# On the metrizability of $F_{pp}$-spaces and its relationship to the normal Moore space conjecture

by

**Z. Balogh** (Debrecen)

**Abstract.** A space each subspace of which is a paracompact $M$-space is called an $F_{pp}$-space. The main aim of the present paper is to describe the structure of non-metrizable $F_{pp}$-spaces by showing that they always contain certain "typical" subspaces. Many interesting corollaries and an unexpected connection with certain investigations concerning the normal Moore space conjecture follow.

**Introduction.** It turned out in the late sixties that the following conditions are equivalent for a regular $T_1$ space $X$ (see [1], [20], [22]:

(a) $X$ is a paracompact $p$-space.

(b) $X$ is a paracompact $M$-space.

(c) There is a perfect map of $X$ onto a metrizable space.

(d) $X$ is homeomorphic to a closed subspace of the product of a metrizable space and a compact $T_2$ space.

The class of paracompact $M$-spaces has been shown to be countably productive and hereditary with respect to closed subspaces. On the other hand, arbitrary subspaces of paracompact $M$-spaces may fail to be paracompact $M$-spaces. (Take, as an example, any $T_2$ compactification of a non-paracompact Tychonoff space.) Following [2], we shall call a space each subspace of which is a paracompact $M$-space an $F_{pp}$-space.

Metrizable spaces are obviously $F_{pp}$-spaces and the one-point compactification of any uncountable discrete space is an easy example of a non-metrizable $F_{pp}$-space. Concerning the natural question how far $F_{pp}$-spaces are from metrizable spaces, A. V. Arhangel'skiĭ [2] posed the following problems.

A1. Is the Souslin number of an $F_{pp}$-space equal to its weight?

A2. Does every $F_{pp}$-space have a dense metrizable subspace?

A3. What are the "simplest" metrizability criteria for $F_{pp}$-spaces?

In [6] the present author has shown that every $F_{pp}$-space is the union of $\leqslant \omega_1$ metrizable subspaces, and deduced that the answer is "yes" to Problems A1 and A2.

(By using a method different from that of [6], A. V. Arhangel'skiĭ [3] answered Problem A2 independently. For partial answers which preceded the results mentioned above, see [2], [14], [17], [4], [5].) Concerning Problem A3, the author [7] has shown that the class of metrizable spaces coincides with the class of perfectly normal $F_{pp}$-spaces (see Corollary 4.4 in this paper).

The main aim of the present paper is to describe the structure of non-metrizable $F_{pp}$-spaces by showing that they always contain certain „typical" subspaces (Theorems 3.3 and 4.3). In particular, it will turn out that Problem A3 has an unexpected connection with certain investigations concerning the normal Moore space conjecture.

Sections 2 and 3 are essentially devoted to the proof of Theorem 3.3 mentioned above. Numerous corollaries of this result are included is Section 4. In Section 1 we prove a decomposition theorem concerning first countable $F_{pp}$-spaces, which we shall use in the proof of Theorem 3.3 and which is interesting in itself (Theorem 1.7).

**0. Preliminaries.** Throughout the paper, $|A|$ denotes the cardinality of a set $A$. Cardinals are identified with initial ordinals. Given a topological space $(X, \tau)$, $\mathrm{cl}_\tau A$ will denote the closure of $A$ in $(X, \tau)$. However, since $\tau$ will usually by omitted from the term $(X, \tau)$, we shall often write $\mathrm{cl}A$ (or $\mathrm{cl}_X A$) for $\mathrm{cl}_\tau A$. We shall say that $A \subset X$ is a *discrete subset* of $(X, \tau)$ if $A$ has no accumulation points in $(X, \tau)$. (Thus, $A$ is a discrete subset of $(X, \tau)$ iff $(A, \tau|A)$ is a closed discrete subspace of $(X, \tau)$.) Given a subset $A$ of a topological space $X$, $\chi(A, X)$ will denote the *character* of $A$ in $Y$ (i.e., the smallest infinite cardinal number $\varkappa$ such that $A$ has a neighbourhood base of cardinality $\leqslant \varkappa$ in $X$). $\chi(x, X)$ stands for $\chi(\{x\}, X)$. The *pseudo-character* of $X$ at a point $x$, denoted by $\psi(x, X)$, is defined to be the smallest infinite cardinal number $\varkappa$ such that $\{x\}$ is the intersection of $\leqslant \varkappa$ open subsets of $X$. A space $X$ is said to be *of point-countable type* if it can be covered by compact subsets which have countable character in $X$. A cover $\mathfrak{G}$ of $X$ is said to be *separating* if for every pair of distinct points $x, y \in X$ there is a $G$ in $\mathfrak{G}$ with $x \in G$, $y \notin G$.

We shall say that a space $X$ is of *Q-type* if every subset of $X$ is an $F_\sigma$, and $X$ is not $\sigma$-discrete.

Let $(X, \tau)$ be a topological space, and let $R$ be an equivalence relation on $X$. For any point $x$ in $X$, let $[x]_R$ denote the equivalence class of $x$. If $A \subset X$, then let $[A]_R = \bigcup \{[x]_R \colon x \in A\}$. If $R'$ is an equivalence relation on a subset $X'$ of $X$, then $R'$ is said to be *finer* than $R$ if $[x]_{R'} \subset [x]_R$ for every point $x \in X'$. Let $\tau_R$ denote the topology on $X$ consisting of those $\tau$-open subsets which can be represented as unions of some equivalence classes of $R$. $R$ is said to be *closed* if, for every closed subset $A$ of $(X, \tau)$, $[A]_R$ is also a closed subset of $(X, \tau)$. Let us say that $R$ is *perfect* if $R$ is closed and all equivalence classes of $R$ are compact subsets of $(X, \tau)$.

