Constructing exotic retracts, factors of manifolds, and generalized manifolds via decompositions

by

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Abstract. By X satisfies a property (p) we mean that X contains no proper subset of dimension ≥ 2 which is an FAR or an ANR, and X contains no proper compact subset of dimension ≥ 2 with UV^1 in X. For each integer $n \geq 3$, there exists a space X of dimension n such that X satisfies (p), and in addition, any one of the following: (a) X is an AR; (b) $X \times S^1 \sim S^n \times S^1$; or (c) $X \times E^1 \sim E^n \times E^1$. For $n = 3$, there are uncountably many such spaces as above.

1. Introduction, notation and terminology

(1.1) **Introduction.** By an AR (ANR) X we mean that X is a compact metric absolute (neighborhood) retract, see [8, 18] for more information concerning these spaces. By an FAR X we mean that X is a compact fundamental absolute retract in the sense of Borsuk [9; Chapt. VIII]. It is well-known that there exist AR's (ANR's) of arbitrary dimension n, $2 \leq n < \infty$, no one of which contains any proper ANR of dimension ≥ 2, see [5, 7, 8, 10, 23, 24, 25, 26, 30]. The purpose of this note is to extend the known results in this direction. As a sample we state the following:

THEOREM. *For each integer $n \geq 3$, there exists an* AR X *of dimension n such that X does not contain any proper subset which is an* FAR *or an* ANR *of dimension* ≥ 2.

The results of this note can be considered and interpreted in several contexts. The following brief list, which is given without relevant historical details, exemplifies some of these contexts and propagandizes our results: (a) The theory of exotic nonmanifold factors of manifolds which was pioneered by Bing [6]; (b) the theory of AR's, ANR's, and the shape theory of Borsuk [8, 9]; (c) the theory of generalized manifolds studied by Wilder and others (cf. [11, 28, 29]) and the topological properties of carriers of homology classes in generalized manifolds as contrasted with some results of Thom [27]; (d) the classical theory of Peano continua concerning the existence of arcs in these spaces; and (e) cell-like mappings, see Lacher [20] for an excellent survey. Furthermore, our results may be contrasted with some far reaching results of Cannon [12] and Edwards [15].

(1.2) **Notation and terminology.** Let E^n, B^n, and S^{n-1}, respectively, denote the n-dimensional Euclidean space, the closed unit ball in E^n, and the unit sphere in E^n. All maps are continuous. Suppose A and B are two closed subsets of a metric space (X, d). A surjective map $f\colon A \to B$ is called an *ε-displacement*, see [1; p. 7], if $d[a, f(a)] < \varepsilon$ for each a belonging to A. An *n-manifold* M^n (or a *manifold* M^n) is a separable metric space such that each point of M^n has a neighborhood whose closure is an n-cell. Observe that an n-manifold M^n may have a nonempty boundary and may not be connected. Suppose M^k is a k-manifold in E^n with $k<n$. M^k is said to be *flat* in E^n if M^k has a neighborhood U homeomorphic to $M^k \times B^{n-k}$ under a homeomorphism $h\colon U \to M^k \times B^{n-k}$ satisfying $h(M^k) = M^k \times \{0\}$. An n-manifold M^n in E^n is *flat* if the boundary ∂M^n is flat in E^n. *All manifolds considered in this note will be flat in a suitable* E^n (*which will be clear from the context*), *piecewise linear, and orientable unless to the contrary is stated.* By $\check{H}^n(A)$ we denote the nth Čech cohomology group of the space A with coefficients in group of integers (cf. [16]). By a *closed manifold* M^n we mean that M^n is a compact manifold without boundary. We shall often denote by $\operatorname{Int}(M^n)$ and ∂M^n, respectively, the interior and the boundary of a manifold M^n. We assume familiarity with cell-like decompositions. An excellent survey concerning cell-like decompositions is given by Lacher [20] where a definition of the property UV^1 and other related discussions may also be found. A collection C of closed subsets of a metric space X is called a *null collection* if for each $\varepsilon > 0$, there are only a finite number of elements in C each of which has diameter greater than ε. By $X \sim Y$ we mean that X is homeomorphic to Y.

