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The equivalence of definable quantifiers in second order arithmetic

by

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Abstract. In this paper we generalize the notion of equivalent quantifiers considered by M. Dubiel in her paper [2] and show nonequivalent countably additive quantifiers in some model of second order arithmetic.

Let L be the language of second order arithmetic A_2 as described in [1]. If M is a model of A_2, then by L_M we denote the language L with additional constants to denote elements of M.

We consider a mapping which assigns to a variable x and a formula $\varphi(x, x_1, \ldots, x_n)$ of L, with free variables $x, x_1, \ldots, x_n$, another formula $\psi(x_1, \ldots, x_n)$ of L, with free variables $x_1, \ldots, x_n$, which we shall denote by $Qx\varphi(x, x_1, \ldots, x_n)$.

If M is a model of A_2, we shall say that the mapping Q is a definable quantifier in M iff the model M satisfies the following axioms:

$$(\varphi \to \psi) \to (Qx\varphi \to Qx\psi)\,, \tag{1}$$

$$Qx(\varphi \vee \psi) \to Qx\varphi \vee Qx\psi\,, \tag{2}$$

$$Qx(x = x)\,, \tag{3}$$

$$\neg \exists y\, Qx(x = y)\,. \tag{4}$$

We call two quantifiers Q_1 and Q_2 equal in M iff for any formula $\varphi(x_1, \ldots, x_n)$ of L the following equivalence is satisfied in M:

$$\forall x_1 \ldots \forall x_n [Q_1 x\varphi(x, x_1, \ldots, x_n) \equiv Q_2 x\varphi(x, x_1, \ldots, x_n)]\,.$$

The above notion of equality of quantifiers is exactly the notion of equivalence of [2]. Our generalization closely corresponds to the following theorem, due to Krivine and Mc Aloon [4].

DEFINITION 1. A formula $\vartheta(x)$ of the language L_M is *countable-like in M* (for the quantifier Q) iff for any formula $\varphi(x, y)$ of L_M

$$M \models Qy \exists x[\vartheta(x) \,\&\, \varphi(x, y)] \to \exists x\, Qy\varphi(x, y)\,.$$

THEOREM 2. *If M is a countable model of A_2 and Q a definable quantifier in M, then there exists a proper elementary extension $N \succ M$ such that any formula $\vartheta(x)$ of L_M is countable-like in M iff*

$$\{x \in M\colon M \models \vartheta[x]\} = \{x \in N\colon N \models \vartheta[x]\}.$$

DEFINITION 3. Two quantifiers Q_1 and Q_2 are *equivalent in the model M* iff they produce elementary extensions via Theorem 2 with the same formulas preserved and the same formulas enlarged. In other words, Q_1 is *equivalent* to Q_2 iff they have in M the same countable-like formulas.

The proofs of the following facts can be found in [4].

LEMMA 4. *If ϑ is countable-like, then $\neg Qx\vartheta(x)$.*

LEMMA 5. *If the model M satisfies the following axioms*

(5) $$Qy\exists x\varphi \to \exists xQy\varphi \vee Qx\exists y\varphi$$

and $M \models \neg Qx\vartheta(x)$, then ϑ is countable-like in M.

Quantifiers satisfying axiom (5) are called Keisler quantifiers in [2]. Let us observe that equivalent Keisler quantifiers are equal. Obviously equal quantifiers are equivalent. Now we shall produce an example of two quantifiers which are equivalent but different.

Let $Q_c x\varphi(x, x_1, \ldots, x_n)$ denote the formula

$$\neg\exists y\forall x[\varphi(x, x_1, \ldots, x_n) \to \exists i[x = (y)_i]],$$

where $(y)_i = \{n\colon J(n, i) \in y\}$, J being the pairing function $J(n, m) = 2^n(2m+1)-1$ for natural numbers. Then Q_c is a Keisler quantifier which formalizes the notion of uncountability.

Next, let $Q_b x\varphi(x, x_1, \ldots, x_n)$ be the formula

$$\forall y[\mathrm{Bord}(y) \to \exists x[\mathrm{Bord}(x) \,\&\, y \prec x \,\&\, \varphi(x, x_1, \ldots, x_n)]],$$

where $\mathrm{Bord}(x)$ denotes the fact that x is a well-ordering of a set of natural numbers and $x \prec y$ means that the well-ordering x is shorter than the well-ordering y. The quantifier Q_b formalizes the idea that arbitrarily large well-ordering satisfy the formula φ.

