Coincidence of maps in Q-simplicial spaces

by

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Abstract. Let X and Y be compact Hausdorff spaces and let Y be a finite dimensional Q-simplicial space (in the sense of R. Knill) and $\dim H_*(Y, Q) < \infty$. If $p, q\colon X \to Y$ and p is a Vietoris map, and the number of a coincidence of p and q is not zero then there exists a point $x \in X$ such that $p(x) = q(x)$.

The aim of this paper is to give an affirmative answer to the questions of L. Górniewicz ([1], p. 9, Problems 5 and 6) in the particular case where the space is finite dimensional. The following theorem is proved.

THEOREM. *Let X and Y be compact, Hausdorff spaces and let Y be a finite dimensional, Q-simplicial space (in the sense of R. Knill,* [2]), *and* $\dim H_*(Y, Q) < \infty$. *If $p, q\colon X \to Y$, and p is a Vietoris map, and the number of a coincidence $L(p, q) \neq 0$, then there exists a point $x \in X$ such that $p(x) = q(x)$.*

I. Preliminaries.

1. Vietoris and Čech homology theories, [2]. We shall consider only compact, Hausdorff spaces and their continuous mappings.

For a compact Hausdorff space X, by $\mathrm{Cov}(X)$ we shall denote the set of all finite, open coverings of X'. If A is a subset in X and $\alpha \in \mathrm{Cov}(X)$ we shall write $\operatorname{diam} A < \alpha$, if some element of the covering α contains A.

If $f\colon X \to Y$ is a continuous map and $\beta \in \mathrm{Cov}(Y)$, $\beta = \{U_1, \ldots, U_s\}$, then by $f^{-1}(\beta)$ we shall denote the covering $\{f^{-1}(U_1), \ldots, f^{-1}(U_s)\} \in \mathrm{Cov}(X)$.

For $\alpha \in \mathrm{Cov}(X)$ the s-Vietoris simplex $\sigma^s = (y_0, \ldots, y_s)$ is called an *α-Vietoris simplex* if there exists an element U of the covering α containing all of the vertices of the simplex σ^s, i.e. $y_i \in U$, $i = 0, 1, \ldots, s$.

The simplicial complex spanned on all α-Vietoris simplexes in the space Y will be denoted by $Y(\alpha)$. The chain complex of this simplicial complex $Y(\alpha)$ with rational coefficients Q will be denote by $V(Y, \alpha)$.

Suppose $\alpha_1, \alpha_2 \in \mathrm{Cov}(X)$ and $\alpha_1 < \alpha_2$. There is an embedding

$$\pi(\alpha_2, \alpha_1)\colon V(X, \alpha_2) \to V(X, \alpha_1)$$

which is a chain map.

Consider again the mapping $f\colon X\to Y$ and the coverings

$$\beta\in \mathrm{Cov}(Y),\quad f^{-1}(\beta)\in\mathrm{Cov}(X).$$

If $\gamma>f^{-1}(\beta)$ then the map f induces a chain homomorphism $f(\gamma,\beta)\colon V(X,\gamma)\to V(Y,\beta)$, defined as follows. For the γ-Vietoris simplex $\tau^s=(x_0,x_1,\dots,x_s)$

$$f(\gamma,\beta)(\tau^s)=\big(f(x_0),\dots,f(x_s)\big).$$

A collection

$$c^n=\{c^n(\alpha)\colon c^n(\alpha)\in V(X,\alpha),\ \alpha\in\mathrm{Cov}(X)\}$$

of n-dimensional cycles (n-cycles) is an n-V-cycle (an n-dimensional Vietoris cycle) if: a) $c^n(\alpha)$ is a cycle in the chain complex $V(X,\alpha)$, and b) the chains $c^n(\alpha_1)$ and $c^n(\alpha_2)$ are homologous in $V(X,\alpha_1)$, if $\alpha_1<\alpha_2$, i.e. $\pi(\alpha_2,\alpha_1)\big(c^n(\alpha_2)\big)$ is homologous to the chain $c^n(\alpha_1)$ in $V(X,\alpha_1)$.

We shall denote by $H_n^v(X)$ the n-dimensional Vietoris homology group of the space X with rational coefficients.

Let us consider once more the mapping $f\colon X\to Y$. Let h^n be an element of $H_n^v(X)$ and let c^n be an n-V-cycle in the homology class h^n. For each covering $\gamma\in\mathrm{Cov}(Y)$, $\gamma>f^{-1}(\beta)$ consider the simplicial map

$$f(\gamma,\beta)\colon V(X,\gamma)\to V(Y,\beta);$$

we define

$$f(c^n)=\{f\big(f^{-1}(\beta),\beta\big)\big(c^n(f^{-1}(\beta))\big),\ \beta\in\mathrm{Cov}(Y)\}$$

and $f_{*n}(h^n)$ as the homology class of the cycle $f(c^n)$.

