Theorems on common fixed points

by

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Abstract. It is proved that if S and T are continuous mappings of a complete metric space (X, d) into itself satisfying the inequality

$$d\big((ST)^p x, (TS)^q y\big) \leqslant c.\max\big\{d\big((ST)^r, (TS)^s y\big), d\big(S(TS)^{s'} y, T(ST)^{r'} x\big), d\big((ST)^r x, T(ST)^{r'} x\big), \\ d\big(S(TS)^{s'} y, (TS)^s y\big):\ 0 \leqslant r \leqslant p;\ 0 \leqslant r' < p;\ 0 \leqslant s \leqslant q;\ 0 \leqslant s' < q\big\}$$

for all x, y in X, where $0 \leqslant c < 1$ and p, q are fixed positive integers, then S and T have a unique common fixed point z. Further, if $q = 1$, the condition that T be continuous is not necessary.

In a recent paper, see [1], the following theorem was proved

THEOREM 1. *Let T be a continuous mapping of a complete metric space (X, d) into itself satisfying the inequality*

$$d(T^p x, T^q y) \leqslant c.\max\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y): \\ 0 \leqslant r, r' \leqslant p;\ 0 \leqslant s, s' \leqslant q\}$$

for all x, y in X, where $0 \leqslant c < 1$ and p, q are fixed positive integers. Then T has a unique fixed point z.

A generalization of this theorem was given in [2] for bounded metric spaces with the following theorem

THEOREM 2. *Let S and T be continuous, commuting mappings of a complete, bounded metric space (X, d) into itself satisfying the inequality*

$$d(S^p T^{p'} x, S^q T^{q'} y) \\ \leqslant c.\max\{d(S^r T^{r'} x, S^s T^{s'} y), d(S^r T^{r'} x, S^\varrho T^{\varrho'} x), d(S^s T^{s'} y, S^\sigma T^{\sigma'} y): \\ 0 \leqslant r, \varrho \leqslant p;\ 0 \leqslant r', \varrho' \leqslant p';\ 0 \leqslant s, \sigma \leqslant q;\ 0 \leqslant s', \sigma' \leqslant q'\}$$

for all x, y in X, where $0 \leqslant c < 1$ and $p, p', q, q' \geqslant 0$ are fixed integers with $p+p'$, $q+q' \geqslant 1$. Then S and T have a unique common fixed point z. Further, if p' or $q' = 0$, then z is the unique fixed point of S and if p or $q = 0$, then z is the unique fixed pont of T.

It was also shown in [2] that the condition that S and T commute was necessary in this theorem. It is possible however that the condition that X be bounded is not necessary in this theorem.

We now prove a theorem which does not require S and T to commute or X to be bounded.

THEOREM 3. *Let S and T be continuous mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$(1)\quad d((ST)^p x, (TS)^p y) \leqslant c.\max\{d((ST)^r x, (TS)^s y), d(S(TS)^{s'} y, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x), d(S(TS)^{s'} y, (TS)^s y):\ 0\leqslant r, s\leqslant p;\ 0\leqslant r', s'<p\}$$

for all x, y in X, where $0\leqslant c<1$ and p is a fixed positive integer. Then S and T have a unique common fixed point z.

Proof. By increasing the value of c if necessary, we may assume that $\frac{1}{2}<c<1$. Inequality (1) will still hold but we will then have $c/(1-c)>1$.

Let x be an arbitrary point in X and define the points x_n inductively by

$$x_0 = x,\quad x_{2n+1} = Tx_{2n},\quad x_{2n+2} = Sx_{2n+1}$$

for $n = 0, 1, 2, \ldots$ The sequence of points $\{x_n: n = 1, 2, \ldots\}$ is bounded. For if not, the set of real numbers $\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p}): n = 1, 2, \ldots\}$ is unbounded and so there exists an integer n such that

$$(2)\quad (1-c).\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\} > c.\max\{d(x_s, x_{2p}), d(x_s, x_{2p+1}):\ 0\leqslant s\leqslant 2p\}.$$