The quantifiers Q_c and Q_b are different in all models of A_2. Namely, there are uncountably many well-orderings of a given infinite length, and so for y such that $M \models \mathrm{Bord}[y]$ and $M \models$ "$\omega \preccurlyeq y$" we have $M \models Q_c x[x \preccurlyeq y]$ and $M \models \neg Q_b x[x \preccurlyeq y]$. In fact, one can easily show that the quantifier Q_b is never a Keisler quantifier.

The aim of the paper is to show that in some models of A_2 the quantifiers Q_c and Q_b are equivalent and in some models of A_2 they are not equivalent.

Let us observe that the quantifiers Q_c and Q_b are countably additive, i.e. the formula $N(x)$, which says that x is a natural number, is countable-like for each of them.

Now let M be a model of A_2 and let a formula $\beta(x, y)$ of L_M define in M a linear ordering $\leqslant$ of the universe with the property that proper initial segments of M are countable in M.

THEOREM 6. *In the model M all countably additive quantifiers are equivalent.*

In view of Theorem 2, in order to prove the above theorem it suffices to prove the following

LEMMA 7. *If N is a proper elementary extension of the model M with the same natural numbers, then for any formula $\varphi(x)$ of L_M, $M \models \neg Q_c x\varphi(x)$ iff*

$$\{x \in M\colon M \models \varphi[x]\} = \{x \in N\colon N \models \varphi[x]\}.$$

Proof. Let us denote by $\leqslant$ the linear ordering of the model N defined in N by the formula β. Since $M \prec N$, it is an extension of the ordering $\leqslant$ of M. We prove that N is then an end extension of M, i.e. for $x \in M$ and $y \in N-M$ we have $x < y$.

Suppose that $y \leqslant x$. Since $M \prec N$, the proper initial segments of N are countable in N. Thus there exists an $a \in N$ such that

$$N \models \forall z[z \leqslant x \equiv \exists i[z = (a)_i]].$$

Since $M \prec N$, such an a exists in M. But then $y = (a)_i$ for some a, $i \in M$, and so $y \in M$, a contradiction.

Now observe that if φ defines a subset of M that is countable (in the sense of M), then it is bounded in M. Any upper bound of φ in M is an upper bound of φ in N and so φ is preserved. On the other hand, if φ is preserved in the extension, then it is bounded in N by any element $y \in N-M$.

Hence it is bounded in M, and so it is countable in M, Q.E.D.

COROLLARY 8. *If $M \models A_2 + V = L$, then Q_c and Q_b are equivalent in M.*

Now we shall construct a model of A_2 in which the quantifiers Q_c and Q_b are not equivalent. The required model will be the continuum of a transitive model of ZFC. A closer inspection of the proof shows that it is enough to assume the existence of a transitive model of $\mathrm{ZFC}^- + V = \mathrm{HC}$. In the proof we use the method of forcing in the boolean version.

Let M be a countable transitive model of $\mathrm{ZFC} + V = L$. We consider the usual Cohen conditions, which add ω_1 generic reals: $p \in P$ iff $p\colon a \to 2$, $a \subseteq \omega_1 \times \omega$ finite, $p \leqslant q$ iff $p \supseteq q$.

Then P satisfies the countable chain condition. Let $G \subseteq P$ be an M-generic filter and let us consider the model $M[G]$. We define certain elements of $M[G]$ together with their boolean names.

$$a_\xi = \{n \in \omega\colon \bigcup G(\xi, n) = 0\},$$
$$\mathrm{dom}(\boldsymbol{a}_\xi) = \{\hat{n}\colon n \in \omega\},$$
$$\boldsymbol{a}_\xi(\hat{n}) = \sum\{p \in P\colon p(\xi, n) = 0\},$$
$$b = \{a_\xi\colon \xi < \omega_1^M\},$$
$$\mathrm{dom}(\boldsymbol{b}) = \{\boldsymbol{a}_\xi\colon \xi < \omega_1^M\},$$
$$\boldsymbol{b}(\boldsymbol{a}_\xi) = \mathbf{1}.$$

Then b is an uncountable set of reals in $M[G]$. For any real $r \subseteq \omega$ of the model $M[G]$ we take a boolean term $\boldsymbol{r}$ such that $\mathrm{val}_G(\boldsymbol{r}) = r$ and $\mathrm{dom}(\boldsymbol{r}) = \{\hat{n}\colon n \in \omega\}$. For each $n \in \omega$ we choose a countable subset $\{p_{n,m}^{(r)}\colon m \in \omega\}$ of P such that $\boldsymbol{r}(\hat{n}) = \sum \{p_{n,m}^{(r)}\colon m \in \omega\}$.