Making use of the definition of paracompact $M$-spaces with the aid of perfect maps, one can easily prove the following assertion:

PROPOSITION 0.1. *A regular $T_1$ space $(X, \tau)$ is a paracompact $M$-space if and only if there is a perfect equivalence relation $R$ on $(X, \tau)$ such that $\tau_R$ has a $\sigma$-locally finite base.*



Finally, we shall need a special case of Lemma 1.2 in [6].

PROPOSITION 0.2. *Let $(X, \tau)$ be a $T_2$ space, and let $X_1$ and $X_2$ be non-void subsets of $X$ such that $X_1 \subset X_2$. Suppose that $R_1$ and $R_2$ are perfect equivalence relations on $(X_1, \tau|X_1)$ and $(X_2, \tau|X_2)$, respectively. Define an equivalence relation $R$ on $X_1$ by putting*

$$x \sim_R y \quad \text{iff} \quad x \sim_{R_1} y \text{ and } x \sim_{R_2} y .$$

*Then*

(a) *$R$ is a perfect equivalence relation on $(X_1, \tau|X_1)$;*

(b) $(\tau|X_1)_R = \sup\{(\tau|X_1)_{R_1}, (\tau|X_2)_{R_2}|X_1\}$.

**1. A decomposition of first countable $F_{pp}$-spaces into the union of metrizable subspaces.** In this section we shall need the following four well-known results.

LEMMA 1.1 (see [18], p. 524). *Let $X$ be a set with $|X| = 2^\omega$, and let $\mathfrak{C}$ be a family of subsets of $X$ such that $|\mathfrak{C}| \leqslant 2^\omega$ and $|C| = 2^\omega$ for each $C$ in $\mathfrak{C}$. Then there are disjoint subsets $A_1$, $A_2$ of $X$ such that $X = A_1 \cup A_2$ and $C \cap A_i \neq \emptyset$ for each $C$ in $\mathfrak{C}$ and each $i \in \{1, 2\}$.*

LEMMA 1.2 (see [16], p. 33). *If $X$ is a first countable compact $T_2$ space, then either $|X| \leqslant \omega$ or $|X| = 2^\omega$.*

THEOREM 1.3 (J. Nagata [21]). *A paracompact $M$-space with a point-countable separating open cover is metrizable.*

LEMMA 1.4 (A. Hajnal and I. Juhász [11]). *If $X$ is a compact $T_2$ space with no isolated points, then there is a countable subset $S$ of $X$ with $|\mathrm{cl}S| \geqslant 2^\omega$.*

In what follows we shall actually make use of the following corollary of Lemma 1.4, which can be established with the help of the well-known fact that an uncountable compact $T_2$ space satisfying the first axiom of countability has a non-empty compact subspace with no isolated points (see [16], p. 33, e.g.).

COROLLARY 1.5. *If $X$ is a first countable compact $T_2$ space with $|X| > \omega$, then there is a countable subset $S$ of $X$ with $|\mathrm{cl}S| = 2^\omega$.*

LEMMA 1.6. *If $X$ is a first countable compact $T_2$ space, then there are disjoint subspaces $A_1$ and $A_2$ of $X$ such that $X = A_1 \cup A_2$ and neither $A_1$ nor $A_2$ contains an uncountable compact subspace.*

Proof. We may assume $|X| > \omega$. Then, by Lemma 1.2, $|X| = 2^\omega$. Thus, denoting by $\mathfrak{C}$ the family of all compact separable subspaces of cardinality $2^\omega$ of $X$, we have $|\mathfrak{C}| \leqslant |X|^\omega = 2^\omega$. Applying Lemma 1.1, $X$ can be split into two disjoint subspaces, $A_1$ and $A_2$, such that $C \cap A_i \neq \emptyset$ for all $C$ in $\mathfrak{C}$ and $i$ in $\{1, 2\}$. We show that if $C_i$ is an arbitrary compact subspace of $A_i$ ($i \in \{1, 2\}$), then $|C_i| \leqslant \omega$. Indeed, let us suppose indirectly that $|C_i| > \omega$. Then by virtue of Corollary 1.5 there is a compact separable subspace $C_i'$ of $C_i$ with $|C_i'| = 2^\omega$. Then $C_i'$ is a compact separable subspace of $X$ with $C_i' \subset A_i$, $|C_i'| = 2^\omega$, in contradiction with the definition of $A_1$ and $A_2$.

Remarks. 1. The proof of Lemma 1.6 yields the following more general result: If $X$ is a first countable $T_2$ space with $|X| \leqslant 2^\omega$, then $X$ can be split into two disjoint subspaces, $A_1$ and $A_2$, such that neither $A_1$ nor $A_2$ contains an uncountable compact subspace.

2. Lemma 1.6 generalizes a result of A. V. Arhangel'skiĭ [2], which says that the conclusion of Lemma 1.6 is valid for perfectly normal compact $T_2$ spaces.

3. There are first countable compact $T_2$ spaces which have more than $2^\omega$ compact subspaces of cardinality $2^\omega$ (e.g., think of Alexandroff's double circumference); thus in the proof of Lemma 1.6, Lemma 1.1 cannot be directly applied to the family of such subspaces.

THEOREM 1.7. *Every first countable $F_{pp}$-space is the union of countably many metrizable subspaces.*

Proof. Since $X$ has a perfect map onto a metrizable space, we infer that there is a point-countable (moreover, $\sigma$-locally finite) open cover $\mathfrak{G}$ of $X$ such that

(a) $C_x = \bigcap \{G: x \in G \in \mathfrak{G}\}$ is compact for each $x \in X$;

(b) $\mathfrak{C} = \{C_x: x \in X\}$ is a partition of $X$ into disjoint subsets.