Let $\alpha\in\mathrm{Cov}(Y)$ be a covering. By $N(\alpha)$ we shall denote the nerve of α. The chain complex with rational coefficients Q of $N(\alpha)$ is denoted by

$$C_*(N(\alpha))=\{C_n(N(\alpha))\colon n=0,1,\dots\}.$$

If $\alpha_1<\alpha_2$, $\alpha_1,\alpha_2\in\mathrm{Cov}(X)$, then there is an augmentation preserving chain map

$$\sigma(\alpha_2,\alpha_1)\colon C_*(N(\alpha_2))\to C_*(N(\alpha_1))$$

(every two such maps are chain homotopic).

$$s^n=\{s^n(\alpha)\colon s^n(\alpha)\in C_n(N(\alpha))\}$$

is called an *n-C-cycle* (*an n-dimensional Čech cycle*) if $s^n(\alpha)$ is a cycle on $N(\alpha)$, and for $\alpha_1<\alpha_2$ the chains $s^n(\alpha_1)$ and $\sigma(\alpha_2,\alpha_1)s^n(\alpha_2)$ are homologous.

The n-dimensional Čech homology group of the space X with rational coefficients is denoted by $H_n^c(X)$.

It is well known that the groups $H_n^v(X)$ and $H_n^c(X)$ are isomorphic for every compact Hausdorff space. We shall use this isomorphism, but we shall also need a more precise result. To state this result we shall recall two definitions.

Let $\alpha\in\mathrm{Cov}(X)$ and A be is a subset of the space X; then

$$\mathrm{St}(A,\alpha)=\bigcup\{U\colon U\in\alpha,\ U\cap A\neq\emptyset\}.$$

The support of the chain c^n in $V(X,\alpha)$ is defined to be the smallest closed subset W in the space X such that c^n is a chain of $V(W,\alpha|W)$ (if $\alpha=\{U_1,\dots,U_s\}$, then $\alpha|W=\{U_1\cap W,\dots,U_s\cap W\}$).

If $s^n\in C_n(N(\alpha))$ let $\mathrm{car}(s^n)$ be the smallest subfamily β of α such that $s^n\in C_n(N(\beta))$ (β need not be a covering of X), and let

$$\sup(s^n)=\bigcup\{U\colon U\in\mathrm{car}(s^n)\}.$$

Lemma 1.1 (C. H. Dowker, [5]). *Let $\alpha\in\mathrm{Cov}(X)$. Then there are chain maps*

$$k(\alpha)\colon C_*(N(\alpha))\to V(X,\alpha),$$
$$l(\alpha)\colon V(X,\alpha)\to C_*(N(\alpha)).$$

such that

a) *$k(\alpha)$ and $l(\alpha)$ preserve the augmentations,*

b) *$k(\alpha)$ and $l(\alpha)$ are chain homotopies inverse to each other,*

c) *$\sup\big(k(\alpha)(s^n)\big)\subset\sup(s^n)$ for $s^n\in C_n(N(\alpha))$,*

d) *$\sup\big(l(\alpha)(c^n)\big)\subset\mathrm{St}\big(\sup(c^n),\alpha\big)$ for $c^n\in V(X,\alpha)$.*

Now and later on we shall denote $H_n^v(X)$ and $H_n^c(X)$ by $H_n(X)$.

For every $\alpha\in\mathrm{Cov}(X)$ choose a continuous map $\sigma(\alpha)\colon X\to N(\alpha)$ (one of barycentric maps, [1], ch. IV, § 1). This map induces a Q-linear homomorphism

$$\sigma(\alpha)_*=\{\sigma(\alpha)_{*n}\colon n=0,1,\dots\}\colon H_*(X)\to H_*(N(\alpha)).$$

It follows that

$$\sigma(\alpha_2,\alpha_1)_{*n}\sigma(\alpha_2)_{*n}=\sigma(\alpha_1)_{*n}\quad\text{for}\quad\alpha_1<\alpha_2.$$

Here

$$\sigma(\alpha,\beta)_*=\{\sigma(\alpha,\beta)_{*n}\colon n=0,1,\dots\}$$

is the homomorphism in the homology induced by the chain map

$$\sigma(\alpha_2,\alpha_1)\colon N(\alpha_2)\to N(\alpha_1).$$

We shall also use two other simplicial maps. Let $\alpha\in\mathrm{Cov}(X)$ we denote by $\mathrm{St}(\alpha)$ the covering $\{\mathrm{St}(U_1,\alpha),\dots,\mathrm{St}(U_s,\alpha)\}$ where $\alpha=\{U_1,\dots,U_s\}$. If $\alpha_1,\alpha_2\in\mathrm{Cov}(X)$ and $\alpha_1<\mathrm{St}(\alpha_2)$, we say that α_2 is a star refinement of α_1, and we shall write $\alpha_1<*\alpha_2$.

It is well known that every covering of the space X has a star refinement, [20].

For $\alpha_1<*\alpha_2$ there is a chain map

$$v(\alpha_2,\alpha_1)\colon V(X,\alpha_2)\to C_*(N(\alpha_1)).$$

This map is defined as follows. For each vertex x_i of $V(X,\alpha_2)$ choose an element $U_i\in\alpha_2$ which contains it and then choose an element $W_i\in\alpha_1$ which contains the set $\mathrm{St}(U_i,\alpha_2)$. Denote W_i by $v(\alpha_2,\alpha_1)(x_i)$. The map, thus defined is a simplicial map of $X(\alpha_2)$ into $N(\alpha_1)$.