We will suppose that this n is the smallest such n so that

$$(3)\quad \max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\} > \max\{d(x_{2r}, x_{2p+1}), d(x_{2r+1}, x_{2p}):\ 0\leqslant r<n\}$$

and since $c/(1-c)>1$ inequality (2) implies that $n>p$. It now follows from inequalities (2) and (3) that

$$\begin{aligned}(1-c).\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\} &> c.\max\{d(x_{2s+1}, x_{2p+1}), d(x_{2s}, x_{2p}):\ 0\leqslant s\leqslant p\}\\ &\geqslant c.\max\{d(x_{2s+1}, x_{2r})-d(x_{2r}, x_{2p+1}), d(x_{2s}, x_{2r+1})-d(x_{2r+1}, x_{2p}):\\ &\qquad 0\leqslant s\leqslant p;\ 0\leqslant r\leqslant n\}\\ &\geqslant c.\max\{d(x_{2s+1}, x_{2r}), d(x_{2s}, x_{2r+1}):\ 0\leqslant s\leqslant p;\ 0\leqslant r\leqslant n\}-\\ &\qquad -c.\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\}\end{aligned}$$

and so

$$(4)\quad \max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\} \geqslant c.\max\{d(x_{2s+1}, x_{2r}), d(x_{2s}, x_{2r+1}):\ 0\leqslant s\leqslant p;\ 0\leqslant r\leqslant n\}.$$

On applying inequality (1) we now have

$$\begin{aligned}\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\} &\leqslant c.\max\{d(x_{2r}, x_{2s+1}), d(x_{2r+1}, x_{2s}), d(x_{2s'+2}, x_{2r'+1}), d(x_{2s'+1}, x_{2r'+2}),\\ &\qquad d(x_{2r}, x_{2r'+1}), d(x_{2r+1}, x_{2r'+2}), d(x_{2s'+2}, x_{2s+1}), d(x_{2s'+1}, x_{2s}):\\ &\qquad 0\leqslant p+r-n, s\leqslant p;\ 0\leqslant p+r'-n, s'<p\}\\ &\leqslant c.\max\{d(x_{2r}, x_{2s+1}):\ 0\leqslant r, s\leqslant n\}\end{aligned}$$

and so

$$(5)\quad \max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\}\leqslant c^k.\max\{d(x_{2r}, x_{2s+1}):\ 0\leqslant r, s\leqslant n\}$$

when $k = 1$. Now assume that inequality (5) holds for some positive integer k. Then because of inequality (4)

$$\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\}\leqslant c^k.\max\{d(x_{2r}, x_{2s+1}):\ p\leqslant r, s\leqslant n\}.$$

After applying inequality (1) to the right hand side of this inequality it follows that

$$\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\}\leqslant c^{k+1}.\max\{d(x_{2r}, x_{2s+1}):\ 0\leqslant r, s\leqslant n\}.$$

Inequality (5) now follows by induction. However, on letting k tend to infinity in inequality (5) we have

$$\max\{d(x_{2n}, x_{2p+1}), d(x_{2n+1}, x_{2p})\} = 0,$$

contradicting the definition of n. The sequence $\{x_n: n = 1, 2, ..\}$ must therefore be bounded and so

$$\sup\{d(x_r, x_s):\ r, s = 0, 1, 2, \ldots\} = M<\infty.$$

For arbitrary $\varepsilon>0$, choose an integer N so that

$$c^N M<\varepsilon.$$

It follows that for $m, n\geqslant 2Np$ and on using inequality (1) N times

$$d(x_m, x_n)\leqslant c^N M<\varepsilon.$$

Thus the sequence $\{x_n: n = 1, 2, \ldots\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Since S and T are continuous it follows that

$$Sz = Tz = z$$

and so z is a common fixed point of S and T.

Now suppose that S and T have a second common fixed point w. Then

$$d(z, w) = d((ST)^p z, (TS)^p w)\leqslant cd(z, w)$$

on using inequality (1). Since $c<1$, $z = w$ and so z is the unique common fixed point of S and T. This completes the proof of the theorem.

COROLLARY. *Let S and T be continuous mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$(6)\quad d((ST)^p x, (TS)^q y)$$
$$\leqslant c.\max\{d((ST)^r x, (TS)^s y), d(S(TS)^{s'} y, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x),$$
$$d(S(TS)^{s'} y, (TS)^s y)\colon 0\leqslant r\leqslant p;\ 0\leqslant r'<p;\ 0\leqslant s\leqslant q;\ 0\leqslant s'<q\}$$

for all x, y in X, where $0\leqslant c<1$ and p, q are fixed positive integers. Then S and T have common fixed point z.

Proof. Suppose that $p>q$. Then

$$d((ST)^p x, (TS)^p y)$$
$$\leqslant c.\max\{d((ST)^r x, (TS)^s y), d(S(TS)^{s'} y, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x),$$
$$d(S(TS)^{s'} y, (TS)^s y)\colon 0\leqslant r\leqslant p;\ 0\leqslant r'<p;\ p-q\leqslant s\leqslant p;\ p-q\leqslant s'<p\}$$
$$\leqslant c.\max\{d((ST)^r x, (TS)^s y), d(S(TS)^{s'} y, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x),$$
$$d(S(TS)^{s'} y, (TS)^s y)\colon 0\leqslant r, s\leqslant p;\ 0\leqslant r', s'<p\}$$

for all x, y in X. The result now follows from the theorem. The same result holds if $q>p$.

