sets $\{\pi;\ 0 < e(\pi) \leqslant n\}$ are finite ($n = 1, 2, 3, \ldots$).*

We say that two p.i. knot products $\tilde{\varkappa} = \prod \pi^{e(\pi)}$, $\tilde{\varkappa}' = \prod \pi^{e'(\pi)}$ are *almost equal* if $e(\pi) = e'(\pi)$ for almost all prime knots π and if $e(\pi) = \infty$ if and only if $e'(\pi) = \infty$ ("almost all" means "all with a finite number of exceptions"). We say that $\tilde{\varkappa}$ divides $\tilde{\varkappa}'$ if $e(\pi) \leqslant e'(\pi)$ for all π and that $\tilde{\varkappa}$ almost divides $\tilde{\varkappa}'$ if $\tilde{\varkappa}$ is almost equal to a divisor of $\tilde{\varkappa}'$.

Let C be a simple closed curve in S^3 which is definable by solid tori and let B be a normal neighborhood of a point in C. Then the arc $L = B \cap C$ defines in B a p.i. knot product $\tilde{\varkappa}(B, L)$: We can easily find a defining sequence $T_1, T_2, \ldots$ of solid tori for C such that each intersection $T_i \cap B$ is a tunnel in B (distinction of entrance and exit is given by the orientation of C). If $\varkappa(B, T_i \cap B) = \prod \pi^{e_i(\pi)}$, it is clear that $e_1(\pi) \leqslant e_2(\pi) \leqslant \ldots$ for each π, and we define $\tilde{\varkappa}(B, L) = \prod \pi^{e(\pi)}$ where $e(\pi) = \lim_{i\to\infty} e_i(\pi)$. It is easily seen that $\tilde{\varkappa}(B, L)$ divides the p.i. knot product $\tilde{\varkappa}(C)$ of C. If B_1, B_2 are normal neighborhoods as above where $B_1 \cap C = L_1$, $B_2 \cap C = L_2$ and $B_1 \subset B_2$, then $\tilde{\varkappa}(B_1, L_1)$ divides $\tilde{\varkappa}(B_2, L_2)$. This simple remark leads easily to the following assertion:

LEMMA (3.2). *Let C be a simple closed curve in S^3 which is definable by solid tori, and let B_1, B_2 be normal neighborhoods of points in C. Then $B_1 \cap C \subset \operatorname{Int}(B_2 \cap C)$ implies that $\tilde{\varkappa}(B_1, B_1 \cap C)$ almost divides $\tilde{\varkappa}(B_2, B_2 \cap C)$.*

For a p.i. knot product $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ we define a new p.i. knot product $\tilde{\varkappa}^* = \prod \pi^{e^*(\pi)}$ by

$$e^*(\pi) = \begin{cases} e(\pi) & \text{if } e(\pi) = 0, \infty, \\ 1 & \text{if } 0 < e(\pi) < \infty. \end{cases}$$

Now let C be a locally homogeneous simple closed curve in S^3 which pierces a disk. By Lemma (2.8) each point of C has arbitrarily small normal neighborhoods. For a p.i. knot product $\tilde{\varkappa}$ we say that C is locally at least $\tilde{\varkappa}$-knotted, if for each normal neighborhood B of a point in C the p.i. knot product $\tilde{\varkappa}$ almost divides $\tilde{\varkappa}(B, B \cap C)$.

LEMMA (3.3). *C is locally at least $\tilde{\varkappa}^*(C)$-knotted.*

SUBLEMMA. *If $\tilde{\lambda}$ is an infinite divisor of $\tilde{\varkappa}(C)$, then there is an infinite divisor $\tilde{\mu}$ of $\tilde{\lambda}$ such that C is locally at least $\tilde{\mu}$-knotted.*

Proof of Lemma (3.3). Assume that C is not locally at least $\tilde{\varkappa}^*(C)$-knotted. Then there is a normal neighborhood B of a point in C and an infinite divisor $\tilde{\lambda}$ of $\tilde{\varkappa}^*(C)$ such that $\tilde{\lambda} = \pi^\infty$ and π has a finite exponent in $\tilde{\varkappa}(B, B \cap C)$ or $\tilde{\lambda}$ is a product of different prime knots which appear in $\tilde{\varkappa}^*(C)$ with exponents equal to 1 but do not appear in $\tilde{\varkappa}(B, B \cap C)$. In both cases the sublemma leads to a contradiction.

Proof of the sublemma. Let $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ be a defining sequence of n.s.t. neighborhoods for C where $\mathfrak{C}_i = \{Z_{i,1}, \ldots, Z_{i,r_i}\}$ and $T_i = \bigcup_k Z_{i,k}$. If we denote the knot $\varkappa(Z_{i,k}, T_j \cap Z_{i,k})$ by $\varkappa_{i,k,j}$ $(j > i)$, we have

$$\varkappa(T_{i+1}) = \varkappa(T_i) \prod_{k=1}^{r_i} \varkappa_{i,k,i+1}$$

where $\varkappa(T_i)$ denotes the knot type of T_i. Under the hypothesis of the sublemma we can find a sequence $Z_{i_1,k_1} \supset Z_{i_2,k_2} \supset \ldots$ of balls such that $\varkappa_{i_m,k_m,i_m+1}$ contains a prime factor of $\tilde{\lambda}$. Let $\tilde{\mu}$ be the product $\prod_{m=1}^{\infty} \varkappa_{i_m,k_m,i_m+1}$, and let p be the common point of all balls Z_{i_m,k_m}. Then for each normal neighborhood B of p we see that $\tilde{\mu}$ almost divides $\tilde{\varkappa}(B, B \cap C)$. This proves the sublemma.

Proof of (3.1). Let $\mathfrak{C} = \{Z_1, \ldots, Z_r\}$ be an element of a defining sequence of n.s.t. neighborhoods for C where $r > n$. By Lemma (3.3) almost all prime factors of $\tilde{\varkappa}(C)$ are divisors of $\tilde{\varkappa}(Z_i, Z_i \cap C)$. Since $\prod_{i=1}^{r} \tilde{\varkappa}(Z_i, Z_i \cap C)$ divides $\tilde{\varkappa}(C)$ and $r > n$ the proof is finished.