Applying Lemma 1.6, it follows that for each $C$ in $\mathfrak{C}$, $C = A_1(C) \cup A_2(C)$ holds, so that neither $A_1(C)$ nor $A_2(C)$ contains an uncountable compact subspace. Let

$$X_i = \bigcup \{A_i(C): C \in \mathfrak{C}\} \quad (i = 1, 2).$$

We are going to show that each $X_i$ $(i = 1, 2)$ is the union of countably many metrizable subspaces. Indeed, since $X_i$ $(i = 1, 2)$ has a perfect map onto a metrizable space, (by writing $X_i$, $\mathfrak{G}_i$, $C_{xi}$ and $\mathfrak{C}_i$ instead of $X$, $\mathfrak{G}$, $C_x$ and $\mathfrak{C}$, respectively) we infer that there is a point-countable open cover $\mathfrak{G}_i$ of $X_i$ satisfying (a) and (b).

Let us now define

$$\mathfrak{G}_i^* = \mathfrak{G}_i \cup \{G \cap X_i: G \in \mathfrak{G}\}.$$

Clearly $\mathfrak{G}_i^*$ is a point-countable open cover of $X_i$ and it can easily be seen that

(a′) $K_{xi} = \bigcap \{G: x \in G \in \mathfrak{G}_i^*\}$ is compact for all $x \in X_i$;

(b′) $\mathfrak{K}_i = \{K_{xi}: x \in X_i\}$ is a partition of $X_i$ into disjoint subsets.

Since for each $x \in X_i$ we have $K_{xi} \subset X_i \cap C_x = A_i(C_x)$ and $A_i(C_x)$ contains no uncountable compact subspaces, we infer that each $K_i \in \mathfrak{K}_i$ is countable, i.e., $K_i = \{x_n(K_i): n < \omega\}$, with repetitions permitted. Let

$$X_{ni} = \{x_n(K_i): K_i \in \mathfrak{K}_i\}.$$

Now the trace of $\mathfrak{G}_i^*$ on $X_{ni}$ is a point-countable separating open cover of $X_{ni}$; thus $X_{ni}$ is metrizable by Theorem 1.3. Since $X = \bigcup \{X_{ni}: n < \omega, i = 1, 2\}$, this completes the proof.

Besides Theorem 1.7, we shall also need the following result, which was independently found by A. V. Arhangel'skiĭ [3] and the author [6], by using different methods.



THEOREM 1.8. *Every $F_{pp}$-space has a dense metrizable subspace.*

**2. $F_{pp}$-spaces of the form $A(X)$.** Let us first recall the construction of the Alexandroff duplicate of a topological space due to R. Engelking [8].

Given an arbitrary topological space $X$, we can topologize the set $A(X) = X \cup X'$, where $X'$ is a disjoint copy of $X$, in the following way. For every $x \in X$ let $x'$ denote the point corresponding to $x$, and for every subset $S$ of $X$ let us define $S' = \{x': x \in S\}$. Now, let $\mathfrak{a}$ denote the topology of $A(X)$ generated by the base

(*) $\mathfrak{B} = \{\{x'\}: x \in X\} \cup \{U \cup U' - A': U$ is an open subset and $A$ is a finite subset of the space $X\}$.

The resulting space, $(A(X), \mathfrak{a})$, is called the *Alexandroff duplicate* of $X$. If there is no danger of confusion, then we shall briefly write $A(X)$ for $(A(X), \mathfrak{a})$.

We shall need the following simple fact from [8]:

PROPOSITION 2.1. *If $X$ is a $T_1$ space and $B$ is an arbitrary subset of $X$, then*

$$\mathrm{cl}_{A(X)} B' = B' \cup B^d$$

*where $B^d$ denotes the set of all accumulation points of $B$ in $X$.*

The easy proofs of the following two propositions are left to the reader.

PROPOSITION 2.2. *Let $(A(X), \mathfrak{a})$ be the Alexandroff duplicate of a $T_2$ space $X$, and let $\tau$ be a $T_2$ topology on $A(X)$ such that $\tau|X = \mathfrak{a}|X$, $\tau|X' = \mathfrak{a}|X'$, and $X$ is a closed subset in $\tau$. Further, let $R_2$ denote the equivalence relation on $A(X)$ the equivalence classes of which are $\{\{x, x'\}: x \in X\}$. Then the following conditions are equivalent:*

(a) *$R_2$ is closed (thus, perfect) with respect to $\tau$;*

(b) *$\tau = \mathfrak{a}$.*

PROPOSITION 2.3. *The Alexandroff duplicate of a paracompact $M$-space is a paracompact $M$-space.*

In what follows we shall need the following two well-known facts.

PROPOSITION 2.4 ([1]). *Every $G_\delta$ subspace and every closed subspace of a $p$-space is a $p$-space.*

LEMMA 2.5. *If a paracompact $M$-space $X$ is the union of countably many closed, metrizable subspaces, then $X$ is metrizable.*

Proof. This lemma is a special case of Theorem 7.1 in [13].

THEOREM 2.6. *Let $A(X)$ be the Alexandroff duplicate of a metrizable space $X$. Then the following conditions are equivalent:*

(a) *Every subset of $X$ is an $F_\sigma$.*

(b) *$A(X)$ is an $F_{pp}$-space.*

Proof. (a) $\Rightarrow$ (b). Considering that the class of all paracompact $p$-spaces and the class of all paracompact $M$-spaces coincide, it is enough to prove that $A(X)$ is hereditarily paracompact and is a $p$-space hereditarily.

In order to prove that every subspace of $A(X)$ is paracompact, we shall first show that subspaces of the form $Y = U \cup X'$, where $U$ is an open subset of $X$, are paracompact. Indeed, by the definition of $A(X)$, $U \cup U'$ and $X' - U'$ are clopen subspaces of $Y$. By Proposition 2.3 $U \cup U'$ is paracompact. (One easily verifies that the topology on $U \cup U'$ induced by the topology of $A(X)$ is the same as the topology of $A(X)$ considered as the Alexandroff duplicate of the subspace $U$ of $X$.) Since the (discrete) subspace $X' - U'$ of $Y$ is clearly paracompact, we conclude that $Y = U \cup U' \cup (X' - U')$ is also paracompact.