We call boolean terms of form $\boldsymbol{a}_\xi$, $\boldsymbol{b}$, $\hat{x}$, for $x \in M$ and $\boldsymbol{r}$ as above, acceptable parameters.

An acceptable formula or sentence is a formula or sentence of the forcing language such that every term occurring in it is acceptable.

We define supports of acceptable parameters.

$$\mathrm{supp}(\boldsymbol{r}) = \{\xi \in \omega_1^M\colon \exists n, m, k[\langle \xi, k\rangle \in \mathrm{dom}(p_{n,m}^{(r)})]\},$$
$$\mathrm{supp}(\boldsymbol{a}_\xi) = \{\xi\},$$
$$\mathrm{supp}(\boldsymbol{b}) = \mathrm{supp}(\hat{x}) = 0.$$

Then for any acceptable parameter t we have

$$M \models |\mathrm{supp}(t)| \leqslant \omega .$$

Next we consider permutations $\pi\colon \omega_1^M \to \omega_1^M$, which move only finitely many ordinals. They extend in a natural way to automorphisms of P

$$\pi p(\pi\xi, n) = p(\xi, n),$$

and thence to automorphisms of the boolean model $M^{(P)}$. We have

$$\pi(\boldsymbol{a}_\xi) = \boldsymbol{a}_{\pi\xi}, \quad \pi\boldsymbol{b} = \boldsymbol{b} \quad \text{and} \quad \pi\hat{x} = \hat{x} .$$

LEMMA 9 (Permutation Lemma). *If $p \Vdash \varphi(x_1, \ldots, x_n)$ then $\pi p \Vdash \varphi(\pi x_1, \ldots, \pi x_n)$.*

For a proof see e.g. [5].

Now we define

$$\mathrm{fix}(A) = \{\pi\colon \forall \xi \in A[\pi\xi = \xi]\} \quad \text{for} \quad A \subseteq \omega_1^M$$

and observe that for any acceptable parameter t and $\pi \in \mathrm{fix}(\mathrm{supp}(t))$ we have $\pi t = t$.

LEMMA 10 (Restriction Lemma). *If φ is an acceptable sentence and $A \subseteq \omega_1^M$, $A \in M$ such that $\mathrm{supp}(t) \subseteq A$ for any acceptable parameter t occurring in φ, then for any condition p*

$$p \Vdash \varphi \to p \restriction A \times \omega \Vdash \varphi .$$

Proof. Suppose that $p \Vdash \varphi$ and $p \restriction A \times \omega \nVdash \varphi$. We take a condition $q \leqslant p \restriction A \times \omega$ such that $q \Vdash \neg\varphi$ and a permutation π which makes πq and p compatible. Then $\pi q \Vdash \neg\varphi$, contradicting $p \Vdash \varphi$, Q.E.D.

By an open interval in $P(\omega)$ we mean a finite sequence $s \in \bigcup_{n \in \omega} 2^n$ and write $r \in s$ for a real $r \subseteq \omega$ in the case where

$$\forall i \in \mathrm{dom}(s)[i \in r \equiv s(i) = 0].$$

LEMMA 11 (Continuity Lemma). *Let $\varphi(x_1, \ldots, x_n)$ be an acceptable formula with n free variables $x_1, \ldots, x_n$ and let $A \subseteq \omega_1^M$, $A \in M$ be such that $\mathrm{supp}(t) \subseteq A$ for any acceptable parameter t occurring in φ. Then for any $\xi_1, \ldots, \xi_n \notin A$ such that $M[G] \models \varphi[a_{\xi_1}, \ldots, a_{\xi_n}]$ there exist pairwise disjoint open intervals $s_1, \ldots, s_n$ such that $a_{\xi_1} \in s_1, \ldots, a_{\xi_n} \in s_n$ and, for all $\eta_1, \ldots, \eta_n \notin A$, if $a_{\eta_1} \in s_1, \ldots, a_{\eta_n} \in s_n$, then $M[G] \models \varphi[a_{\eta_1}, \ldots, a_{\eta_n}]$.*