Now let $c^n = \{c^n(\alpha)\colon \alpha \in \mathrm{Cov}(X)\}$ be n-V-cycle. For each covering $\alpha \in \mathrm{Cov}(X)$ let $\alpha <_* \alpha^*$ (we choose one such covering α^*) and define $z^n(\alpha)$ to be

$$z^n(\alpha) = v(\alpha^*, \alpha)(c^n(\alpha^*)) .$$

It is known that $z^n = \{z^n(\alpha)\colon \alpha \in \mathrm{Cov}(X)\}$ is a Čech cycle and the correspondence of c^n to z^n induces also an isomorphism of $H_n^v(X)$ onto $H_n^c(X)$. We shall denote this isomorphism by

$$v(\alpha^*, \alpha)_* = \{v(\alpha^*, \alpha)_{*n}\colon n = 0, 1, \ldots\} .$$

Let us define also a chain map

$$\bar{v}(\alpha^*, \alpha)\colon C_*(N(\alpha^*)) \to V(X, \alpha) .$$

Let $U \in \alpha^*$ choose $x(U) \in U$ and by definition $\bar{v}(\alpha^*, \alpha)(U) = x(U)$. By $\bar{v}(\alpha^*, \alpha)_* = \{\bar{v}(\alpha^*, \alpha)_{*n}\colon n = 0, 1, \ldots\}$ we shall denote the homomorphism of $H_*(N(\alpha))$ into $H_*(X(\alpha))$ induced by $\bar{v}(\alpha^*, \alpha)$.

The projection

$$\pi(\alpha)_*\colon H_*(X) \to H_*(X(\alpha))$$

is defined as follows

$$\pi(\alpha)_* = \bar{v}(\alpha^*, \alpha)_* \sigma(\alpha^*)_* .$$

The following lemmas are well known.

Lemma 1.2. *Suppose X is a compact, Hausdorff space and $\dim H_*(X) < \infty$. Then there exists a covering $\alpha \in \mathrm{Cov}(X)$ such that the projection $\pi(\alpha)_*\colon H_*(Y) \to H_*(Y(\alpha))$ is a monomorphism.*

Lemma 1.3. *Let X and Y be two compact Hausdorff spaces and let f be a map of X in Y. Let $\beta \in \mathrm{Cov}(Y)$ and $\alpha \in \mathrm{Cov}(X)$ such that $\alpha > f^{-1}(\beta)$. Then*

$$\pi(\beta)_* f_* = f(\alpha, \beta)_* \pi(\alpha)_* . \tag{1.1}$$

2. The Vietoris–Begle theorem, [2]. Let us consider the map $p\colon X \to Y$. We assume that this map is a Vietoris map, i.e. $p(X) = Y$ and for every point $y \in Y$ the space $p^{-1}(y)$ is connected and $H_i(p^{-1}(y)) = 0$ for $i = 1, 2, \ldots$

It [2] it is proved that every Vietoris map is a n-Vietoris map for every $n = 1, 2, \ldots$ Let us recall that the map p of the space X onto the space Y is a n-Vietoris map if for every covering $\alpha \in \mathrm{Cov}(X)$ and each point $y \in Y$ there is a covering $\beta = \beta(\alpha, y) \in \mathrm{Cov}(X)$ such that $\beta > \alpha$ and any k-cycle on $V(p^{-1}(y), \beta)$ bounds on $V(p^{-1}(y), \alpha)$ for $0 \leqslant k \leqslant n$.

Lemma 1.4. *For each covering $\beta \in \mathrm{Cov}(Y)$ and each covering $\alpha \in \mathrm{Cov}(X)$, $\alpha > p^{-1}(\beta)$, there is a covering $R = R(\alpha, \beta) \in \mathrm{Cov}(Y)$ such that*

a) $R(\alpha, \beta) > \beta$,

b) *there exists an augmentation preserving chain map $T(\alpha, \beta)$ of the $(n+1)$-skeleton of $V(Y, R(\alpha, \beta))$ in $V(X, \alpha)$ such that for any k-simplex σ^k of $V(Y, R(\alpha, \beta))$ $0 \leqslant k \leqslant n+1$, the chain $p(\alpha, \beta)T(\alpha, \beta)(\sigma^k)$ is a barycentric subdivision, $\delta\sigma^k$, of σ^k with $\sup(\delta\sigma^k) < \beta$.*

c) *for any k-simplex σ^k of $V(Y, R(\alpha, \beta))$ there exists a point $y(\sigma^k) \in Y$ for which*

$$\mathrm{St}(y(\sigma^k), R(\alpha, \beta)) \supset \sup(\sigma^k) , \tag{1.2}$$

$$\sup(T(\sigma^k)) \subset \mathrm{St}(p^{-1}(y(\sigma^k)), \alpha . \tag{1.3}$$

This lemma is Lemma 2 from [2]. Statement c) is not explicitly formulated in [2] but it follows (and is proved) from the proof of Lemma 2, [2].

We shall choose the point $y(\sigma^k)$ once for all.