We note that although the mappings S and T in Theorem 3 and its corollary have a unique common fixed point it is possible for S and T to have other fixed points. This is easily seen by letting $X = \{x, y, z\}$ with the discrete metric and defining continuous mapping S and T by

$$Sx = x, \quad Sy = Sz = z, \quad Ty = y, \quad Tx = Tz = z.$$

Then

$$STx = TSx = STy = TSy = STz = TSz = z$$

and so inequality (1) is trivially satisfied with $c = \frac{1}{2}$, but S and T each have two fixed points.

It is also necessary that both the mappings S and T be continuous in Theorem 3 if $p>1$ and in its corollary if $p, q>1$. To see this let X be the closed interval $[0, 1]$ with the usual metric. Define a continuous mapping S by

$$Sx = \tfrac{1}{2}x$$

for all x in X and a discontinuous mapping T by

$$Tx = \begin{cases} \frac{1}{2}x, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Inequalities (1) and (6) are satisfied with $c = \frac{1}{2}$, but T has no fixed point.

In the next theorem it is not necessary for the mapping T to be continuous.

THEOREM 4. *Let S be a continuous mapping and T be a mapping of a complete metric space (X, d) into itself satisfying the inequality*

$$(7)\quad d((ST)^p x, TSy)$$
$$\leqslant c.\max\{d((ST)^r x, (TS)^s y), d(Sy, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x),$$
$$d(Sy, (TS)^s y)\colon 0\leqslant r\leqslant p;\ 0\leqslant r'<p;\ s = 0, 1\}$$

for all x, y in X, where $0\leqslant c<1$ and p is a fixed positive integer. Then S and T have a unique common fixed point z.

Proof. Let x be an arbitrary point in X and let the sequence $\{x_n\colon n = 1, 2, \ldots\}$ be as defined in the proof of Theorem 3. Then since inequality (1) holds if inequality (7) holds, the sequence $\{x_n\colon n = 1, 2, \ldots\}$ is again a Cauchy sequence with a limit z in the complete metric space X. Since S is continuous, z is a fixed point of S. Further

$$d(z, Tz) = d(z, TSz)$$
$$\leqslant d(z, x_{2n}) + d(x_{2n}, TSz)$$
$$\leqslant d(z, x_{2n}) + c.\max\{d(x_{2r}, (TS)^s z), d(Sz, x_{2r'+1}), d(x_{2r}, x_{2r'+1}),$$
$$d(Sz, (TS)^s z)\colon 0\leqslant p+r-n\leqslant p;\ 0\leqslant p+r'-n<p;\ s = 0, 1\}$$

and on letting n tend to infinity we have

$$d(z, Tz)\leqslant cd(z, Tz).$$

It follows that z is a common fixed point of S and T. The uniqueness of z follows as before. This completes the proof of the theorem.

It is still necessary for S to be continuous in this theorem. To see this let X be the closed interval $[0, 1]$ with the usual metric. Define discontinuous mappings S and T on X by

$$S0 = T0 = 1,$$
$$Sx = \tfrac{1}{3}x, \quad Tx = \tfrac{1}{2}x, \quad \text{if } x \neq 0.$$

Inequality (7) is satisfied with $c = \frac{1}{2}$ but neither S nor T have a fixed point.

In the following theorem it is not necessary for either S or T to be continuous.

THEOREM 5. *Let S and T be mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$(8)\quad d(STx, Ty)\leqslant c.\max\{d(Tx, y), d(x, Ty), d(y, Ty), d(x, Tx), d(Tx, STx)\}$$

for all x, y in X, where $0\leqslant c<1$. Then S and T have a unique common fixed point z. Further z is the unique fixed point of T.

Proof. Let x be an arbitrary point in X and let the sequence $\{x_n\colon n = 1, 2, \ldots\}$ be as defined in the proof of Theorem 3. Then since inequality (1) holds if inequality (8) holds, the sequence $\{x_n: n = 1, 2, \ldots\}$ is again a Cauchy sequence with a limit z in the complete metric space X. Thus

$$\begin{aligned} d(z, Tz) &\leqslant d(z, x_{2n}) + d(x_{2n}, Tz) \\ &\leqslant d(z, x_{2n}) + c.\max\{d(x_{2n-1}, z), d(x_{2n-2}, Tz), d(z, Tz), \\ &\qquad d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n})\} \end{aligned}$$

and on letting n tend to infinity we have

$$d(z, Tz) \leqslant cd(z, Tz).$$

It follows that z is a fixed point of T and so

$$\begin{aligned} d(Sz, z) &= d(STz, Tz) \\ &\leqslant c.\max\{d(Tz, z), d(z, Tz), d(z, Tz), d(z, Tz), d(Tz, STz)\} \\ &= cd(z, Sz). \end{aligned}$$

Hence z is a common fixed point of S and T.