4. Subarcs of locally homogeneous simple closed curves. In this section we define an equivalence relation $\sim$ for p.i. knot products (Definition (4.6)) and show that for the corresponding classes $[\tilde{\varkappa}]$ and positive real numbers t powers $[\tilde{\varkappa}]^t$ can be defined which are $\sim$-classes of p.i. knot product again (Definition (4.7)). If $\tilde{\varkappa}$ satisfies the hypothesis of Theorem 3 (i.e. if for each positive natural number n there are only finitely many prime knots whose exponents in $\tilde{\varkappa}$ are equal to n), then we have

$$[\tilde{\varkappa}]^{s+t} = [\tilde{\varkappa}]^s \cdot [\tilde{\varkappa}]^t \tag{4.1}$$

where products of classes are defined by multiplication of representants. Later we show how

(4.2) *each subarc L of a locally homogeneous simple closed curve C in S^3 which pierces a disk determines a $\sim$-class $[\tilde{\varkappa}](L)$ of p.i. knot products, where this class depends only on the embedding of L in S^3 i.e. on the topological type of the pair (S^3, L) in which both S^3 and L are oriented.*

Then we assume that in $\tilde{\varkappa}(C) = \prod \pi^{e(\pi)}$ infinitely many positive finite exponents appear and show that

(4.3) *for each subarc L of C there is exactly one positive real number ϱ such that $[\tilde{\varkappa}](L) = [\tilde{\varkappa}(C)]^\varrho$.*

If we denote by L_{pq} the subarc of C with end points p, q which runs in positive direction from p to q, then each pair $p, q \in C$ defines a real number $\varrho(p, q)$ by $[\tilde{\varkappa}](L_{pq}) = [\tilde{\varkappa}(C)]^{\varrho(p,q)}$ if $p \neq q$ and $\varrho(p, p) = 0$. Let o be an arbitrarily chosen point on C. Then by $\varphi(p) = e^{2\pi i \varrho(o,p)}$ a map $\varphi\colon C \to S^1$ is defined. We shall prove that

(4.4) *$\varphi\colon C \to S^1$ is a homeomorphism.*

As an immediate consequence of these facts we get

(4.5) *If $h\colon S^3 \to S^3$ is an orientation preserving homeomorphism such that $h(C) = C$ respecting the orientation of C, then $\varphi h|_C \varphi^{-1}\colon S^1 \to S^1$ is a rotation of the unit circle S^1, i.e. $\varphi h|_C \varphi^{-1}(e^{2\pi i \alpha}) = e^{2\pi i(\alpha+\beta)}$ where β depends only on $h|_C$.*

DEFINITION (4.6). Two p.i. knot products $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ and $\tilde{\varkappa}' = \prod \pi^{e'(\pi)}$ are called *equivalent* if for each $\varepsilon > 0$

$$(1-\varepsilon) e'(\pi) \leqslant e(\pi) \leqslant (1+\varepsilon) e'(\pi)$$

holds for almost all prime knots π, and $e(\pi) = \infty$ if and only if $e'(\pi) = \infty$. The class of $\tilde{\varkappa}$ will be denoted by $[\tilde{\varkappa}]$.

Definition (4.7). If $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ and t is a positive real number, then $[\tilde{\varkappa}]^t$ is defined by $[\tilde{\varkappa}]^t = [\prod \pi^{d(\pi)}]$ where $d(\pi) = \infty$ if $e(\pi) = \infty$ and $d(\pi)$ is the largest integer $\leqslant e(\pi)^t$ if $e(\pi)$ is finite.

This definition is independent of the representant $\tilde{\varkappa}$ of $[\tilde{\varkappa}]$. If $\tilde{\varkappa}$ satisfies the hypothesis of Theorem 3, then the equation (4.1) is an immediate consequence of this definition.

To prove (4.2) we take a defining sequence $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ of n.s.t. neighborhoods for C such that there are balls $Z_{1,1}, \ldots, Z_{1,n}$ in $\mathfrak{C}_1$ for which $L = Z \cap C$ where $Z = Z_{1,1} \cup \ldots \cup Z_{1,n}$. Then we define $[\tilde{\varkappa}](L) = [\tilde{\varkappa}(Z, L)]$. To make this definition correct, we have to prove the following

Lemma (4.8). *The class $[\tilde{\varkappa}(Z, L)]$ depends only on the oriented arc L and not on C or Z.*

Proof. We consider two sequences $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ and $\mathfrak{C}'_1, \mathfrak{C}'_2, \ldots$ with the properties mentioned in the definition of $[\tilde{\varkappa}](L)$, where $\mathfrak{C}_i = \{Z_{i,1}, \ldots, Z_{i,r_i}\}$, $\mathfrak{C}'_i = \{Z'_{i,1}, \ldots, Z'_{i,r'_i}\}$, $Z = Z_{1,1} \cup \ldots \cup Z_{1,n}$, $Z' = Z'_{1,1} \cup \ldots \cup Z'_{1,n'}$, $Z \cap C = Z' \cap C = L$. For each index i we consider balls B_i^-, B_i^+, B_i each of which is a union of balls in $\mathfrak{C}_i$ such that one end point of L lies in $\operatorname{Int} B_i^-$ and the other in $\operatorname{Int} B_i^+$, $B_i^- \cup B_i^+ \cup L \subset B_i$ and B_i^-, B_i^+, B_i are minimal with respect to these properties. If i is sufficiently large, then $B_i^- \cap B_i^+ = \emptyset$ and $B_i^* = \operatorname{Cl}(B_i \setminus (B_i^- \cup B_i^+))$ is also a ball. We define the following p.i. knot products

$$\tilde{\varkappa}(B_i, B_i \cap C) = \prod \pi^{d_i(\pi)}, \qquad \tilde{\varkappa}(B_i^*, B_i^* \cap C) = \prod \pi^{d_i^*(\pi)}.$$