Now, let $Z$ be an arbitrary open subspace of $A(X)$, and let $U = Z \cap X$. Then

$$Z = U \cup X' - (X' - Z)$$

is a closed subspace of the paracompact subspace $U \cup X'$; thus, $Z$ is paracompact. Since a $T_2$ space is hereditarily paracompact iff each of its open subspaces is paracompact, we conclude that $A(X)$ is hereditarily paracompact.

To prove that every subspace $Y$ of $A(X)$ is a $p$-space, let $Y_1 = Y \cap X$ and $Y_2' = Y \cap X'$. Since $X - Y_1$ is an $F_\sigma$ subset of the space $X$, it follows that $X' \cup Y_1$ is a $G_\delta$ subspace of $A(X)$; thus $X' \cup Y_1$ is a $p$-space by Proposition 2.4. Further, since $Y = X' \cup Y_1 - (X' - Y_2')$ is a closed subspace of the subspace $X' \cup Y_1$, $Y$ is a $p$-space, again by Proposition 2.4.

(b) $\Rightarrow$ (a). Let $H$ be an arbitrary subset of the space $X$, and let $Y = H' \cup (X - H)$. Then the natural map $f\colon Y \to X$ defined by formulas

$$f(y) = \begin{cases} y, & \text{if } y \in X - H, \\ x, & \text{if } y = x' \in H' \end{cases}$$

is a one-to-one continuous map of $Y$ onto the metrizable space $X$. Thus, by Theorem 1.3, $Y$ is metrizable. Therefore $H'$ is an $F_\sigma$ subset in $Y$, i.e. $H' = \bigcup_{n<\omega} F_n'$ such that $\mathrm{cl}_Y F_n' = F_n' \cup F_n^d \cap (X - H) = F_n'$. Then $\mathrm{cl}_X F_n = F_n \cup F_n^d \subset H$ for every $n<\omega$, which implies that $H = \bigcup_{n<\omega} F_n$ is an $F_\sigma$ subset of $X$.

Remark. It can easily be seen that the Alexandroff duplicate of a paracompact $T_2$ space is again a paracompact $T_2$ space. Using this instead of Proposition 2.3, the first part of the proof of (a) $\Rightarrow$ (b) in Theorem 2.6 gives the following more general result:

The Alexandroff duplicate of a hereditarily paracompact $T_2$ space is hereditarily paracompact.

PROPOSITION 2.7. *Let $A(X)$ be the Alexandroff duplicate of a metrizable space $X$. Then the following conditions are equivalent:*

(a) *$A(X)$ is metrizable.*

(b) *$X'$ is a $G_\delta$ subset of $A(X)$.*

(c) *$X$ is $\sigma$-discrete.*

Proof. (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (c). Let $X' = \bigcup_{n<\omega} D_n'$, where $D_n'$ is a closed subspace of $A(X)$ for each $n<\omega$. By virtue of Proposition 2.1, $D_n^d = \emptyset$, so that $D_n$ is a closed discrete subspace of $X$ $(n<\omega)$. Then (c) follows by $X = \bigcup_{n<\omega} D_n$.

(c) $\Rightarrow$ (a). Let $X$ be the union of a family $\{D_n\colon n<\omega\}$ of closed discrete subspaces. Then, for each $n<\omega$, $D_n \cup D_n'$ is a closed discrete (and thus metrizable) subspace of $X$. Then $A(X)$ is metrizable by Proposition 2.3 and Lemma 2.5.



The following result, which derives from Theorem 2.6 and Proposition 2.7, plays an important role in what follows.

THEOREM 2.8. *The Alexandroff duplicate of a metrizable space $X$ is a non-metrizable $F_{pp}$-space if and only if $X$ is a space of $Q$-type.*

**3. Non-metrizable, first countable $F_{pp}$-spaces.** In order to prove the main result in this section, we shall need the following two lemmas.

LEMMA 3.1. *If a paracompact $M$-space $X$ is the union of a family $\{X_n\colon n<\omega\}$ of metrizable subspaces, then $X^* = \bigcap_{n<\omega} \mathrm{cl}\, X_n$ is a metrizable subspace.*

In particular, *if a paracompact $M$-space $X$ is the union of countably many dense metrizable subspaces, then $X$ is metrizable.*

Proof. By virtue of Theorem 1.3 it is enough to show that $X^*$ has a point-countable base, and this results from the following theorem of E. Michael and M. E. Rudin [19]:

If $X$ is a regular $T_1$ space, $X = \bigcup_{n<\omega} X_n$, and each subspace $X_n$ has a $\sigma$-disjoint base, then $X^* = \bigcap_{n<\omega} \mathrm{cl}\, X_n$ also has a $\sigma$-disjoint base.

LEMMA 3.2. *Let $(X, \tau)$ be a hereditarily paracompact regular $T_1$ space, and let $R$ be a perfect equivalence relation on $(X, \tau)$ such that $\tau_R$ has a $\sigma$-locally finite base. Suppose that $X$ is the union two metrizable subspaces, $X_1$ and $X_2$, such that $[X_i]_R = X_i$ $(i = 1, 2)$. Then $(X, \tau)$ is metrizable.*

Proof. By virtue of Proposition 0.1 and Theorem 1.3 it is enough to show that $(X, \tau)$ has a point-countable separating open cover.

To prove this, let $\mathfrak{B}_i = \bigcup_{n<\omega} \mathfrak{B}_{in}$ be a base for $(X_i, \tau|X_i)$ such that, for each $n<\omega$, $\mathfrak{B}_{in}$ is a discrete family in $(X_i, \tau|X_i)$. Further, let $\mathfrak{B}$ be a $\sigma$-locally finite base of the space $(X, \tau_R)$.