2. Linking

Suppose $f\colon Y \to S^1$ is an essential map where Y is a compactum. We say Y is *irreducible with respect to the map* $f\colon Y \to S^1$ if the restriction of f to any proper closed subset of Y is inessential. The following lemma will be useful in the sequel.

(2.1.0) LEMMA. *Suppose X is a compact subset of E^n such that* $\dim(X) \geqslant 2$ *and* $n \geqslant 2$. *Then, there exist a continuum A and an essential map $f\colon A \to S^1$ such that* $A \subset X$ *and* $\dim(A) \leqslant 2$.

Proof. Choose a subset B of X with $\dim(B) = 2$. By Theorem VIII 3 of [17; p. 151], there exists a compactum C contained in B such that the sequence $\check{H}^1(B) \to \check{H}^1(C) \to 0$ fails to be exact, where the unlabelled homomorphism $\check{H}^1(B) \to \check{H}^1(C)$ is induced by the inclusion map $i\colon C \to B$. This implies that $\check{H}^1(C)$ is not the zero group. Choose a nonzero element γ^1 belonging to $\check{H}^1(C)$ and represent γ^1 by an essential map $g\colon C \to S^1$ where the homotopy class of g corresponds to γ^1 [19; p. 59]. Let $S = \{D\colon D$ is a closed subset of C with $g|\colon D \to S^1$ essential$\}$. Since the Čech cohomology is continuous, we may apply Brouwer's Reduction Theorem [17; p. 161] to find a closed subset A of C such that A is irreducible with respect to the essential map $f\colon A \to S^1$ where $f = g|$. A simple argument proves that A is connected and our proof is finished.

(2.2.0) Remarks on Lemma 2.1.0. The conclusions of Lemma 2.1.0 can be easily satisfied if X contains a continuum C having the shape of S^1. In this case, one may choose an essential map $f\colon C \to S^1$ such that the homotopy class of f corresponds to a generator of $\check{H}^1(C)$. It is not known to us whether every compactum X of dimension $\geqslant 2$ contains a continuum having the shape of S^1; we believe that this is not true in general.

(2.3) **Linking 1-cocycles in E^n, $n \geqslant 5$.** We shall assume familiarity with the classical theory of the intersection (or the linking) numbers for chains (cycles) in E^n. This theory appears in [2, 3] and in the well-known books of Seifert–Threlfall, and Lefschetz (see [29] for a bibliographic reference to these books). Throughout this note we shall assume that the coefficients for the homology are taken to be in the ring of integers for various calculations concerning the intersection (or the linking) numbers. Suppose M^p and M^q are two closed and connected manifolds in E^n such that $p+q = n-1$. The *linking number* $\operatorname{Lk}(M^p, M^q)$ is defined to be the absolute value of the linking number of arbitrary orientations of the manifold M^p and M^q. The manifolds M^p and M^q are said to be *linked* if $\operatorname{Lk}(M^p, M^q) \neq 0$.

(2.3.0) DEFINITION. Suppose A is a compact subset of E^k, $k \geqslant 3$, and M^{k-2} is a (connected) closed $(k-2)$-manifold in E^k. We say *A links M^{k-2} in a homological sense* (Abbreviate: A *h-links* M^{k-2}) if each neighborhood of A in E^n contains a simple closed curve M^1 such that M^1 and M^{k-2} are linked, i.e., $\operatorname{Lk}(M^1, M^{k-2}) \neq 0$.

Suppose X and Y are two disjoint subsets of E^k. *X links Y* in the sense of Wright [30] if each neighborhood of X contains a loop in $(E^k - Y)$ which is essential in $(E^k - Y)$. It is clear that A h-links M^{k-2}, where A and M^{k-2} are as above in Definition 2.3.0, implies A links M^{k-2} in the sense of [30]. We do not know how this concept of "h-linking" behaves with respect to linking in the sense of Vietoris–Čech homology theory (cf. [2, 3]). The proof of the following technical lemma will occupy the remainder of this section.