Lemma 1.5 (proof of Theorem 1, pp. 541–542, [2]). *Suppose $\alpha_1, \alpha_2 \in \mathrm{Cov}(X)$ and $\alpha_1 < \alpha_2$. Suppose also that $\beta_1, \beta_2 \in \mathrm{Cov}(Y)$ and $\alpha_1 > p^{-1}(\beta_1)$, $\alpha_2 > p^{-1}(\beta_2)$. Let $T(\alpha_1, \beta_1)$ and $T(\alpha_2, \beta_2)$ be chain maps from Lemma 1.4, and let*

$$c^n = \{c^n(\gamma)\colon \gamma \in \mathrm{Cov}(Y)\}$$

be a Vietoris cycle. Then the chain $T(\alpha_2, \beta_2)(c^n(R(\alpha_2, \beta_2)))$ is homologous to the chain $T(\alpha_1, \beta_1)(c^n(R(\alpha_1, \beta_1)))$ on $V(X, \alpha_1)$.

Therefore for any covering $\gamma \in \mathrm{Cov}(Y)$ for which $\gamma > R(\alpha_2, \beta_2)$ and $\gamma > R(\alpha_1, \beta_1)$ the chain $T(\alpha_2, \beta_2)$ $(c^n(\gamma))$ is homologous to the chain $T(\alpha_1, \beta_1)(c^n(\gamma))$ on $V(X, \alpha_1)$.

From this result it follows that

$$\pi(\alpha_2, \alpha_1)_{*i} T(\alpha_2, \beta_2)_{*i} = T(\alpha_1, \beta_1)_{*i} \quad \text{for} \quad 0 \leqslant i \leqslant n . \tag{1.4}$$

Here $T(\alpha_2, \beta_2)_{*i}$, $T(\alpha_1, \beta_1)_{*i}$, $\pi(\alpha_2, \alpha_1)_{*i}$ are the homomorphisms induced by the chain maps $T(\alpha_2, \beta_2)$, $T(\alpha_1, \beta_1)$, $\pi(\alpha_2, \alpha_1)$, respectively, and we choose $R(\alpha_1, \beta_1) = R(\alpha_2, \beta_2)$.

Theorem V-B ([2]). *Let p be a Vietoris map of a compact X onto Y, then the homomorphism*

$$p_* = \{p_{*i}\colon i = 0, 1, \ldots\}\colon H_*(X) \to H_*(Y)$$

is an isomorphism.

Let p, q be maps of a compact space X in Y and let p be a Vietoris map. Suppose also that $\dim H_*(Y) < \infty$ (as a vector space over Q). The number of a coincidence $L(p, q)$ of the mappings p and q is defined as follows

$$L(p, q) = \sum (-1)^i \operatorname{tr} q_{*i} p_{*i}^{-1} .$$

Here $\operatorname{tr} q_{*i} p_{*i}^{-1}$ is the trace of the homomorphism $q_{*i} p_{*i}^{-1}$, [7].

Lemma 1.6 (The proof of Theorem 1, p. 542, [2]). *Let $\alpha \in \mathrm{Cov}(X)$ and $\beta = p^{-1}(\alpha)$. Let*

$$c^n = \{c^n(\gamma)\colon \gamma \in \mathrm{Cov}(Y)\}$$

be a Vietoris cycle. Then the chain $c^n(\beta)$ is homologous to the chain

$$p(\beta, \alpha) T(\alpha, \beta)(c^n(R(\alpha, \beta)))$$

on $V(Y, \beta)$.

Therefore, for every $\gamma \in \mathrm{Cov}(Y)$, $\gamma > R(\alpha, \beta)$ the chain $c^n(\gamma)$ is homologous to the chain $p(\beta, \alpha)T(\alpha, \beta)(c^n(\gamma))$ on $V(Y, \beta)$ and also

$$\pi(\alpha)_{*i} p_{*i}^{-1} = T(\alpha, \beta)_{*i} \pi(\gamma)_{*i} \tag{1.5}$$

for $0 \leqslant i \leqslant n$.

LEMMA 1.7 (2.2, [12]). *For any $\beta \in \mathrm{Cov}(Y)$ there exists a covering $\gamma(\beta) \in \mathrm{Cov}(Y)$ such that $\beta <^* \gamma(\beta)$ and if $\delta \in \mathrm{Cov}(Y)$, $\delta > \gamma(\beta)$ then any augmentation preserving chain map ω of $V(Y, \delta)$ into $V(Y, \beta)$ which is subordinate to $\gamma(\beta)$ is a chain homotopic to $\pi(\delta, \beta)$.*

Let us recall that the chain map ω is subordinate to $\gamma(\beta)$ if for every simplex $\sigma^k \in V(Y, \gamma(\beta))$

$$\sup(\omega(\sigma^k)) \subset \mathrm{St}(\sup(\sigma^k), \beta). \tag{1.6}$$

3. Q-simplicial spaces, [12]. The compact, Hausdorff space Y is called a *Q-simplicial space* if for any $\beta \in \mathrm{Cov}(Y)$ there exists a covering $\beta(Y) \in \mathrm{Cov}(Y)$ such that: a) $\beta(Y) > \beta$, and b) for every $\gamma \in \mathrm{Cov}(Y)$ there is an augmentation preserving chain map

$$\omega\colon V(Y, \beta(Y)) \to V(Y, \gamma)$$

which is subordinate to the covering β.