For every $B$ in $\mathfrak{B}_{in}$ let us choose an open subset $\tilde{B}$ of $(X, \tau)$ such that $\tilde{B} \cap X_i = B$, and let us define

$$\tilde{\mathfrak{B}}_{in} = \{\tilde{B}\colon B \in \mathfrak{B}_{in}\} \quad (n<\omega, i = 1, 2).$$

Since $(X, \tau)$ is hereditarily paracompact, there is a $\sigma$-point finite refinement $\mathfrak{G}_{in}$ of $\tilde{\mathfrak{B}}_{in}$ with $\bigcup \mathfrak{G}_{in} = \bigcup \tilde{\mathfrak{B}}_{in}$. Now, let us put

$$\mathfrak{G} = \mathfrak{B} \cup \bigcup \{\mathfrak{G}_{in}\colon n<\omega, i = 1, 2\}.$$

$\mathfrak{G}$ is clearly a point-countable (moreover, a $\sigma$-point finite) open cover of $(X, \tau)$. It remains to show that $\mathfrak{G}$ is also separating.

To verify this, let $x, y$ be a pair of distinct points of $X$. Now, if $y \notin [x]_R$, then there is a $B$ in $\mathfrak{B}(\subset\mathfrak{G})$ with $x \in B$, $y \notin B$. So suppose $y \in [x]_R$. Then $[X_i]_R = X_i$ $(i = 1, 2)$ implies that $x$ and $y$ are in the same subspace $X_i$. Thus, there is a $B_i$ in $\mathfrak{B}_i$ with $x \in B_i$, $y \notin B_i$. Let $n<\omega$ be such that $B_i \in \mathfrak{B}_{in}$. Then $\bigcup \mathfrak{G}_{in} = \bigcup \mathfrak{B}_{in}$ implies that there is a $G_i \in \mathfrak{G}_{in}$ with $x \in G_i$. Since $\mathfrak{G}_{in}$ is a refinement of $\tilde{\mathfrak{B}}_{in}$ and $\tilde{B}_i$ is the only member of $\mathfrak{B}_{in}$ containing $x$, it follows that $x \in G \subset \tilde{B}_i$. Moreover, $y \in X_i$ and $y \notin B_i$ imply $y \notin \tilde{B}_i$, and thus $y \notin G_i$.

THEOREM 3.3. *A first countable $F_{pp}$-space $(X, \tau)$ is metrizable if and only if it does not contain a subspace homeomorphic to the Alexandroff duplicate of a metrizable space of $Q$-type.*

Proof. If $(X, \tau)$ has a subspace homeomorphic to the Alexandroff duplicate of a metrizable space of $Q$-type, then $(X, \tau)$ is non-metrizable by Theorem 2.8.

Conversely, suppose that $(X, \tau)$ is a first countable, non-metrizable $F_{pp}$-space. By virtue of Theorem 1.7 $X$ is the union of a family $\{A_n\colon n<\omega\}$ of metrizable subspaces. (From now on, by a subspace we shall mean a subspace of $(X, \tau)$, and we shall briefly write $Y$ for a subspace $(Y, \tau|Y)$.)

It follows from the non-metrizability of $(X, \tau)$ that there is an $n_0<\omega$ such that $A_{n_0}$ is not contained in a dense metrizable subspace of $(X, \tau)$. Indeed, if there were no such index $n_0$, then $X$ would be the union of countably many dense, metrizable subspaces, and Lemma 3.1 would imply that $(X, \tau)$ is metrizable.

Let $U_{n_0} = X - \mathrm{cl}_\tau A_{n_0}$. Since by Theorem 1.8 the subspace $U_{n_0}$ has a dense metrizable subspace, there is a dense subspace $S = \bigcup\limits_{k<\omega} S_k$ of $U_{n_0}$ such that $S_k$ is a closed discrete subspace of $S$ for every $k<\omega$. By the definition of $U_{n_0}$ and $S$, $Z_k = A_{n_0} \cup S_k$ is a closed subspace of the subspace $Z = A_{n_0} \cup S$. Since, by the assumption on $n_0$, $A_{n_0} \cup S$ is non-metrizable, we infer by Lemma 2.5 that there is a $k_0<\omega$ such that $Z_{k_0}$ is non-metrizable.

Now, let $R_0$ be a perfect equivalence relation on the subspace $Z_{k_0} = A_{n_0} \cup S_{k_0}$ such that $(\tau|Z_{k_0})_{R_0}$ has a $\sigma$-locally finite base. (In what follows, we shall call such an equivalence relation a *good* equivalence relation.) Since $A_{n_0}$ is closed in the subspace $Z_{k_0}$, Lemma 1.6 implies that, for each equivalence class $C$ of $R_0$, there is a decomposition $C \cap A_{n_0} = M_1(C) \cup M_2(C)$ such that neither $M_1(C)$ nor $M_2(C)$ contains an uncountable compact subset. Let $M_i = \bigcup \{M_i(C)\colon C$ is an equivalence clases of $R_0\}$, $M_i^* = M_i \cup S_{k_0}$ $(i = 1, 2)$, and let $R_i$ be a good equivalence relation on the subspace $M_i^*$. By virtue of Proposition 0.2, we may assume that each $R_i$ $(i = 1, 2)$ is finer than $R_0$, and thus, since $M_i$ is closed in the subspace $M_i^*$, it follows that

$$|M_i \cap C_i| \leqslant \omega \text{ for each equivalence class } C_i \text{ of } R_i\,.$$

For every equivalence class $C_i$ of $R_i$ let $M_i \cap C_i = \bigcup\limits_{m<\omega} X_m(C_i)$, where $|X_m(C_i)| \leqslant 1$ for each $m<\omega$. Further, let

$$A_{mi} = \bigcup \{X_m(C_i)\colon C_i \text{ is an equivalence class of } R_i\}\,,$$



and

$$Y_{mi} = A_{mi} \cup S_{k_0} \quad (m<\omega,\ i = 1, 2)\,.$$

We shall prove that there is a non-metrizable subspace among the subspaces $Y_{mi}$.