(2.3.1) LEMMA. *Suppose $f\colon A \to S^1$ is an essential map from a continuum A contained in E^n with* $\dim(A) \leqslant 2$ *and* $n \geqslant 5$. *Then there exists a closed $(n-2)$-manifold M^{n-2} contained in E^n such that A h-links M^{n-2}.*

Proof. Since S^1 is an ANR-space, there exists an extension $\tilde{f}\colon N_0 \to S^1$ of the map $f\colon A \to S^1$ where N_0 is a neighborhood of A in E^n. We require that N_0 is a compact, connected, and smooth n-manifold. Since $\tilde{f}$ can be approximated by a smooth map, we shall assume from now on, without loss of generality, that $\tilde{f}$ is smooth. Assume $\dim(A) = 2$. The case when $\dim(A) = 1$ will trivially follow our arguments. Let $N_1 \supset N_2 \supset N_3 \supset \ldots$ be a nest of (compact) neighborhoods of A such that $N_1 \subset N_0$ and $A = \bigcap_{i=0}^{\infty} N_i$. For each i, $0 \leqslant i < \infty$, choose a positive number ε_i such that each linear segment in E^n of length less than ε_i and with one endpoint in A is contained in N_i. Note that $\varepsilon_i \to 0$ as $i \to \infty$. For each i, $0 \leqslant i < \infty$, there exists an ε_i-displacement $\varphi_i\colon A \to P_i$ of A onto a 2-dimensional polyhedron $P_i \subset N_i$, see Aleksandrov [1] concerning ε-displacements where other references may also be found. For each i, $0 \leqslant i < \infty$, let K_i denote the complex corresponding to a triangulation of P_i. For each i, $0\leqslant i<\infty$, let $j_i\colon A\to N_i$ and $k_i\colon P_i\to N_i$ denote the respective inclusion maps, and let $f_i\colon N_i\to S^1$ denote the restriction of $\tilde{f}\colon N_0\to S^1$. Since $k_i\varphi_i\colon A\to S^1$, $0\leqslant i<\infty$, is homotopic to $j_i\colon A\to N_i$, it follows that $f_ij_i\colon P_i\to S^1$ is essential. For convenience of reference we shall summarize several facts concerning this setting as follows.

(2.3.2) The set $P_0(A)=[A\cup(\bigcup_{i=0}^{\infty}P_i)]$ is a compact subset of E^n contained in the neighborhood N_0 of A in E^n. Furthermore, we stipulate that $P_i\cap P_j=\emptyset$ whenever $i\neq j$, $0\leqslant i<\infty$, and $0\leqslant j<\infty$; although, this will not be necessary we prefer that the construction of $P_0(A)$ has this property. For each i, $0\leqslant i<\infty$, put $P_i(A)=[A\cup(\bigcup_{j=i}^{\infty}P_j)]$.

(2.3.3) Every neighborhood N of A contains a compact set $P_n(A)$, for some n with $0\leqslant n<\infty$. This is clear since one may choose an n such that $i\geqslant n$ implies that N_i is contained in N.

(2.3.4) Suppose N is a neighborhood of A in E^n. By (2.3.3), there exists an integer n such that $P_n(A)$ is contained in N and we may assume, without loss of generality, that $n=0$ and $N=N_0$. Now the restriction $\hat{f}\colon P_0(A)\to S^1$ of $\tilde{f}\colon N_0\to S^1$ has that property that for each i, $0\leqslant i<\infty$, the restriction $f_i\colon P_i\to S^1$ of $\hat{f}\colon P_0(A)\to S^1$ is essential.