The proof of the following assertion is an easy consequence of the local homogeneity of C: For each $\varepsilon > 0$ there is an index i_0 such that for each $i \geqslant i_0$ and every π we have

$$d_i^*(\pi) \leqslant d_i(\pi) \leqslant (1+\varepsilon) d_i^*(\pi).$$

Now $[\tilde{\varkappa}(Z, L)] = [\tilde{\varkappa}(Z', L)]$ can easily be proved by the following fact which is a consequence of Lemma (3.2): For each i which is sufficiently large $\tilde{\varkappa}(B_i^*, B_i^* \cap C)$ almost divides $\tilde{\varkappa}(Z, L)$, $\tilde{\varkappa}(Z', L)$, and $\tilde{\varkappa}(Z, L)$, $\tilde{\varkappa}(Z', L)$ almost divide $\tilde{\varkappa}(B_i, B_i \cap C)$. This proves that $[\tilde{\varkappa}](C)$ is independent of Z. That $[\tilde{\varkappa}](C)$ does not depend on C becomes clear by the fact that it is determined by the sequence $\tilde{\varkappa}(B_i^*, B_i^* \cap C) = \tilde{\varkappa}(B_i^*, B_i^* \cap L)$ $(i = i_0, i_0+1, \ldots)$.

Remark (4.9). If L is subdivided by a point p in the subarcs L_1, L_2, then $[\tilde{\varkappa}](L) = [\tilde{\varkappa}](L_1) \cdot [\tilde{\varkappa}](L_2)$.

We assume now that in $\tilde{\varkappa}(C) = \prod \pi^{e(\pi)}$ infinitely many prime knots with positive, finite exponents appear and denote these by $\pi_1, \pi_2, \ldots$

Proof of (4.3). Let $[\tilde{\varkappa}](L) = [\prod \pi^{g(\pi)}]$. We have to prove that

$$\lim_{i\to\infty} \big(g(\pi_i)/e(\pi_i)\big) = \varrho$$

exists.

Let $\sigma_1, \sigma_2, \ldots$ and $\sigma'_1, \sigma'_2, \ldots$ be subsequences of $\pi_1, \pi_2, \ldots$ If H is any subarc of C and $[\tilde{\varkappa}](H) = [\prod \pi^{h(\pi)}]$, then we denote the limits $\lim_{i\to\infty} \big(h(\sigma_i)/e(\sigma_i)\big)$ and $\lim_{i\to\infty} \big(h(\sigma'_i)/e(\sigma'_i)\big)$ — if they exist — by $\nu(H)$ and $\nu'(H)$ respectively. We note some properties of $\nu(H)$, $\nu'(H)$.

(4.10) $0 \leqslant \nu(H),\ \nu'(H) \leqslant 1$.

(4.11) If H is divided by an interior point in two subarcs H_1, H_2 and if the limits exist for two of the arcs H, H_1, H_2, then they exist for the third, and we have $\nu(H) = \nu(H_1)+\nu(H_2)$, $\nu'(H) = \nu'(H_1)+\nu'(H_2)$.

(4.12) If H' is the complementary arc of H in C, then the existence of $\nu(H)$, $\nu'(H)$ implies the existence of $\nu(H')$, $\nu'(H')$ and $\nu(H)+\nu(H') = \nu'(H)+\nu'(H') = 1$.

(4.13) If H and $H_1 \subset H_2 \subset \ldots$ are subarcs of C for which the limits exist and for which H ist the closure of $H_1 \cup H_2 \cup \ldots$, then $\nu(H_i) \leqslant \nu'(H_i)$ $(i = 1, 2, \ldots)$ implies $\nu(H) \leqslant \nu'(H)$.

(4.14) If $\nu(L_{pq}) < \nu'(L_{pq})$, then there is a point r in $\operatorname{Int} L_{pq}$ such that $\nu(L_{pr}) \geqslant \nu'(L_{pr})$. (As above, L_{pq} is the arc on C running in positive direction from p to q.)

The properties (4.10), (4.11), (4.12) are obvious, and (4.13) is a consequence of the local homogeneity of C. (As in the proof of Lemma (4.8) $[\tilde{\varkappa}](H) = [\prod \pi^{h(\pi)}]$, $[\tilde{\varkappa}](H_i) = [\prod \pi^{h_i(\pi)}]$, and $h(\pi) = \sup h_i(\pi)$, where $\prod \pi^{h(\pi)}$, $\prod \pi^{h_i(\pi)}$ are obtained by normal neighborhoods B, B_i of points in C for which $B \cap C = H$, $B_i \cap C = H_i$, $B_i \subset B$.)

For the proof of (4.14) we need the following fact:

(4.15) Let L_{uv}, L_{xy} be subarcs of C. Then there is a point $w \in L_{uv} \setminus \{u\}$ or a point $z \in L_{xy} \setminus \{x\}$ such that $[\tilde{\varkappa}](L_{uw}) = [\tilde{\varkappa}](L_{xy})$ in the first case or $[\tilde{\varkappa}](L_{xz}) = [\tilde{\varkappa}](L_{uv})$ in the second case.

This can be proved as follows: Call finite sequences $u_1, u_2, \ldots, u_n$ in L_{uv} and $x_1, x_2, \ldots, x_n$ in L_{xy} equivalent, if $u_1 = u$, $x_1 = x$, $u_{i+1} \in L_{u_i v} \setminus \{u_i\}$, $x_{i+1} \in L_{x_i y} \setminus \{x_i\}$ and $[\tilde{\varkappa}](L_{u_i u_{i+1}}) = [\tilde{\varkappa}](L_{x_i x_{i+1}})$ $(i = 1, \ldots, n-1)$. By the local homogeneity of C there are equivalent sequences for which $x_n = y$ or $u_n = v$. Define $w = u_n$ in the first case and $z = x_n$ in the second case.