Proof. Let $B = A \cup \{\xi_1, \ldots, \xi_n\}$ and take $p \in G$ such that $p \Vdash \varphi(\boldsymbol{a}_{\xi_1}, \ldots, \boldsymbol{a}_{\xi_n})$. By the Restriction Lemma we may assume that $p = p \restriction B \times \omega$. By extending p if necessary we may also assume that it has the following properties:

$$\langle \xi, m\rangle \in \mathrm{dom}(p) \;\&\; m' < m \to \langle \xi, m'\rangle \in \mathrm{dom}(p),$$
$$\xi_1 \neq \xi_2 \in B - A \to \exists m[p(\xi_1, m) \neq p(\xi_2, m)].$$

The above properties allow us to define pairwise disjoint open interwals $s_1, \ldots, s_n$ as follows:

$$s_1(m) = p(\xi_1, m), \; \ldots, \; s_n(m) = p(\xi_n, m).$$

Then of course $a_{\xi_1} \in s_1, \ldots, a_{\xi_n} \in s_n$. Let us take $\eta_1, \ldots, \eta_n \notin A$ such that $a_{\eta_1} \in s_1, \ldots, a_{\eta_n} \in s_n$. We define a condition q as follows:

$$q(\eta_1, m) = s_1(m), \; \ldots, \; q(\eta_n, m) = s_n(m).$$

Then $q \in G$. We take a permutation π such that $\pi \in \mathrm{fix}(A)$ and $\pi\xi_1 = \eta_1, \ldots, \pi\xi_n = \eta_n$. Then $\pi p \Vdash \varphi(\boldsymbol{a}_{\eta_1}, \ldots, \boldsymbol{a}_{\eta_n})$ and $\pi p \in G$ because $q \cup \pi p = q \cup p \restriction A \times \omega \in G$, Q.E.D.

COROLLARY 12. *Let $\varphi(x, x_1, \ldots, x_n)$ be a set-theoretical formula. If $x_1, \ldots, x_n \in M$ are either reals or ordinals or $x_i = b$, then there exists a countable subset $a_{x_1,\ldots,x_n} \subseteq b$, $a_{x_1,\ldots,x_n} \in M[G]$ such that for any $x \in b - a_{x_1,\ldots,x_n}$ there exists an open interval s with the property*

$$M[G] \models \varphi(x, x_1, \ldots, x_n) \to \forall y \in b \cap s - a_{x_1,\ldots,x_n}\, \varphi(y, x_1, \ldots, x_n).$$

The proof follows immediately from the Continuity Lemma.

Now we shall observe that Corollary 12 remains valid for a large class of generic extensions of the model $M[G]$. Suppose that in $M[G]$ we are given a notion of forcing Q with the following properties:

(a) Both Q and $\leqslant_Q$ are definable in $M[G]$ by formulas with parameters which are reals, ordinals or the set b.

(b) The elements of Q can be definably coded by reals (in the definition we again allow only parameters mentioned in (a).)

(c) Q satisfies ccc.

Then Corollary 12 is satisfied in every extension $M[G][F]$ for an $M[G]$-generic filter $F \subseteq Q$. For a proof let us observe that under assumptions (a), (b), (c) on the notion of forcing Q there exists a coding of names of reals of the model $M[G][F]$

by reals of $M[G]$. Namely for a boolean term $t \in M[G]^{(Q)}$ such that $\mathrm{dom}(t) = \{\hat{n}: n \in \omega\}$ we put $t(\hat{n}) = \sum\{q^{(t)}_{n,m} \in Q: m \in \omega\}$ for some countable antichain $\{q^{(t)}_{n,m}: m \in \omega\} \subseteq Q$. Since each $q^{(t)}_{n,m}$ may be treated as a real, the double sequence $\langle q^{(t)}_{n,m}: n, m \in \omega\rangle$ can be coded by a single real. We also observe that the assignment $x \to \hat{x}$ is $M[G]$-definable. Now it is enough to observe that for any formula φ the relation $\{\langle q, x_1, \ldots, x_n\rangle: q \Vdash \varphi(x_1, \ldots, x_n)\}$ becomes an $M[G]$-definable relation between reals and standard elements $\hat{x}$. We apply Corollary 12 and for any formula $\varphi(x, x_1, \ldots, x_n)$ there exists a countable subset $c_{q,x_1,\ldots,x_n} \subseteq b$ in the model $M[G][F]$ with the following property:

for each $x \in b - c_{q,x_1,\ldots,x_n}$ there exists an open interval s such that if $q \Vdash \varphi(\hat{x}, x_1, \ldots, x_n)$ then $\forall y \in b \cap s - c_{q,x_1,\ldots,x_n}[q \Vdash \varphi(\hat{y}, x_1, \ldots, x_n)]$.