(2.3.5) Our notation and terminology is as above in (2.3.4). Recall that for each i, $0\leqslant i<\infty$, K_i is a 2-complex corresponding to a triangulation of P_i. We let Q_i denote the underlying polytope of the 1-skeleton of K_i and we let $g_i\colon Q_i\to S^1$ denote the restriction of the map $f_i\colon P_i\to S^1$, for $0\leqslant i<\infty$. For each i, $0\leqslant i<\infty$, it follows that $g_i\colon Q_i\to S^1$ is essential. This is straightforward since $f_i\colon P_i\to S^1$ is essential. Furthermore, for each i, $0\leqslant i<\infty$, the polytope Q_i contains a circle C_i such that the restriction $h_i\colon C_i\to S^1$ of the map $g_i\colon Q_i\to S^1$ is essential. This is elementary and we omit details.

Choose two distinct points s and t in S^1 and keept them fixed for the remainder of our discussions. Let $S^1=I\cup J$, where I and J are two arcs such that $I\cap J=\{s,t\}$. Choose an n-manifold M^n in E^n such that M^n contains $f^{-1}(s)$ in its interior and $f^{-1}(t)\cap M^n=\emptyset$. Let M^{n-1} denote the boundary of M^n. Now $A=f^{-1}(I)\cap M^{n-1}$ and $B=f^{-1}(J)\cap M^{n-1}$ are two disjoint compact subsets of M^{n-1}. Choose an $(n-1)$-manifold M_1^{n-1} contained in M^{n-1} with $M^{n-2}=\partial M_1^{n-1}$ such that M_1^{n-1} contains A in its interior and $B\cap M_1^{n-1}=\emptyset$. Clearly, $A\cap M^{n-2}=\emptyset$. We shall prove in the next few paragraphs that A h-links M^{n-2}.

Let N be a (compact) neighborhood of A such that $N\cap M^{n-2}=\emptyset$. Choose an extension $\hat{f}\colon P_0(A)\to S^1$ of $f\colon A\to S^1$ as defined in (2.3.4). Let $f_i\colon P_i\to S^1$, $g_i\colon Q_i\to S^1$, and $h_i\colon C_i\to S^1$ be the suitable restrictions of $\hat{f}\colon P_0(A)\to S^1$ as defined in (2.3.4) and (2.3.5). In this setting and with notation and terminology of (2.3.3)-(2.3.5) we have the following.

(2.3.6) There exists a positive integer n_0 such that $i\geqslant n_0$ implies that $f_i^{-1}(s)$ is contained in the interior of M^n and $f_i^{-1}(t)\cap M^n=\emptyset$.

(2.3.7) There exists an integer m_0 such that $i\geqslant m_0$ implies that $f_i^{-1}(I)\cap M^{n-1}$ is contained in the interior or M_1^{n-1} and $[f_i^{-1}(J)\cap M^{n-1}]$ misses M_1^{n-1}.

(2.3.8) There exist a positive integer m such that $i\geqslant m$ implies the following: (a) $g_i^{-1}(s)$ is contained in the interior of M^n; (b) $g_i^{-1}(t)$ does not meet M^n; (c) $[g_i^{-1}(I)\cap M^{n-1}]$ is contained in the interior of M_1^{n-1}; and (d) $g_i^{-1}(J)\cap M_1^{n-1}=\emptyset$. Moreover, for each $i\geqslant m$ the following is clear: (a) $h_i^{-1}(s)$ is contained in the interior of M^n; (b) $h_i^{-1}(t)$ does not meet M^n; (c) $[h_i^{-1}(I)\cap M^{n-1}]$ is contained in the interior of M_1^{n-1}; and (d) $[h_i^{-1}(J)\cap\cap M_1^{n-1}]=\emptyset$.

These assertions can be proved by using point-set theoretic arguments. We omit details. We observe that for each i, $0\leqslant i<\infty$, the sets $h_i^{-1}(s)$ and $h_i^{-1}(t)$ are nonempty since the map $h_i\colon C_i\to S^1$ is essential. Let m be a positive integer such that (2.3.8) holds. It follows from some basic considerations concerning intersection numbers that for each $i\geqslant m$, the linking number $\mathrm{Lk}(C_i,M^{n-2})$ of C_i with the manifold M^{n-2} is equal to $|\deg(h_i)|$, where $|\deg(h_i)|$ denotes the absolute value of the degree of h_i; and hence, C_i and M^{n-2} are linked since the map h_i is essential. We shall sketch a proof of this fact in the next paragraph.