Now we consider L_{pq} as in (4.14). For $L_{uv} = L_{pq}$, $L_{xy} = L_{qp}$ we get by (4.15) a point $w \in L_{pq}$ or a point $z \in L_{qp}$. In the first case define $r = w$, and (4.14) is proved. In the second case we have $\nu(L_{zp}) > \nu'(L_{zp})$ and we apply (4.15) once more for $L_{uv} = L_{pq}$, $L_{xy} = L_{zp}$. After a finite number of steps we get in this way a point $w = r$ in L_{pq}.

Now we apply the properties (4.10)–(4.14) to prove (4.3). Let p, q be the end points of the arc L in (4.3) such that $L = L_{pq}$. Assume that $\lim_{i\to\infty} \big(g(\pi_i)/e(\pi_i)\big)$ does not exist. Since $0 \leqslant g(\pi_i)/e(\pi_i) \leqslant 1$ there are sequences $\sigma_1, \sigma_2, \ldots$ and $\sigma'_1, \sigma'_2, \ldots$ such that $\nu(L_{pq}) < \nu'(L_{pq})$. We consider the union M of all subarcs L_{pr} of L_{pq} for which $\nu(L_{pr}) \geqslant \nu'(L_{pr})$ holds. By (4.14) M is not empty, and by (4.14) and (4.11) M is open.

But (4.13) implies that M is closed, and we have $M = L_{pq}$ and therefore $\nu(L_{pq}) \geqslant \nu'(L_{qp})$ which is a contradiction.

The proof of (4.4) is easy and left to the reader.

5. Special defining sequences. In this section we introduce a special kind of defining sequences of n.s.t. neighborhoods for simple closed curves in S^3 and reduce the proofs for Theorem 2 and the remaining part of Theorem 3 to a lemma about the existence of such special defining sequences (Lemma (5.8)).

(5.1) Two defining sequences $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ and $\mathfrak{C}'_1, \mathfrak{C}'_2, \ldots$ of n.s.t. neighborhoods for simple closed curves C, C' respectively will be called *isomorphic*, if the following conditions are satisfied:

(a) $\mathfrak{C}_i = (Z_{i,1}, \ldots, Z_{i,r_i})$ and $\mathfrak{C}'_i = (Z'_{i,1}, \ldots, Z'_{i,r_i})$ consist of the same number of cells ($i = 1, 2, \ldots$).

(b) $Z_{i,k} \subset Z_{j,l}$ if and only if $Z'_{i,k} \subset Z'_{j,l}$.

(c) $\varkappa(Z_{i,k}, Z_{i,k} \cap T_{i+1}) = \varkappa(Z'_{i,k}, Z'_{i,k} \cap T'_{i+1})$, and the knots $\varkappa(T_1)$, $\varkappa(T'_1)$ represented by the tori T_1, T'_1 are equal too.

The following lemma is trivial.

Lemma (5.2). *If simple closed curves C, C' have isomorphic defining sequences of n.s.t. neighborhoods, then there is an orientation preserving homeomorphism $h\colon S^3 \to S^3$ mapping C onto C' respecting the orientations of these curves.*

For a p.i. knot product $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ we denote by $\sigma_1, \sigma_2, \ldots$ the sequence of all prime knots for which $e(\sigma_i) = \infty$. This sequence may be empty, finite, or infinite. If it is finite, we denote by σ the product of all knots σ_i. If the sequence is empty, σ is the trivial knot ν. Now we assume that $\tilde{\varkappa}$ satisfies the hypothesis of Theorem 3 concerning the exponents $e(\pi)$ and define a sequence $\lambda_1, \lambda_2, \ldots$ of finite knots as follows: λ'_i is the product of all prime knots π for which $e(\pi) = i$ and $\lambda_i = \lambda'_i \cdot \sigma$ if σ is defined or $\lambda_i = \lambda'_i \cdot \sigma_1 \ldots \sigma_i$ if we have an infinite sequence $\sigma_1, \sigma_2, \ldots$ Then we have $\tilde{\varkappa} = \prod_{i=1}^{\infty} \lambda_i^i$.

Now let C be a simple closed curve in S^3 which has a defining sequence of n.s.t. neighborhoods and which represents a p.i. knot product $\tilde{\varkappa} = \prod \pi^{e(\pi)}$ satisfying the hypothesis of Theorem 3. A special defining sequence for C is a defining sequence $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ of n.s.t. neighborhoods which satisfies the following conditions where $\mathfrak{C}_i = \{Z_{i,1}, \ldots, Z_{i,r_i}\}$, $T_i = Z_{i,1} \cup \ldots \cup Z_{i,r_i}$ (for technical reasons we use indices $3, 4, \ldots$ instead of $1, 2, \ldots$).

(i) $r_i = 4 \cdot 5 \cdot \ldots \cdot (i+1)$.

(ii) $Z_{i-1,k}$ contains $Z_{i,l}$ for $(k-1)(i+1) < l \leqslant k(i+1)$.

(iii) T_3 represents the knot $\varkappa(T_3) = \lambda_1 \cdot \lambda_2^2 \cdot \lambda_3^3$, and $\varkappa(Z_{i,k}, Z_{i,k} \cap T_{i+1})$ is λ_{i+1} if r_{i-1} divides k and the trivial knot ν if r_{i-1} does not divide k ($i = 3, 4, \ldots$, where $r_2 = 1$).

(iv) If W_{ik} denotes the union of all $Z_{i,l}$ for which $(k-1)r_{i-1} < l \leqslant kr_{i-1}$, then $\lim_{i\to\infty} \max_{1\leqslant k\leqslant i+1} \operatorname{diam} W_{i,k} = 0$.