Suppose indirectly that $Y_{mi}$ is metrizable for every $m<\omega$ and $i = 1, 2$. Then, by Lemma 3.1,

$$Y = \bigcap \{\mathrm{cl}_{\tau|Z_{k_0}} Y_{mi}\colon m<\omega,\ i = 1, 2\} \subset S_{k_0}$$

is a metrizable subspace of the subspace $Z_{k_0}$. Since metrizable spaces are perfectly normal, the subset $S_{k_0}$ (consisting of isolated points of $Y$) is the union of a countable family $\{F_l\colon l<\omega\}$ of subsets closed in $Y$ (and thus in $Z_{k_0}$). Then, for each $l<\omega$, $N_l = A_{n_0} \cup F_l$ is a closed subspace of $Z_{k_0}$, and, since $N_l$ is the topological sum of the discrete subspace $F_l$ and the metrizable subspace $A_{n_0}$, $N_l$ is metrizable. Thus, by Lemma 2.5, $Z_{k_0} = \bigcup\limits_{l<\omega} N_l$ is metrizable, in contradiction with the choice of $k_0$.

Thus we can choose an $m_0<\omega$ and $i_0 \in \{1, 2\}$, so that the subspace $Y_{m_0 i_0} = A_{m_0 i_0} \cup S_{k_0}$ is non-metrizable. Let $R^*$ be a good equivalence relation on $Y_{m_0 i_0}$. By Proposition 0.2 we may assume that $R^*$ is finer than $R_{i_0}$. Therefore, for each equivalence class $C^*$ of $R^*$, $|A_{m_0 i_0} \cap C^*| \leqslant 1$ holds. Considering that each $C^*$ is a first countable compact subspace, and $S_{k_0} \cap C^* = C^* - A_{m_0 i_0} \cap C^*$ is an open discrete subspace of $C^*$, we conclude that $|S_{k_0} \cap C^*| \leqslant \omega$ for each equivalence class $C^*$ of $R^*$, i.e.,

$$S_{k_0} \cap C^* = \bigcup_{j<\omega} X_j(C^*), \quad \text{where} \quad |X_j(C^*)| \leqslant 1$$

for every $j<\omega$. Let us now define

$$D_j = \bigcup \{X_j(C^*)\colon C^* \text{ is an equivalence class of } R^*\}$$

and

$$W_j = A_{m_0 i_0} \cup D_j \quad (j<\omega)\,.$$

Then, by virtue of Lemma 2.5 and of $Y_{m_0 i_0} = \bigcup\limits_{j<\omega} W_j$, there is a $j_0<\omega$ such that the subspace $W_{j_0} = A_{m_0 i_0} \cup D_{j_0}$ is non-metrizable.

Let us now consider a good equivalence relation $R_1^*$ on $W_{j_0}$. By virtue of Proposition 0.2 we may assume that $R_1^*$ is finer than $R^*$. Then by the construction of $W_{j_0}$ we infer that

(1) $A_{m_0 i_0} \cap D_{j_0} = \emptyset$;

(2) $A_{m_0 i_0}$ is a closed metrizable subspace of $W_{j_0}$;

(3) $D_{j_0}$ is an open discrete subspace of $W_{j_0}$;

(4) for each equivalence class $C_1^*$ of $R_1^*$ $|C_1^* \cap A_{m_0 i_0}| \leqslant 1$ and $|C_1^* \cap D_{j_0}| \leqslant 1$ hold.

Let us define

$$T = \{t \in A_{m_0 i_0}\colon |[t]_{R_1^*}| = 2\}\,,$$

and let

$$T' = \{t' \colon t \in T\}, \quad \text{where} \quad \{t'\} = [t]_{R_1^*} \cap D_{j_0}.$$

Further, let $L = \{w \in W_{j_0} \colon |[w]_{R_1^*}| = 1\}$. It follows from (4) that $t'$ uniquely exists for all $t$ in $T$ and that $W_{j_0}$ is the union of the pairwise disjoint sets $T$, $T'$ and $L$. It can easily be seen that the restriction of a good equivalence relation $R$ to a subspace which is the union of some equivalence classes of $R$ (in particular, the restriction $R_2^*$ of $R_1^*$ to the subspace $A(T) = T \cup T'$ and the restriction $R_L$ of $R_1^*$ to the subspace $L$) is again a good equivalence relation. Since $R_L$ is the relation "=" on $L$ and $R_L$ is closed, it follows that

$$(\tau|L)_{R_L} = \tau|L.$$

Since, by the definition of a good equivalence relation, $(\tau|L)_{R_L}$ has a $\sigma$-locally finite base, we find that the subspace $L$ is metrizable. Then the subspace $A(T)$ is non-metrizable, since otherwise by Lemma 3.2 $W_{j_0} = A(T) \cup L$ would also be metrizable, in contradiction with the choice of $j_0$.

Now, let us equip $A(T) = T \cup T'$ with the topology $\mathfrak{a}$ of the Alexandroff duplicate of the subspace $T$. By conditions (1)-(3), and since $R_2^*$ is a closed equivalence relation on the subspace $A(T)$, Proposition 2.2 implies that $\tau|A(T) = \mathfrak{a}$, i.e., the subspace $A(T)$ is homeomorphic to the Alexandroff duplicate of the subspace $T$.

Finally, since $A(T)$ is a non-metrizable $F_{pp}$-space, it follows from Theorem 2.8 that the subspace $T$ is a metrizable space of $Q$-type.