Choose an index $j\geqslant m$. For simplicity of notation, put $C=C_j$ and denote $h_j\colon C_j\to S^1$ by $h\colon C\to S^1$. Recall that $\tilde{f}\colon N_0\to S^1$ is smooth. Without loss of generality, we assume that s and t are regular values (cf. [22]) of the map $h\colon C\to S^1$. Let us recall, for convenience, the following facts: (a) The set $h^{-1}(s)$ is contained in $\mathrm{Int}(M^n)$ and $h^{-1}(t)\cap M^n=\emptyset$; (b) the compact set $[h^{-1}(I)\cap M^{n-1}]$ is contained in $\mathrm{Int}(M_1^{n-1})$ and $[h^{-1}(J)\cap M^{n-1}]$ misses M_1^{n-1}; and (c) $M^{n-2}=\partial M_1^{n-1}$ and $M^{n-1}=\partial M^n$. We want to show that $\mathrm{Lk}(C,M^{n-2})=|\deg(h)|$. Construct an n-manifold M_2^n as a disjoint union of n-balls $B_1,B_2,\ldots$, and B_k such that for each i, $1\leqslant i\leqslant k$, $(B_i\cap C)$ is connected, h restricted to $(B_i\cap C)$ has no critical points, and each B_i contains exactly one point of $h^{-1}(s)$; and furthermore, we require that $h^{-1}(s)$ is contained in $\mathrm{Int}(M_2^n)$. Assume that the sets $h^{-1}(I)$ and ∂M_2^n are in relative general position. Choose an $(n-1)$-manifold N^{n-1} inside ∂M_2^n such that: (a) N^{n-1} is a disjoint union of $(n-1)$-balls where each ball contains exactly one point of the set $[h^{-1}(I)\cap\partial M_2^n]$; (b) the set $[h^{-1}(I)\cap\partial M_2^n]$ is contained in the $\mathrm{Int}(N^{n-1})$; and (c) N^{n-1} misses the set $h^{-1}(J)$. In the following, we let $A.B$ denote the absolute value of the intersection number between A and B. Notice that it follows from the definition of the degree and the choice of ∂M_2^{n-1} that $|\deg(h)|=C.N^{n-1}$. By definition, $\mathrm{Lk}(C,\partial N^{n-1})$ equals to $C.N^{n-1}$. The equality $C.N^{n-1}=h^{-1}(I).N^{n-1}=h^{-1}(I).\partial M_2^n$ is clear. Since the boundary $\partial[h^{-1}(I)]$ is disjoint from $\mathrm{cl}(M^n-M_2^n)$, it follows that $h^{-1}(I).\partial[\mathrm{cl}(M^n-M_2^n)]=0$ where $\mathrm{cl}(M^n-M_2^n)$ denotes the closure of $(M^n-M_2^n)$. Therefore, $h^{-1}(I).\partial M^n=h^{-1}(I).\partial M_2^n=h^{-1}(I).N^{n-1}=|\deg(h)|$. This finishes our proof.

(2.4) **Linking 1-cocycles in E^n, $n = 3$ or 4.** Suppose A is a continuum of dimension at most 2 in E^n, where $n = 3$ or 4, and $f: A \to S^1$ is an essential map. Our proof of Lemma 2.3.1 can be adapted to handle this situation as follows. Identify E^3 and E^4 with the subsets $E^3 \times \{0\} \times \{0\}$ and $E^4 \times \{0\}$ of E^1, respectively. Clearly, the set $P_0(A)$, see (2.3.2), can be constructed in E^5 satisfying an additional requirement that for each i, $0 \leqslant i < \infty$, the set Q_i is contained in E^n with $n = 3$ or 4. Apply the arguments of Lemma 2.3.1 by suitably restricting to the set $[A \cup (\bigcup_{i=0}^{\infty} Q_i)]$ and E^3 to find a manifold M^{n-2} such that A links M^{n-2}. Note that M^{n-2} is a circle when $n = 3$.