As a simple consequence of these conditions we have $\varkappa(T_i) = \lambda_1 \cdot \lambda_2^2 \cdot \ldots \cdot \lambda_i^i$ and $\varkappa(W_{i,k}, W_{i,k} \cap T_{i+1}) = \lambda_{i+1}$. Since for special defining sequences the numbers r_i are fixed and the knots $\varkappa(Z_{i,k}, Z_{i,k} \cap T_{i+1})$ are determined by $\tilde{\varkappa}$, we have the following lemma:

Lemma (5.3). *Two special defining sequences for C, C' respectively corresponding to the same p.i. knot product are isomorphic.*

Corollary (5.4). *If C, C' have special defining sequences and if the corresponding p.i. knot products are equal, then there is an orientation preserving homeomorphism $h\colon S^3 \to S^3$ such that $h(C) = C'$ respecting the orientations of these curves.*

Lemma (5.5). *If a p.i. knot product $\tilde{\varkappa}$ satisfies the hypothesis of Theorem 3 concerning the exponents, then there is a simple closed curve C in S^3 which represents $\tilde{\varkappa}$ and has a special defining sequence.*

This can easily be proved by constructing a sequence $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ which satisfies (i)–(iv). (In order to obtain (iv) make all balls $Z_{i,k}$ so small that $\operatorname{diam} Z_{i,k} < (i+1)!^{-1}$ and that there is a polygonal arc in $Z_{i,k}$ of length $<(i+1)!^{-1}$ which connects $Z_{i,k-1} \cap Z_{i,k}$ with $Z_{i,k} \cap Z_{i,k+1}$.) The curve C is obtained as the intersection of all the corresponding solid tori T_i.

Lemma (5.6). *Each simple closed curve in S^3 which has a special defining sequence is homogeneously embedded.*

Proof. Let $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ be a special defining sequence for C ($\mathfrak{C}_i = \{Z_{i,1}, \ldots, Z_{i,r_i}\}$, $T_i = T_{i,1} \cup \ldots \cup Z_{i,r_i}$, $W_{i,k} = \bigcup Z_{i,l}$ where $(k-1)r_{i-1} < l \leqslant kr_{i-1}$) and let p be any point on $\mathfrak{C}$. We prove the lemma by constructing an orientation preserving homeomorphism $h\colon S^3 \to S^3$ such that $h(C) = C$ respecting the orientation of C and $h(p) = p_0$ where p_0 is the point in which C pierces $Z_{3,r_3} \cap Z_{3,1}$. We denote by $k(i)$ the index for which $p \in W_{i,k(i)} \setminus W_{i,k(i)+1}$ (second indices modulo $i+1$). For each index $i = 3, 4, \ldots$ we define an orientation preserving homeomorphism $h_i\colon \operatorname{Bd} T_i \to \operatorname{Bd} T_i$ for which there is an integer m such that

$$h_i(Z_{i,k} \cap \operatorname{Bd} T_i) = Z_{i,k+m} \cap \operatorname{Bd} T_i\,. \tag{5.7}$$

The number m is determined by the condition

$$h_i(W_{i,k(i)} \cap \operatorname{Bd} T_i) = W_{i,1} \cap \operatorname{Bd} T_i\,.$$

Since the balls $W_{i,k}$ become smaller and smaller as $i \to \infty$ all these homeomorphisms h_i define a uniformly continuous homeomorphism $h'\colon B \to B$ where $B = \bigcup_{i=3}^{\infty} \operatorname{Bd} T_i$. By the uniform continuity h' can be extended to a continuous mapping $h''\colon B \cup C \to B \cup C$, and we see by (5.7) that h'' is one-to-one and therefore a homeomorphism. Now we mention that by the definition of special defining sequences we have $\varkappa(W_{i,k}, W_{i,k} \cap T_{i+1}) = \lambda_{i+1}$. Therefore we can define for each index $i = 3, 4, \ldots$ a homeomorphism $g_i\colon T_i\setminus\mathrm{Int}T_{i+1} \to T_i\setminus\mathrm{Int}T_{i+1}$ which extends $h_i\colon \mathrm{Bd}T_i \to \mathrm{Bd}T_i$ and for which there are integers m', m'' such that

$$g_i(W_{i,k}\setminus\mathrm{Int}T_{i+1}) = W_{i,k+m'}\setminus\mathrm{Int}T_{i+1},$$

$$g_i(Z_{i+1,k}\cap\mathrm{Bd}T_{i+1}) = Z_{i+1,k+m''}\cap\mathrm{Bd}T_{i+1}.$$

Now we modify g_i by a shift near $\mathrm{Bd}T_{i+1}$ to get a homeomorphism $g_i'\colon T_i\setminus\mathrm{Int}T_{i+1} \to T_i\setminus\mathrm{Int}T_{i+1}$ which coincides on $\mathrm{Bd}T_i$ with h_i and on $\mathrm{Bd}T_{i+1}$ with h_{i+1}. These homeomorphisms can be chosen so that for $i\to\infty$ we have $g_i'\to g_i$. Then by $h = g_i'$ on $T_i\setminus\mathrm{Int}T_{i+1}$, $h = h''$ on C a homeomorphism $h\colon T_3\to T_3$ is defined which can be extended over S^3 and which satisfies $h(p) = p_0$.

LEMMA (5.8). *Each locally homogeneously embedded simple closed curve in S^3 has a special defining sequence.*

This lemma will be proved in the next section.

It is clear that Lemma (5.5) and Lemma (5.6) give the remaining part of Theorem 3. Theorem 2 is obtained by Corollary (5.4) and Lemma (5.8).

6. Proof of Lemma (5.8). We start with two simple remarks.

(6.1) Let T be a tunnel in a ball Z where $\varkappa(Z, T)$ is the product $\varkappa_1\cdot\varkappa_2$ of two knots. Then there is a tunnel T' in Z with the same entrance and exit as T such that $T\subset T'$ and $\varkappa(Z, T') = \varkappa_1$ (see Fig. 2).