**4. On the structure of non-metrizable $F_{pp}$-spaces.** In order to prove a characterization of non-metrizable $F_{pp}$-spaces, we shall need a result of Arhangel'skiĭ [2] quoted as Lemma 4.2 in our paper. It is claimed in [2] that this result can be proved by a slight change of argument in the proof of Theorem 2 in [2]. Since it is not clear from [2] that Theorem 2 has nothing to do with the continuum hypothesis (cf. [13], [4]), and the changes needed are nontrivial, it seems worthwhile to give a proof of Lemma 4.2 here.

When proving Lemma 4.2, we shall make use of the following proposition, the easy proof of which is left to the reader.

PROPOSITION 4.1. *Let $X$ be a regular $T_1$ space. Then the following assertions hold*:

(a) *If $x \in X$ and there is a compact subspace $C$ of $X$ with $x \in C$ and $\chi(C, X) = \omega$, then $\chi(x, X) = \psi(x, C)$.*

(b) *If $Z$ is a dense subspace of $X$ and $x \in Z$, then $\chi(x, Z) = \chi(x, X)$.*

LEMMA 4.2. *Suppose that $X$ is a regular $T_1$ space such that each subspace of $X$ is of point-countable type (in particular, $X$ is an $F_{pp}$-space). Further, suppose that $x \in X$ and $\chi(x, X) = \varkappa > \omega$. Then there is a discrete subspace $D$ of $X$ such that $|D| = \varkappa$ and the subspace $D^* = D \cup \{x\}$ of $X$ is homeomorphic to the one-point compactification of $D$.*



Proof. By virtue of Zorn's lemma there is a maximal family $\mathfrak{G}$ of pairwise disjoint open subsets of $X$ such that

$$x \notin \operatorname{cl} G \quad \text{for each } G \in \mathfrak{G}.$$

By the maximality of $\mathfrak{G}$, $U = \bigcup \mathfrak{G}$ is a dense open subspace of $X$. Let $Z = U \cup \{x\}$, then by Proposition 4.1 (b) $\chi(x, Z) = \chi(x, X) = \varkappa > \omega$. Now, let $C$ be a compact subspace of $Z$ with $x \in C$ and $\chi(C, Z) = \omega$, and let

$$\mathfrak{G}_1 = \{G \in \mathfrak{G} \colon C \cap G \neq \emptyset\}.$$

Since by

$$\{x\} = \bigcap \{C - \operatorname{cl} G \colon G \in \mathfrak{G}_1\}$$

$\psi(x, C) \leqslant |\mathfrak{G}_1|$, Proposition 4.1 (a) implies that

$$|\mathfrak{G}_1| \geqslant \chi(x, X) = \varkappa.$$

For each $G \in \mathfrak{G}_1$ let $x_G$ be an arbitrary point of $G \cap C$ and let us define $D = \{x_G \colon G \in \mathfrak{G}_1\}$, $D^* = D \cup \{x\}$. It follows from the compactness of the subspace $C$ that every open subset $V$ of $C$ with $x \in V$ covers all but finitely many of the pairwise disjoint open subsets $G \cap C$ $(G \in \mathfrak{G}_1)$ of $C$. Thus, by the definition of $D$, every point of $D$ is isolated in the subspace $D^*$, and for every open subset $U$ of $D$ with $x \in U$, $|D^* - U| < \omega$ holds, i.e., $D^*$ is homeomorphic to the one-point compactification of $D$.

Finally, $|\mathfrak{G}_1| \geqslant \varkappa$ implies $|D| \geqslant \varkappa$, and $|D| \leqslant \varkappa$ holds by $\chi(x, D^*) \leqslant \chi(x, X) = \chi$. Thus $|D| = \varkappa$.

By virtue of Theorems 2.8, 3.3 and Lemma 4.2 we find the following main result concerning the structure of nonmetrizable $F_{pp}$-spaces.

THEOREM 4.3. *An $F_{pp}$-space is metrizable if and only if it contains neither a subspace homeomorphic to the one-point compactification of an uncountable discrete space nor a subspace homeomorphic to the Alexandroff duplicate of a metrizable space of $Q$-type.*

COROLLARY 4.4. *A topological space is metrizable if and only if it is a perfectly normal $F_{pp}$-space.*

Proof. Only the "if" part needs proof.

To prove it, note first that neither the one-point compactification of an uncountable discrete space nor the Alexandroff duplicate of a metrizable space of $Q$-type is perfectly normal. (For the latter, see Proposition 2.7.) Since every subspace of a perfectly normal space is perfectly normal, we can apply Theorem 4.3.

Remark. For another proof of this corollary, see [7].

COROLLARY 4.4.1 (A. V. Arhangel'skiĭ [2]). *A regular $T_1$ space is a separable metrizable space if and only if it is a Lindelöf $M$-space hereditarily.*

Proof. The "only if" part is trivial. The "if" part follows from Corollary 4.4 and the fact that hereditarily Lindelöf regular $T_1$ spaces are hereditarily paracompact and perfectly normal.

Corollary 4.5. *The existence of a first countable, non-metrizable $F_{pp}$-space is equivalent to the existence of a metrizable space of $Q$-type.*

Proof. This corollary follows from Theorems 3.3 and 2.8.