(2.5) **Linking and some results of Thom.** The results of Thom [27] and the Alexander duality may be considered in relation to our present setting. We shall discuss this as follows. Suppose γ^1 is a nonzero element of $\check{H}^1(A)$ where A is a continuum of dimension at most 2 and A is contained in E^n with $n \geqslant 3$. (We are not interested in generality!) By the Alexander duality, there exists an element z of $H_{n-2}(S^n - A)$ which is dual to γ^1. Thom [27] has studied the problem of representing a homology class by a manifold which may not be flat. It follows from Thom [27] that there exists a nonzero integer k such that $k.z$ is represented by an $(n-2)$-manifold M^{n-2} in $(E^n - A)$ where M^{n-2} may not be flat; however, for E^3 the 1-manifold M^1 obtained in this manner is a smoothly embedded circle in E^3 which is clearly flat. Hence for E^3, this provides an alternative proof of our results concerning linking given in (2.4). For $n \geqslant 4$, the manifold M^{n-2} found above, does not suffice for our purposes. This is primarily due to the fact that a "Cantor-set replacement technique", which is discussed by Daverman–Edwards [14] in the case when M^{n-2} is flat in E^n, remains unknown when M^{n-2} is not flat. Our linking techniques avoid these difficulties.

3. Decompositions of B^n, S^n, or E^n

(3.1.0) Theorem. *For each integer $n \geqslant 3$, there exists an u.s.c. decomposition G of B^n such that*: (a) *G is a null collection of arcs and singletons sets*; (b) *the decomposition space B^n/G is an* AR *of dimension n; and* c) *B^n/G does not contain any proper* FAR *or any proper* ANR *of dimension $\geqslant 2$. Furthermore, B^n/G does not contain any proper closed subset of dimension $\geqslant 2$ with the property* UV^1.

Proof. We shall consider the case when $n = 3$ separately. Suppose n is a fixed integer with $n \geqslant 4$. Let G be the decomposition of B^n defined by Wright [30] and let $p: B^n \to B^n/G$ denote the projection onto the decomposition space B^n/G. We shall prove B^n/G satisfies our requirements. It suffices to show that B^n/G does not contain any proper closed subset of dimension $\geqslant 2$ and satisfying UV^1.

Suppose B is a closed subset of B^n/G of dimension $\geqslant 2$ and satisfying UV^1. By Armentrout [4; Lemma 5.9], the set $A = p^{-1}(B)$ satisfies UV^1. It is easy to see that $\dim(A) \geqslant 2$. By Lemma 2.1.0 and Lemma 2.3.1, there exists a continuum C contained in A such that C h-links an $(n-2)$-manifold M^{n-2}. Clearly, C h-links M^{n-2} implies that C links M^{n-2} in the sense of Wright [30]. The remainder of the proof is the usual "backing-up argument" which is used in Singh [24] and Wright [30]; we proceed as in Wright [30; p. 126] to find a pair of the form (M^j, V^j) which is described in [30; p. 125–126] such that C links M^j in the sense of [30] and A does not meet the closure of V^j. The remainder of the proof is the same as in [30; p. 126] and we omit details.

Suppose $n = 3$. Consider the u.s.c. decomposition of B^3 defined in [24]. The arguments given in [24] also prove the desired result. This finishes our proof.

The following are some immediate corollaries of our methods. The proofs for these corollaries are analogous to the proof of Theorem 3.1, and therefore, omitted. Some results of Meyer [21] will also be needed.

(3.1.1) Corollary. *For each integer $n \geqslant 3$, there exists an u.s.c. decomposition G of S^n such that*: (a) *G is a null collection of arcs and singletons sets*; (b) *the decomposition space S^n/G is an n-dimensional* ANR *satisfying $S^n/G \times S^1 \sim S^n \times S^1$; and* (c) *$S^n/G$ does not contain any proper* ANR *or any* FAR *of dimension $\geqslant 2$. Furthermore, S^n/G does not contain any proper closed subset of dimension $\geqslant 2$ with* UV^1.