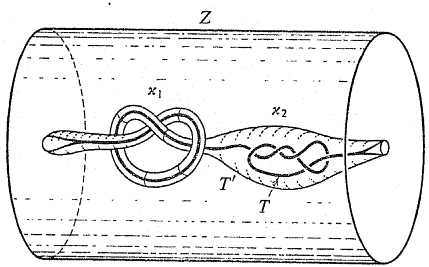

Fig. 2

(6.2) Let Z, Z', Z'' be oriented balls with tunnels T, T', T'' respectively such that $Z'\subset Z$, $Z'\cap\mathrm{Bd}Z = D$ is a disk, $Z''\cap Z = \mathrm{Bd}Z''\cap\mathrm{Bd}Z = D$, $T' = T\cap Z'$, $T''\cap Z = T'\cap\mathrm{Bd}Z\subset\mathrm{Int}D$. Then $A = D\setminus\mathrm{Int}(D\cap T)$ is an annulus. Let $\varkappa = \prod\pi^{e(\pi)}$ be a (finite) knot. We assume that $\varkappa(Z, T) = \prod\pi^{f(\pi)}$, $\varkappa(Z', T') = \prod\pi^{f'(\pi)}$, $\varkappa(Z'', T'') = \prod\pi^{f''(\pi)}$, and $-f'(\pi)\leqslant e(\pi)-f(\pi)\leqslant f''(\pi)$ for all π. Then there is an annulus A' with $\mathrm{Bd}A' = \mathrm{Bd}A$ and $\mathrm{Int}A'\subset\mathrm{Int}(Z'\cup Z'')\setminus(T'\cup T'')$ such that if Z^* denotes the ball bounded by $(\mathrm{Bd}Z\setminus A)\cup A'$ we have $\varkappa(Z^*, T) = \varkappa$ (see Fig. 3.)

Now we prove Lemma (5.8). Let C be a locally homogeneous simple closed curve in S^3. By (3.1) and (5.5) we can find a special defining sequence $\mathfrak{C}_3^*$, $\mathfrak{C}_4^*$, ... for a simple closed curve C^* such that $\tilde{\varkappa}(C^*) = \tilde{\varkappa}(C)$. As usual the balls in $\mathfrak{C}_i^*$ are denoted by $Z_{i,k}^*$, and T_i^* are the corresponding toral neighborhoods of C^*. Using

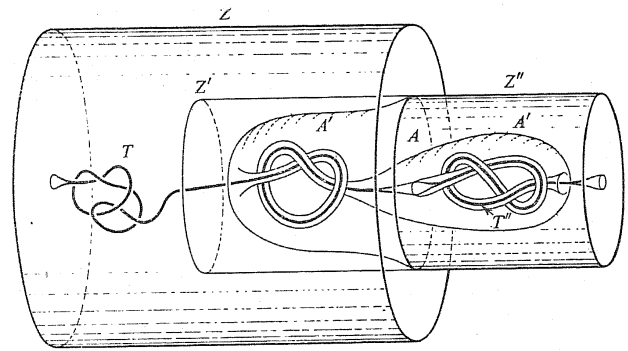

Fig. 3

$\varkappa(Z^*, T) = \varkappa(Z, T)\varkappa(Z', T')^{-1}\varkappa(Z'', T'')$

a homeomorphism $\varphi\colon C\to S^1$ which is defined by (4.4) if in $\tilde{\varkappa}(C)$ infinitely many prime knots with positive, finite exponent appear and which is arbitrary in the other case we define subarcs $L_{i,k}$ of C by

$$L_{j,k} = \{\varphi^{-1}(e^{2\pi it});\ (k-1)/r_j\leqslant t\leqslant k/r_j\}$$

($j = 3, 4, \ldots$; $k = 1, 2, \ldots, r_j$ where $r_j = 4\cdot 5\cdot\ldots\cdot(i+1)$ is the number of balls in $\mathfrak{C}_j^*$). By the definition of φ we have $[\tilde{\varkappa}](L_{j,k}) = [\tilde{\varkappa}(C)]^{1/r_j} = [\tilde{\varkappa}](Z_{j,k}^*\cap C^*)$.

Now it is not difficult to find a defining sequence $\mathfrak{C}_3^0$, $\mathfrak{C}_4^0$, ... of n.s.t. neighborhoods for C such that

(1) $\mathfrak{C}_i^0$ consists of the same number r_i of balls $Z_{i,1}^0, \ldots, Z_{i,r_i}^0$ as $\mathfrak{C}_i^*$, and $Z_{i,k}^0\subset Z_{j,l}^0$ if and only if $Z_{i,k}^*\subset Z_{j,l}^*$.

(2) $Z_{i,k}^0\cap C = L_{i,k}$ ($i = 3, 4, \ldots$; $k = 1, \ldots, r_i$).

After applying (6.1) we may assume that $\mathfrak{C}_3^0$, $\mathfrak{C}_4^0$, ... has the following additional properties:

(3) $\varkappa(T_i^0) = \varkappa(T_i^*)$ ($i = 3, 4, \ldots$) where $T_i^0 = Z_{i,1}^0\cup\ldots\cup Z_{i,r_i}^0$.

(4) If π is a prime knot which has in $\tilde{\varkappa}(C)$ the exponent ∞, then the exponent of π in $\varkappa(Z_{i,k}^0, Z_{i,k}^0\cap T_{i+1}^0)$ and the exponent of π in $\varkappa(Z_{i,k}^*, Z_{i,k}^*\cap T_{i+1}^*)$ are equal.

Possibly (6.1) must be applied infinitely often to obtain (3) and (4), and a little bit care is necessary in order that after these modifications we have still $\operatorname{diam} Z^0_{i,k} \to 0$, if $i \to \infty$.

To get a special defining sequence $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ for C we shall perform an infinite series of modifications on the sequence $\mathfrak{C}^0_3, \mathfrak{C}^0_4, \ldots$ In this connection we use the following notation: Let $\mathfrak{C}^u_3, \mathfrak{C}^u_4, \ldots$ be a defining sequence of n.s.t. neighborhoods (where $u = 0, 1, 2, \ldots$ or $u = *$). Then, if $\mathfrak{C}^u_i = \{Z^u_{i,1}, \ldots, Z^u_{i,r_i}\}$, we denote the balls in $\mathfrak{C}^u_3 \cup \mathfrak{C}^u_4 \cup \ldots$ by $Z^u_1, Z^u_2, \ldots$ where $Z^u_1 = Z^u_{3,1}$, $Z^u_2 = Z^u_{3,2}, \ldots, Z^u_{r_1+1} = Z^u_{4,1}, \ldots$, i.e. for $Z^u_{i,k} = Z^u_m$, $Z^u_{j,l} = Z^u_n$ we have $m<n$ if and only if $i<j$ or $i=j$ and $k<l$.