It is well known (see [24]) that Martin's axiom plus the negation of the continuum hypothesis implies the existence of a subspace of the real line which is a space of $Q$-type. (Note that a subspace $A$ of $R$ is of $Q$-type if and only if it is a $Q$-set in the sense of [24].) On the other hand, G. M. Reed deduced from a celebrated result of W. Fleissner [9] (which states that every normal $T_2$ space with character $\leqslant 2^{\omega}$ is collectionwise Hausdorff in the constructible universe) that under Gödel's axiom of constructibility (V = L) there are no metrizable (not even first countable normal) spaces of $Q$-type at all (see [23], p. 46). Therefore the following corollary holds:

Corollary 4.6. *The existence of a first countable, non-metrizable $F_{pp}$-space is consistent with and independent of the usual axioms of set theory.*

It follows from a result of M. Ismail [15] that every $F_{pp}$-space (moreover, every space which is hereditarily of point-countable type) has an open, dense, first countable subspace (see also in [10], [3]). Thus Corollary 4.5 and the result of Fleissner and Reed mentioned above imply the following corollary:

Corollary 4.7 (V = L). *Every $F_{pp}$-space has an open, dense, metrizable subspace.*

The following two problems seem to be of special interest:

Problem 1. *Is it true that every $F_{pp}$-space is the union of countably many metrizable subspaces?*

(Note that, by Theorem 1.7, the answer is "yes" for first countable $F_{pp}$-spaces.)

Problem 2. Does every $F_{pp}$-space have an open, dense, metrizable subspace?

(Note that, by Corollary 4.7, the answer is "yes" if Gödel's axiom of constructibility holds.)

Remark. Finally, we would like to point out the relationship between certain restricted versions of the normal Moore space conjecture and the metrizability of $F_{pp}$-spaces.

It was shown by R. W. Heath [12] that if there is a normal, separable, non-metrizable Moore-space, then there is a subspace of the real line of $Q$-type. Thus, Corollary 4.5 gives us the following result:

If there is a normal, separable, non-metrizable Moore space, then there is a first countable, non-metrizable $F_{pp}$-space.

On the other hand, let us consider the following construction, which is a generalization of the familiar "bubble space". Let $X$ be a metrizable space of $Q$-type, $\varrho$ a metric inducing the topology of $X$, and let us define the metric

$$d((x_1, r_1), (x_2, r_2)) = \sqrt{\varrho^2(x_1, x_2)+(r_1-r_2)^2} \quad (x_1, x_2 \in X, r_1, r_2 \in [0, \infty))$$



on the product set $X\times[0, \infty)$. Let $\tau$ denote the topology on $X\times[0, \infty)$ induced by all open sets in the product topology of $X\times[0, \infty)$ and all sets of the form

$$B(x, r) = \{(x', r')\in X\times[0, \infty):\ d((x', r'), (x, r))<r\} \cup \{(x, 0)\} \quad (x\in X, r\in(0, \infty))$$

as a subbase. By a similar argument to that needed in the case of the (classical) bubble space one can easily show that the space we get in this way is a normal, non-collectionwise Hausdorff Moore space. Then Corollary 4.5 implies the following result:

If there is a first countable, non-metrizable $F_{pp}$-space, then there is normal, non-metrizable (not even collectionwise Hausdorff) Moore space.

The author wishes to express his thanks to the referee for his helpful suggestions including a shorter proof of (b) ⇒ (a) in Theorem 2.6.

**Added in proof.** 1. H. R. Bennett and D. Lutzer (Fund. Math. 107 (1980), pp. 71–84), proved that a GO space is metrizable iff it is an $F_{pp}$-space. (Note that in the class of GO spaces hereditarily $p$-spaces are hereditarily paracompact.) This result also follows from Theorem 4.3 since neither of the two special subspaces mentioned there is a GO space.

2. Independently of the author's work, E. G. Pitkeev (Math. Zametki 28 (1980), pp. 603–618) gave a complete characterization of non-metrizable $F_{pp}$-spaces by proving that they can be "put together" from the two special subspaces mentioned in Theorem 4.3. His characterization gives affirmative answers to Problems 1 and 2.

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# The equivalence of definable quantifiers in second order arithmetic

by

**Wojciech Guzicki** (Warszawa)

**Abstract.** In this paper we generalize the notion of equivalent quantifiers considered by M. Dubiel in her paper [2] and show nonequivalent countably additive quantifiers in some model of second order arithmetic.

Let $L$ be the language of second order arithmetic $A_2$ as described in [1]. If $M$ is a model of $A_2$, then by $L_M$ we denote the language $L$ with additional constants to denote elements of $M$.

We consider a mapping which assigns to a variable $x$ and a formula $\varphi(x, x_1, \ldots, x_n)$ of $L$, with free variables $x, x_1, \ldots, x_n$, another formula $\psi(x_1, \ldots, x_n)$ of $L$, with free variables $x_1, \ldots, x_n$, which we shall denote by $Qx\varphi(x, x_1, \ldots, x_n)$.

If $M$ is a model of $A_2$, we shall say that the mapping $Q$ is a definable quantifier in $M$ iff the model $M$ satisfies the following axioms:

$$(\varphi \to \psi) \to (Qx\varphi \to Qx\psi)\,, \tag{1}$$

$$Qx(\varphi \vee \psi) \to Qx\varphi \vee Qx\psi\,, \tag{2}$$

$$Qx(x = x)\,, \tag{3}$$

$$\neg\exists y\, Qx(x = y)\,. \tag{4}$$

We call two quantifiers $Q_1$ and $Q_2$ equal in $M$ iff for any formula $\varphi(x_1, \ldots, x_n)$ of $L$ the following equivalence is satisfied in $M$:

$$\forall x_1 \ldots \forall x_n [Q_1 x\varphi(x, x_1, \ldots, x_n) \equiv Q_2 x\varphi(x, x_1, \ldots, x_n)]\,.$$

The above notion of equality of quantifiers is exactly the notion of equivalence of [2]. Our generalization closely corresponds to the following theorem, due to Krivine and Mc Aloon [4].

Definition 1. A formula $\vartheta(x)$ of the language $L_M$ is *countable-like in* $M$ (for the quantifier $Q$) iff for any formula $\varphi(x, y)$ of $L_M$

$$M \models Qy\exists x[\vartheta(x) \,\&\, \varphi(x, y)] \to \exists x\, Qy\varphi(x, y)\,.$$