Consider the homomorphisms $\omega_* \pi(\beta)_*$ and $\pi(\gamma)_*$ (they are the homomorphisms induced by chain maps ω, $\pi(\beta)$ and $\pi(\gamma)$, respectively). In general, the homomorphism $\omega_* \pi(\beta)_*$ is different from the homomorphism $\pi(\gamma)_*$. But if $\beta(Y) > \gamma(\beta)$ (see Lemma 1.7) and $\gamma > \gamma(\beta)$ then

$$\pi(\gamma, \beta)_* \omega_* = \pi(\beta(Y), \beta)_*. \tag{1.7}$$

LEMMA 1.8. *Suppose that Y is a compact Hausdorff space. Then Y is a Q-simplicial space if either:*

1. *Y is a convex set in a locally convex linear topological space,* [12].
2. *Y is a neighborhood extensor for the class of compact spaces. In particular, Y is* ANR *for normal spaces,* [12].
3. *Y is a compact topological group,* [13].
4. *Y is a product of Q-simplicial spaces,* [12].
5. *Y is a weak semicomplex,* [19].
6. *Y is a compact generalized manifold (n-gm),* [21, 19].
7. *Y is a snace-like continuum,* [6, 19].
8. *Y is a hyperspace of a snake-like continuum,* [15, 19].

4. Approximation of compact spaces in I^τ. By I^τ we shall denote the product of τ intervals $I = [0, 1]$ and by $\mathfrak{U}(I^\tau)$ the family of all symmetric neighbourhoods of the diagonal of the product $I^\tau \times I^\tau$, [11], Ch. 6, 29.

For the space Y there exists an embedding in I^τ (for some τ). We suppose that $Y \subset I^\tau$.

LEMMA 1.9. *Let $Y \subset I^\tau$ and $\mathfrak{U}_0 \in \mathfrak{U}(I^\tau)$. There exists a covering $\beta \in \mathrm{Cov}(Y)$ and an embedding $i\colon N(\beta) \to I^\tau$ such that $(iv(\alpha)(y), y) \in \mathfrak{U}_0$ for every $y \in Y$.*

This lemma is also well known, see for example [17].

For a set $A \subset I^\tau$ the set $\mathfrak{U}_0^i(A)$ is defined as follows:

$$\mathfrak{U}_0^{i+1}(A) = \mathfrak{U}_0(\mathfrak{U}_0^i(A))$$

and

$$\mathfrak{U}_0(A) = \{z \in I^\tau\colon \text{there exists a } y \in A \text{ such that } (z, y) \in \mathfrak{U}_0\}.$$

Let $\beta \in \mathrm{Cov}(Y)$ and $\mathfrak{U} \in \mathfrak{U}(I^\tau)$. We shall write $\beta < \mathfrak{U}$ if $U \times U \subset \mathfrak{U}$ for every $U \in \beta$. Supposing $\dim Y = n < \infty$, let $\mathrm{Cov}(Y, n)$ be the set of all finite open coverings α, such that $\dim N(\alpha) \leqslant n$. The set $\mathrm{Cov}(Y, n)$ is confinal in the set of all coverings $\mathrm{Cov}(Y)$.

II. Proof of the theorem

Recall that X, Y and $p, q\colon X \to Y$ satisfy the following conditions

1. p is a Vietoris map,
2. Y is a Q-simplicial space,
3. $\dim H_*(Y, Q) < \infty$,
4. $n = \dim Y < \infty$,
5. X and Y are compact, Hausdorff spaces.

The theorem is a consequence of the following

LEMMA 2.1. *If for every $x \in X$ we have $p(x) \neq q(x)$ then $L(p, q) = 0$.*

The condition $p(x) \neq q(x)$ is equivalent to $y \notin q(p^{-1}(y))$ for every $y \in Y$.

Suppose that $y \notin q(p^{-1}(y))$ for all $y \in Y$. Provided the space Y is a closed subset in I^τ and $y \notin q(p^{-1}(y))$ for every $y \in Y$, then there exists a $\mathfrak{U}_0 \in \mathfrak{U}(I^\tau)$ such that

$$y \notin \mathfrak{U}_0^4(q(p^{-1}(y))) \quad \text{for every } y \in Y. \tag{2.1}$$

The space X is compact and $q\colon X \to Y$ is continuous. For $\mathfrak{U}_0 \in \mathfrak{U}(I^\tau)$ there exists an $\alpha_0 \in \mathrm{Cov}(X)$ such that for every $U \in \alpha_0$ we have $q(U) \times q(U) \subset \mathfrak{U}_0$.

Let $n = \dim Y$. Then as soon as the map p is a Vietoris map, it follows that p is a n-Vietoris map, [2].

Let $\beta \in \mathrm{Cov}(Y, n)$ be such that $\beta < \mathfrak{U}_0$ and $\pi(\beta)_*\colon H_*(Y) \to H_*(Y(\beta))$ be a monomorphism (Lemma 1.2).