Now suppose that T has a second fixed point w. Then

$$d(z, w) = d(STz, Tw) \leqslant cd(z, w)$$

and it follows that z is the unique fixed point of T.

We now prove a theorem for compact metric spaces.

THEOREM 6. *Let S and T be continuous mappings of a compact metric space (X, d) into itself satisfying the inequality*

$$\begin{aligned} (9)\quad & d((ST)^p x, (TS)^q y) \\ &< \max\{d((ST)^r x, (TS)^s y), d(S(TS)^{s'} y, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x), \\ &\qquad d(S(TS)^{s'} y, (TS)^s y): 0 \leqslant r \leqslant p;\ 0 \leqslant r' < p;\ 0 \leqslant s \leqslant q;\ 0 \leqslant s' < q\} \end{aligned}$$

for all x, y in X if the right hand side of the inequality is positive and

$$d((ST)^p x, (TS)^q y) = 0$$

otherwise, where p, q are fixed positive integers. Then S and T have a unique common fixed point z.

Proof. If S and T satisfy inequality (6) for some c, with $0 \leqslant c < 1$, the result follows from the corollary to Theorem 3.

If no such c exists let $\{c_n: n = 1, 2, \ldots\}$ be a monotonically increasing sequence of real numbers converging to 1. Then there exist sequences $\{x_n: n = 1, 2, \ldots\}$ and $\{z_n: n = 1, 2, \ldots\}$ in X such that

$$\begin{aligned} & d((ST)^p x_n, (TS)^q z_n) \\ &> c_n.\max\{d((ST)^r x_n, (TS)^s z_n), d(S(TS)^{s'} z_n, T(ST)^{r'} x_n), \\ &\qquad d((ST)^r x_n, T(ST)^{r'} x_n), d(S(TS)^{s'} z_n, (TS)^s z_n): \\ &\qquad 0 \leqslant r \leqslant p;\ 0 \leqslant r' < p;\ 0 \leqslant s \leqslant q;\ 0 \leqslant s' < q\} \end{aligned}$$

for $n = 1, 2, \ldots$ Since X is compact there exist convergent subsequences $\{x_{n_k} = x'_k: k = 1, 2, \ldots\}$ and $\{z_{n_k} = z'_k: k = 1, 2, \ldots\}$ of $\{x_n: n = 1, 2, \ldots\}$ and $\{z_n: n = 1, 2, \ldots\}$ converging to x and z respectively. Letting $c_{n_k} = c'_k$ for $k = 1, 2, \ldots$ we have

$$\begin{aligned} & d((ST)^p x'_k, (TS)^q z'_k) \\ &> c'_k.\max\{d((ST)^r x'_k, (TS)^s z'_k), d(S(TS)^{s'} z'_k, T(ST)^{r'} x'_k), \\ &\qquad d((ST)^r x'_k, T(ST)^{r'} x'_k), d(S(TS)^{s'} z'_k, (TS)^s z'_k): \\ &\qquad 0 \leqslant r \leqslant p;\ 0 \leqslant r' < p;\ 0 \leqslant s \leqslant q;\ 0 \leqslant s' < q\}. \end{aligned}$$

Letting k tend to infinity we have

$$\begin{aligned} & d((ST)^p x, (TS)^q z) \\ &\geqslant \max\{d((ST)^r x, (TS)^s z), d(S(TS)^{s'} z, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x), \\ &\qquad d(S(TS)^{s'} z, (TS)^s z): 0 \leqslant r \leqslant p;\ 0 \leqslant r' < p;\ 0 \leqslant s \leqslant q;\ 0 \leqslant s' < q\} \end{aligned}$$

which implies that

$$\begin{aligned} & d((ST)^p x, (TS)^q z) = 0 \\ &= \max\{d((ST)^r x, (TS)^s z), d(TS)^{s'} z, T(ST)^{r'} x), d((ST)^r x, T(ST)^{r'} x), \\ &\qquad d(S(TS)^{s'} z, (TS)^s z): 0 \leqslant r \leqslant p;\ 0 \leqslant r' < p;\ 0 \leqslant s \leqslant q;\ 0 \leqslant s' < q\}. \end{aligned}$$

It follows that $z = x$ is a common fixed point of S and T.

Now suppose that S and T have a second distinct common fixed point w. Then

$$0 < d(z, w) = d((ST)^p z, (TS)^q w) < d(z, w)$$

on using inequality (9), giving a contradiction. The common fixed point z must therefore be unique. This completes the proof of the theorem.

References

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