The sequence which we get after v modifications will be denoted by $\mathfrak{C}^v_3, \mathfrak{C}^v_4, \ldots$ It will have in addition to (1)–(4) the following properties:

(5) If $n \leqslant v$, then $\varkappa(Z^v_n, Z^v_n \cap T^v_j) = \varkappa(Z^*_n, Z^*_n \cap T^*_j)$ for all $j>i$ where i is determined by $Z^v_n = Z^v_{i,k}$ and $Z^*_n = Z^*_{i,k}$.

(6) If $n \leqslant v$, then $Z^v_n = Z^n_n$.

(7) $T^v_i = T^0_i$ $(i = 3, 4, \ldots; v = 1, 2, \ldots)$.

(8) $Z^v_n \cap \operatorname{Bd} T^0_i = Z^0_n \cap \operatorname{Bd} T^0_i$ $(i = 3, 4, \ldots; v = 1, 2, \ldots; n = 1, 2, \ldots)$.

By (1), (2), (6), and (7) we see that the sequence $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ where

$$\mathfrak{C}_i = \{Z^v_{i,1}, \ldots, Z^v_{i,r_i}\}$$

for large v is a defining sequence of n.s.t. neighborhoods for C. That the diameters of the balls $Z_{j,1}$ tend to zero for $j \to \infty$ is implied by (6) and (7) and the fact that $Z^v_{i,k} \cap T^0_m$ is near $Z^v_{j,k} \cap C = L_{i,k}$ for large m. Then by (1) $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ satisfies conditions (i) and (ii) for special defining sequences (see Section 5), and (5) implies (iii). Property (iv) can be obtained by the remark that $W_{i,k} \cap C$ is the union of all $L_{i,l}$ for which $(k-1)r_{i-1} < l \leqslant kr_{i-1}$ and that this union is mapped by the homeomorphism $\varphi\colon C \to S^1$ of (4.4) onto an subarc of S^1 with length $2\pi/(i+1)$. Therefore, $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ is a special defining sequence.

Now we describe the construction which leads from the sequence $\mathfrak{C}^{v-1}_3, \mathfrak{C}^{v-1}_4, \ldots$ which is assumed to satisfy (1)–(8) to $\mathfrak{C}^v_3, \mathfrak{C}^v_4, \ldots$ $(v = 1, 2, \ldots)$. We consider the ball $Z^{v-1}_v = Z^{v-1}_{i,k}$ and the annuli $A_j = D^{v-1}_{i,k} \cap (T^0_j \setminus \operatorname{Int} T^0_{j+1})$ where $D^{v-1}_{i,k} = Z^{v-1}_{i,k} \cap Z^{v-1}_{i,k+1}$ $(j = i, i+1, \ldots)$. By (3), (5), and (7) we can apply (6.2) in order to obtain new annuli A'_j $(j = i, i+1, \ldots)$ which have the following properties:

(a) $\operatorname{Bd} A'_j = \operatorname{Bd} A_j = A'_j \cap (\operatorname{Bd} Z^{v-1}_v \setminus \operatorname{Int} A_j)$.

(b) $\operatorname{Int} A'_j \subset \operatorname{Int} T^0_j \setminus T^0_{j+1}$.

(c) $\varkappa(Z'_j \cap T^0_j, Z'_j \cap T^0_{j+1}) = \varkappa(Z^*_{i,k} \cap T^*_j, Z^*_{i,k} \cap T^*_{j+1})$ $(j = i, i+1, \ldots)$ where Z'_j denotes the ball which is bounded by $(\operatorname{Bd} Z^{v-1}_{i,k} \setminus A_j) \cup A'_j$.

(d) $A'_j \subset \operatorname{Int} Z^{v-1}_n$ provided $n<v$ and $A_j \subset \operatorname{Int} Z^{v-1}_n$. $A'_j = A_j$ if there is a ball Z^{v-1}_n with $n<v$ such that $A_j \subset \operatorname{Bd} Z^{v-1}_n$.

If $j>i$ let $Z^{v-1}_{j,l}, Z^{v-1}_{j,l+1}$ be the two balls in $\mathfrak{C}^{v-1}_j$ which intersect $D^{v-1}_{i,k}$. Then by the definition of the arcs $L_{m,n}$ in the triplet

$$\tilde{\varkappa}_1 = \tilde{\varkappa}(\operatorname{Cl}(Z^{v-1}_{i,k} \setminus Z^{v-1}_{j,l}), \operatorname{Cl}(Z^{v-1}_{i,k} \setminus Z^{v-1}_{j,l}) \cap C),$$
$$\tilde{\varkappa}_2 = \tilde{\varkappa}(Z^*_{i,k}, Z^*_{i,k} \cap C^*),$$
$$\tilde{\varkappa}_3 = \tilde{\varkappa}(Z^{v-1}_{i,k} \cup Z^{v-1}_{j,l+1}, (Z^{v-1}_{i,k} \cup Z^{v-1}_{j,l+1}) \cap C),$$

$[\tilde{\varkappa}_1] = [\tilde{\varkappa}(C)]^{\varrho_1}$, $[\tilde{\varkappa}_2] = [\tilde{\varkappa}(C)]^{\varrho_2}$, $[\tilde{\varkappa}_3] = [\tilde{\varkappa}(C)]^{\varrho_3}$ holds with $\varrho_1 < \varrho_2 < \varrho_3$. This implies that $\tilde{\varkappa}_1$ almost divides $\tilde{\varkappa}_2$ and $\tilde{\varkappa}_2$ almost divides $\tilde{\varkappa}_3$. Therefore, by (6.2) we can choose A'_m in $Z^{v-1}_{j,l} \cup Z^{v-1}_{j,l+1}$ provided m is sufficiently large. This remark allows to assume that — for large m — the annuli A'_m are near A_m and that $(\operatorname{Bd} Z^{v-1}_{i,k} \setminus D^{v-1}_{i,k}) \cup \bigcup_{m=i}^{\infty} A'_m$ is a 2-sphere which bounds a ball $Z^v_v = Z^v_{i,k}$.