This immediately implies that Corollary 12 holds in $M[G][F]$.

We are particulary interested in the case where the set b is definable in $M[G][F]$. In order to do it we apply Harrington's notion of forcing $Q(b)$ as described in [3]. It is proved in [3] that $Q(b)$ satisfies ccc and for any $M[G]$-generic filter $F \subseteq Q(b)$ the set b is Π^1_2 in $M[G][F]$. Therefore b is definable in the model $P(\omega) \cap M[G][F] \models A_2$.

We leave it to the reader to verify that the notion of forcing $Q(b)$ satisfies conditions (a) and (b) as well. As a consequence of that we infer that there exists a model $M^* \supseteq M$ of ZFC with a set of reals b such that $b \cap s$ is uncountable in M^* for any open interval s and, such that M^* satisfies Corollary 12.

Now let us suppose that a formula $\varphi(x, \xi)$ (possibly with parameters being reals) defines in M^* a relation in $b \times \omega_1$, such that the set $\{\xi \in \omega_1^{M^*}: \exists x \varphi(x, \xi)\}$ is uncountable in M^*.

We claim that there exists an $x \in b$ such that the set $\{\xi: \varphi(x, \xi)\}$ is uncountable in M^*.

We take a set $A \subseteq b$ countable in M^* and such that for any $x \in b - A$ there exists an open interval S with the property

$$M^* \models \varphi(x, \xi) \to \forall y \in b \cap s - A \varphi(y, \xi)\,.$$

There are two possible cases:

(1) There are uncountably many ordinals ξ such that $\{x \in b: \varphi(x, \xi)\} \subseteq A$.

Since A is countable, there exists an $x \in A$ such that the set $\{\xi: \varphi(x, \xi)\}$ is uncountable.

(2) There is an ordinal $\xi_0 \in \omega_1^{M^*}$ such that for $\xi \geqslant \xi_0$ we have $\{x \in b: \varphi(x, \xi)\} \not\subseteq A$.

We take $\xi \geqslant \xi_0$ and $z \in b - A$ such that $\varphi(z, \xi)$.

There exists an open interval s_ξ such that

$$M^* \models \forall y \in b \cap s_\xi - A \varphi(y, \xi)\,.$$

Since there are uncountably many ordinals ξ such that $\xi \geqslant \xi_0$ and only countably many open intervals s, there exists an open interval s such that the set

$$\{\xi \in \omega_1^{M^*}: M^* \models \forall y \in b \cap s - A \varphi(y, \xi)\}$$

is uncountable in M^*. The set $b \cap s$ is uncountable; therefore there exists an $x \in b \cap s - A$ and thence there are uncountably many ordinals ξ such that $\varphi(x, \xi)$, which proves the claim.

Now we are ready to prove

THEOREM 13. *In the model $P(\omega) \cap M^*$ of A_2 the quantifiers Q_c and Q_b are not equivalent.*

Proof. We show that the set b, which is definable in $P(\omega) \cap M^*$, is countable-like for the quantifier Q_b. Since b is uncountable, it cannot be countable-like for Q_c.

Let the formula $b(y)$ define the set b and suppose that for some formula $\psi(x, y)$ of the language $L_{P(\omega) \cap M^*}$ we have

$$P(\omega) \cap M^* \models Qx\exists y[b(y) \,\&\, \varphi(x, y)]\,.$$

Let us consider a formula $\varphi(\xi, y)$ such that $M^* \models \varphi(\xi, y)$ iff $y \in b$ and there exists a well ordering x of type ξ such that $P(\omega) \cap M^* \models \psi(x, y)$. Then the set

$$\{\xi \in \omega_1^{M^*}: \exists y \in b \varphi(\xi, y)\}$$

is un countable in M^*. By the claim there exists a $y \in b$ such that the set $\{\xi: \varphi(\xi, y)\}$ is uncountable in M^*, i.e. $P(\omega) \cap M^* \models Qx\psi(x, y)$. Therefore

$$P(\omega) \cap M^* \models \exists y Qx\psi(x, y)\,. \quad \text{Q.E.D.}$$

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