(3.1.2) Corollary. *For each integer $n \geqslant 3$, there exists an u.s.c. decomposition G of E^n such that*: (a) *G is a null collection of arcs and singleton sets*; (b) *the decomposition space E^n/G is an absolute retract for metric spaces satisfying $E^n/G \times E^1 \sim E^{n+1}$; and* (c) *$E^n/G$ does not contain any* ANR *or* FAR *of dimension $\geqslant 2$. Furthermore, E^n/G does not contain any compact subset with* UV^1.

These corollaries provides examples of rather exotic Cartesian factors of the manifolds $S^n \times S^1$ and E^{n+1}, with $n \geqslant 3$. A *generalized n-manifold M* is an ENR (Euclidean neighborhood retract, see [20]) such that for each $x \in M$, $H_*(M, M - \{x\}) \approx H_*(E^n, E^n - \{0\})$, where the homology is taken with integral coefficients. It is well-known that a finite dimensional cell-like image of a generalized manifold is a generalized manifold [28]. *It is interesting to note that the decomposition spaces S^n/G and E^n/G satisfying the conclusions of Corollaries* (3.1.1) *and* (3.1.2), *respectively, are generalized n-manifolds.*

(3.2) **A family of 3-dimensional** ANR's **(AR's).** There is quite a bit more known concerning decompositions of B^3, S^3, or E^3 [26]. We remark that all the results given in [26] can be suitably restated to obtain stronger results. This is an immediate consequence of our results on linking given in (2.4)–(2.5). As a sample we have the following.

(3.2.1) Theorem. *There exists an uncountable family F of topologically distinct 3-dimension* ANR's *such that each X belonging to F satisfies*: (a) *$X \times S^1 \sim S^3 \times S^1$; and* (b) *$X$ does not contain any proper* ANR *or* FAR *of dimension $\geqslant 2$. Furthermore, each X in F contains no proper closed subset of dimension $\geqslant 2$ with* UV^1. *Each X in F is a generalized 3-manifold.*

(3.2.2) Theorem. *There exists an uncountable family E of topologically distinct 3-dimensional* AR's *such that*: *If X belongs to E, then X does not contain any proper*

ANR *or any proper* FAR *of dimension* $\geqslant 2$; *and furthermore,* X *does not contain any proper closed subset of dimension* $\geqslant 2$ *with* UV^1.

(3.2.3) THEOREM. *There exists an uncountable family* D *of topologically distinct* 3-*dimensional absolute retracts for metric spaces* (cf. [8]) *such that each* X *belonging to* D *satisfies*: (a) $X \times E^1 \sim E^3 \times E^1$; *and* (b) X *does not contain any* ANR *or* FAR *of dimension* $\geqslant 2$. *Furthermore, each* X *in* D *contains no compact subset of dimension* $\geqslant 2$ *with* UV^1. *Each* X *in* D *is a* (*noncompact*) *generalized* 3-*manifold.*

(3.2.4) Remark. Suppose X is an AR (ANR) such that X does not contain any proper closed subset of dimension $\geqslant 2$ with UV^1. Suppose G is a cell-like u.s.c. decomposition of X. Then, it can be easily shown that the decomposition space X/G does not contain any proper closed subset of dimension $\geqslant 2$ with UV^1, see [4; Lemma 5.9]. This can be interpreted as the stability of this property under cell-like images. Of course the decomposition spaces constructed in this note are stable in this sense. A topological space X is *strongly locally simply connected* if each point of X has arbitrarily small simply connected open neighborhoods. This definition appears in the work of Armentrout (cf. [26]). The decomposition spaces considered in this note are not strongly locally simply connected at any point, however, they are locally contractible [8]. The following problem of Armentrout (see [26] for a reference to Armentrout's paper on toroidal decompositions) remains open: *Does there exist a cell-like u.s.c. decomposition* G *of* E^n, $n \geqslant 3$, *such that the decomposition space* E^n/G *is strongly locally simply connected and* E^n/G *is not homeomorphic to* E^n?

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