Now for every $\delta \in \mathrm{Cov}(Y, n)$, $\delta > \beta$ the homomorphism $\pi(\delta)_*\colon H_*(Y) \to H_*(Y(\delta))$ is also a monomorphism and the homomorphism $\pi(\delta, \beta)_*\colon H_*(Y(\delta)) \to H_*(Y(\beta))$ is a monomorphism on the image of $\pi(\delta)_*$. This follows from the equality $\pi(\beta)_* = \pi(\delta, \beta)_* \pi(\delta)_*$.

Suppose $\beta_1 \in \mathrm{Cov}(Y, n)$ and $\alpha_1, \alpha_2 \in \mathrm{Cov}(X)$ are such that:

a) $\alpha_2 > p^{-1}(\beta)$, $\alpha_2 > q^{-1}(\beta)$, $\alpha_2 > \alpha_0$,

b) $\beta_1 > \gamma(\beta)$, $\beta_1 > \beta(Y)$, $\beta_1 > R(\alpha_2, \beta)$,

c) $\alpha_1 > \alpha_2$, $\alpha_1 > p^{-1}(\beta_1)$.

d) $\mathrm{Im}\,\pi(\beta_1)_* \supset \mathrm{Im}\,q(\alpha_1, \beta_1)_* T(\alpha_1, \beta_1)_*$.

For the definitions of the coverings $\gamma(\beta)$ see Lemma 1.7; for $\beta(Y)$ — the definition of Q-simplicial space; for $R(\alpha_2, \beta)$ — Lemma 1.4 (we suppose that $R(\alpha_2, \beta) \in \mathrm{Cov}(Y, n)$).

Let us consider the coverings α_1 and β_1. Apply Lemma 1.4. There exists a covering $R(\alpha_1, \beta_1) \in \mathrm{Cov}(Y, n)$ such that:

a) $R(\alpha_1, \beta_1) > \beta_1$,

b) there exists a chain map $T(\alpha_1, \beta_1)$ of the $(n+1)$-skeleton of $V(Y, R(\alpha_1, \beta_1))$ in $V(X, \alpha_1)$ such that, for any k-simplex σ^k of $V(Y, R(\alpha_1, \beta_1))$, $0 \leqslant k \leqslant n+1$, the chain $p(\alpha_1, \beta_1)T(\alpha_1, \beta_1)\sigma^k$ is a barycentric subdivision, $\delta\sigma^k$, of σ^k with

$$\mathrm{diam}(\sup \delta\sigma^k) < \beta .$$

c) for any k-simplex σ^k of $V(Y, R(\alpha_1, \beta_1))$ there exists a point $y(\sigma^k)$ such that

$$\mathrm{St}\big(y(\sigma^k), R(\alpha_1, \beta_1)\big) \supset \sup(\sigma^k) , \tag{2.2}$$

$$\mathrm{St}\big(p^{-1}(y(\sigma^k), \alpha_1\big) \supset \sup\big(T(\sigma^k)\big) . \tag{2.3}$$

Let us consider the augmentation preserving chain map

$$\omega\colon\ V(Y, \beta_1) \to V\big(Y, R(\alpha_1, \beta_1)\big)$$

subordinate to β (ω exists since $\beta_1 > \beta(Y)$ and Y is a Q-simplicial space).

Since $\beta_1 > \gamma(\beta)$, and $R(\alpha_1, \beta_1) > \beta_1$ by Lemma 1.7

$$\pi\big(R(\alpha_1, \beta_1), \beta\big)_* \omega_* = \pi(\beta_1, \beta)_* \tag{2.4}$$

(here ω_* is the homomorphism induced by the chain map ω).

Now consider $\alpha_2 \in \mathrm{Cov}(X)$ and $\beta \in \mathrm{Cov}(Y, n)$. Since $\alpha_2 > p^{-1}(\beta)$, by Lemma 1.4 there exists a chain map $T(\alpha_2, \beta)$ of $(n+1)$-skeleton of $V\big(Y, R(\alpha_2, \beta)\big)$ in $V(X, \alpha_2)$ and, by Lemma 1.5, (1.4)

$$\pi(\alpha_1, \alpha_2)_{*i} T(\alpha_1, \beta_1)_{*i} = T(\alpha_2, \beta)_{*i}\,\pi\big(R(\alpha_1, \beta_1), R(\alpha_2, \beta)\big)_{*i} \tag{2.5}$$

for $0 \leqslant i \leqslant n$.

Since $H_*(Y(\beta))$ and $H_*(Y)$ are Q-linear spaces, and $\pi(\beta)_*\colon H_*(Y) \to H_*\big(Y(\beta)\big)$ is monomorphism, there exists a Q-linear map

$$r_* = \{r_i\}\colon\ H_*\big(Y(\beta)\big) \to H_*(Y)$$

such that $r_* \pi(\beta)_* = \mathrm{id}$ (id is the identity of $H_*(Y)$).