If we replace in $\mathfrak{C}^{v-1}_i$ the ball $Z^{v-1}_v = Z^{v-1}_{i,k}$ by Z^v_v, then it may happen that these new balls do not fit together to a cell decomposition of T^0_i, and for $n>v$ balls Z^{v-1}_n may intersect Z^v_v rather wildly. But by (d) it is possible to replace the balls Z^{v-1}_n $(n>v)$ by new balls Z^v_n such that we obtain the new defining sequence $\mathfrak{C}^v_3, \mathfrak{C}^v_4, \ldots$

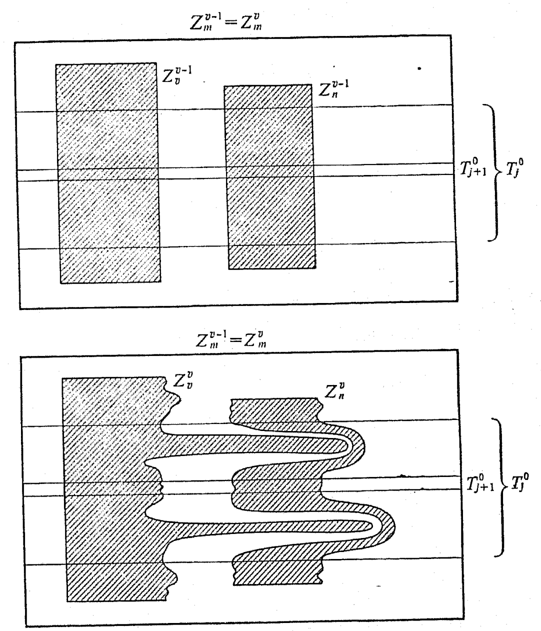

Fig. 4

which has all properties (1)–(8) (see Fig. 4). Indeed $Z_n^v \neq Z_n^{v-1}$ will happen only for balls Z_n^{v-1} which lie in the union of $Z_{i,k}^{v-1}$ and the component of $(T_i^0 \cap Z_m^{v-1}) \setminus Z_{i,\varkappa}^{v-1}$ which intersects Z_{v+1}^{v-1}, where Z_m^{v-1} is the ball of $\mathfrak{C}_{i-1}^{v-1}$ which contains $Z_{i,k}^{v-1}$.

7. Proof of Theorem 4. The second part of this theorem is an immediate consequence (4.5) and (5.6). To prove the first part we proceed as in [2]. There the following lemma is proved:

LEMMA (7.1). *Let C be a simple closed curve in S^3 with the following property: If a subarc L of C and a positive number ε are given, then there is a homeomorphism $f\colon S^3 \to S^3$ which is the identity outside the ε-neighborhood of L such that $f(C) = C$ and $d(f(p), q) < \varepsilon$ where p, q are the end points of L. Then each orientation preserving homeomorphism $h_0\colon C \to C$ can be extended to an orientation preserving homeomorphism $h\colon S^3 \to S^3$.*

Consider a locally homogeneously embedded simple closed curve C in S^3 which pierces a disk and for which in $\tilde{\varkappa}(C) = \prod \pi^{e(\pi)}$ only finitely many prime knots π have a positive finite exponent $e(\pi)$. Then we can find a special defining sequence $\mathfrak{C}_3, \mathfrak{C}_4, \ldots$ for C and an index k such that (if $\mathfrak{C}_i = \{Z_{i,1}, \ldots, Z_{i,r_i}\}$ and $T_i = Z_{i,1} \cup \ldots \cup Z_{i,r_i}$) the "finite part" $\prod \pi^{d(\pi)}$ of $\tilde{\varkappa}(C)$ is a divisor of $\varkappa(T_k)$ where

$$d(\pi) = \begin{cases} e(\pi) & \text{if } 0 \leqslant e(\pi) < \infty, \\ 0 & \text{if } e(\pi) = \infty. \end{cases}$$

Let L be a subarc of C with end points p, q and let U be a neighborhood of L in S^3. We choose an arc L' in $C \cap U$ with end points p', q such that $L \subset L'$ and $p \in \operatorname{Int} L'$. Then we can select balls Z_n $(n = \ldots, -2, -1, 0, 1, 2, \ldots)$ in $\mathfrak{C}_k \cup \mathfrak{C}_{k+1} \cup \ldots$ such that

(1) $Z_n \subset U$ $(-\infty < n < \infty)$.

(2) $Z_n \cap Z_{n+1} = \operatorname{Bd} Z_n \cap \operatorname{Bd} Z_{n+1} = D_n$ is a disk.

(3) $Z_n \cap Z_m = \emptyset$ if $|n-m| \geqslant 2$.

(4) $\bigcup_{n=-\infty}^{\infty} Z_n \cap C = \operatorname{Int} L'$.

(5) $\limsup_{n \to -\infty} Z_n = p'$, $\limsup_{n \to +\infty} Z_n = q$.

Looking at the definition of the special defining sequences and at the choice of k we can prove by the methods of [2] that for each n there is a homeomorphism $f_n\colon Z_n \to Z_{n+1}$ such that $f_n(Z_n \cap C) = Z_{n+1} \cap C$, $f_n(D_{n-1}) = D_n$, $f_n(D_n) = D_{n+1}$, and $f_n(x) = f_{n+1}(x)$ for $x \in D_n$. These homeomorphisms f_n can be extended to a homeomorphism $f\colon S^3 \to S^3$ which is the identity on $S^3 \setminus U$ and on $C \setminus L'$. Then for a given $\varepsilon > 0$ we have $d(f^l(p), q) < \varepsilon$ provided l is sufficiently large. Since $f^l(C) = C$ and $f^l = \mathrm{id}$ on $S^3 \setminus U$, Lemma (7.1) proves the first part of Theorem 4. (For more details of the proof in this section see [2].)

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INSTITUT FÜR MATHEMATIK
DER AKADEMIE DER WISSENSCHAFTEN DER DDR
Berlin

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