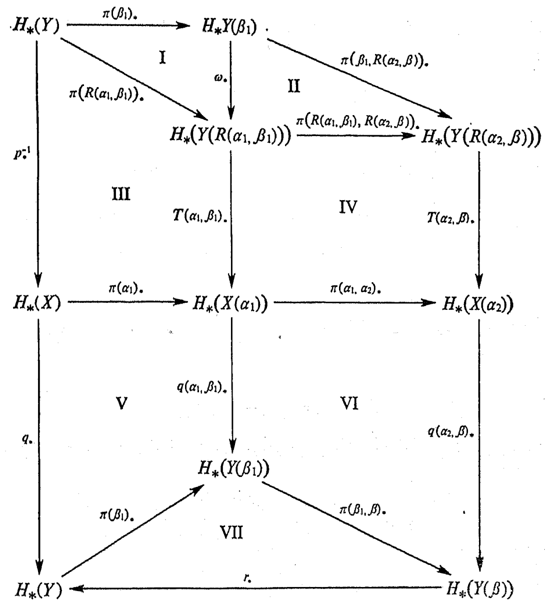

Diagram 1

Let us consider Diagram 1. All the subdiagrams II–VII of this diagram are commutative (this follows from (1.1), (1.4), (1.5) (1.7), (2.3), (2.4)). The diagram is possibly noncommutative.

Now, from Diagram 1 we obtain:

$$\begin{aligned}\mathrm{tr}(q_{*i}p_{*i}^{-1}) &= \mathrm{tr}\big(r_i \pi(\beta)_{*i} q_{*i} p_{*i}^{-1}\big)\\ &= \mathrm{tr}\big(r_i q(\alpha_2, \beta)_{*i} T(\alpha_2, \beta)_{*i} \pi(\beta_1, R(\alpha_2, \beta_1))_* \pi(\beta_1)_{*i}\big)\\ &= \mathrm{tr}\big(r_i \pi(\beta_1, \beta)_{*i} q(\alpha_1, \beta_1)_{*i} T(\alpha_1, \beta_1)_{*i} \omega_{*i} \pi(\beta_1)_{*i}\big)\\ &= \mathrm{tr}\big(\pi(\beta_1)_{*i} r_i \pi(\beta_1, \beta)_{*i} q(\alpha_1, \beta_1)_{*i} T(\alpha_1, \beta_1)_{*i} \omega_{*i}\big) .\end{aligned}$$

Let us recall that on the image of the homomorphism $\pi(\beta_1)_{*i}$ the homomorphism $\pi(\beta_1)_{*i} r_i \pi(\beta_1, \beta)_{*i}$ is the identity.

Also we have

$$\mathrm{Im}\big(q(\alpha_1, \beta_1)_{*i} T(\alpha_1, \beta_1)_{*i} \subset \mathrm{Im}\big(\pi(\beta_1)_{*i}\big) .$$

Therefore

$$\mathrm{tr}(q_{*i}p_{*i}^{-1}) = \mathrm{tr}\big(q(\alpha_1, \beta_1)_{*i} T(\alpha_1, \beta_1)_{*i} \omega_{*i}\big) .$$

Now, let us apply Lemma 1.1. There are augmentation preserving chain maps

$$k(\beta_1)\colon\ C_*\big(N(\beta_1)\big) \to V(Y, \beta_1) ,$$
$$l(\beta_1)\colon\ V(Y, \beta_1) \to C_*\big(N(\beta_1)\big) ,$$

which are chain homotopies inverse to each other and which satisfy the following conditions:

(2.6) $$\sup(k(\beta_1)(c))\subset\sup(c),$$

(2.7) $$\sup(l(\beta_1)(d))\subset\mathrm{St}(\sup(d),\beta_1)$$

for a chain c on $N(\beta_1)$, and a chain d on $Y(\beta_1)$. Therefore

$$\begin{aligned}\mathrm{tr}(q(\alpha_1,\beta_1)_{*i}T(\alpha_1,\beta_1)_{*i}\omega_{*i}) &= \mathrm{tr}(k(\beta)_{*i}l(\beta_1)_{*i}q(\alpha_1,\beta_1)_{*i}T(\alpha_1,\beta_1)_{*i}\omega_{*i})\\ &= \mathrm{tr}(l(\beta_1)_{*i}q(\alpha_1,\beta_1)_{*i}T(\alpha_1,\beta_1)_{*i}\omega_{*i}k(\beta_1)_{*i}).\end{aligned}$$

By Hopf's trace formula

$$L(p,q)=\sum(-1)^i\mathrm{tr}(l(\beta_1)_iq(\alpha_1,\beta_1)_iT(\alpha_1,\beta_1)\omega_ik(\beta_1)_i).$$

So

$$L(p,q)=\sum(-1)^i\mathrm{tr}\,\varphi_i$$

(here $\varphi_i=l(\beta_1)_iq(\alpha_1,\beta_1)_iT(\alpha_1,\beta_1)_i\omega_ik(\beta_1)_i$ is a chain map of $C_*(N(\beta_1))$ in itself, and we use $\dim N(\beta_1)\leqslant n$)!

Supposing $L(p,q)\neq 0$, there exists an i such that $\mathrm{tr}\,\varphi_i\neq 0$. From this it follows that there exists a simplex σ^i which belongs to the chain $\varphi_i(\sigma^i)$.

Therefore

$$\sup(\sigma^i)\subset\sup(\varphi_i(\sigma^i)).$$

Let

$$\sigma_1^i\in q(\alpha_1,\beta_1)T(\alpha_1,\beta_1)\omega k(\beta_1)(\sigma^i)$$

and

(2.8) $$\sigma^i\in l(\beta_1)(\sigma_1^i),\quad \sup(\sigma^i)\subset\mathrm{St}(\sup(\sigma_1^i),\beta_1),$$

(2.9) $$\sigma_2^i\in T(\alpha_1,\beta_1)\omega k(\beta_1)(\sigma^i),$$

(2.10) $$\sigma_1^i=q(\alpha_1,\beta_1)(\sigma_2^i),$$

$$\sigma_3^i\in\omega k(\beta_1)(\sigma^i),$$

$$\sigma_2^i\in T(\alpha_1,\beta_1)(\sigma_3^i),$$

(2.11) $$\sigma_4^i\in k(\beta_1)(\sigma^i),$$

(2.12) $$\sigma_3^i\in\omega(\sigma_4^i),$$

σ_1^i, σ_2^i, σ_3^i, σ_4^i are V-simplexes. From (2.6) and (2.11) we obtain

(2.13) $$\sup(\sigma_4^i)\subset\sup(\sigma^i).$$

The chain map ω is subordinate to β. So from (1.6) and (2.12) we obtain

$$\sup(\sigma_3^i)\subset\mathrm{St}(\sup(\sigma_4^i),\beta)$$

and from (2.13) follows

(2.14) $$\sup(\sigma_3^i)\subset\mathrm{St}(\sup(\sigma^i),\beta).$$

From (2.3) there exists a point $y(\sigma_3^i)$ such that

$$\sup(T(\alpha_1,\beta_1)(\sigma_3^i))\subset\mathrm{St}(p^{-1}(y(\sigma_3^i),\alpha_1)).$$

Therefore from (2.9)

(2.15) $$\sup(\sigma_3^i)\subset\mathrm{St}(p^{-1}(y(\sigma_3^i),\alpha_1))$$

and from (2.2)

(2.16) $$\sup(\sigma_1^i)\subset\mathfrak{A}(qp^{-1}(\sigma_3^i)).$$

From (2.10) and (2.15) and (2.8) we obtain

(2.17) $$\sup(\sigma^i)\subset\mathfrak{A}^2(qp^{-1}(y(\sigma_3^i))).$$

From (2.14) and (2.16) follows

(2.18) $$y(\sigma_3^i)\in\mathfrak{A}^2(\sup(\sigma^i)).$$

From (2.17) and (2.18) we have

$$y(\sigma_3^i)\in\mathfrak{A}^4(qp^{-1}(y(\sigma_3^i)))$$

which contradicts to (2.1). Lemma 2.1 and therefore the theorem are proved.

III. Consequences from the theorem

It is well known that the theorem implies the Lefschetz fixed point theorem for multivalued upper-semicontinuous and acyclic mappings, [8].

COROLLARY 1. *Let Y be a compact, finite dimensional Q-simplicial space and $\dim H_*(Y)<\infty$. Let $F\colon Y\to Y$ be an upper-semicontinuous, acyclic multivalued mapping. If the Lefschetz number $L(F)$ of the mapping F is not zero, then there exists a point $y\in Y$ such that $y\in F(y)$.*

From the theorem and Lemma 1.0 follows

COROLLARY 2. *Let the compact Hausdorff space Y satisfy one of the conditions* a)-e), *and $p,q\colon X\to Y$. Let p be a Vietoris map. If $L(p,q)\neq 0$ then there exists a point $x\in X$ such that $p(x)=q(x)$.*

a) *Y is a finite dimensional group,*

b) *Y is a generalized manifold (n-gm) in the sense of R. L. Wilder*, [21].

c) *Y is a finite dimensional* HLC* *space in the sense of S. Lefschetz*, [14].

d) *Y is finite dimensional, and is an extensor for the class of all normal spaces,*

e) *Y is a finite dimensional weak semicomplex*, [19], *or a quasicomplex*, [14], *or a semicomplex*, [4].

From the theorem and Lemma 1.8 we obtain

COROLLARY 3. *Let the compact Hausdorff space Y satisfy one of the conditions* a), b), *and* $p, q: X \to Y$, *and let p be a Vietoris map. There exists a point $x \in X$ such that* $p(x) = q(x)$.

a) *Y is an acyclic, finite-dimensional Q-simplicial space,*

b) *Y is a snake-like continuum, or a product of a finite number of snake-like continua.*

From Corollary 1 follows.

COROLLARY 4. *Let the compact Hausdorff space Y satisfy one of the conditions* a) -e) *in Corollary* 2, *and let $F: Y \to Y$ be an acyclic, upper-semicontinuous, multivalued mapping. If the Lefschetz number $L(F)$ of the mapping F is not zero, then there exists a point $y \in Y$ such that $y \in F(y)$.*

COROLLARY 5. *Let the compact Hausdorff space Y satisfy one of the conditions* a), b) *in Corollary* 3. *For every acyclic, upper-semicontinuous, multivalued mapping $F: Y \to Y$ there exists a point $y \in Y$ such that $y \in F(y)$.*

The results of this paper were announced in [16].

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Accepté par la Rédaction le 25. 6. 1979