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Local expansions on graphs

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Abstract. A necessary and sufficient condition is proved under which a linear graph admits a local expansion.

§ 1. Introduction. This paper is motivated by a short note of Rosenholtz [11] who studied local expansions on metric continua and proved that every open local expansion on a metric continuum onto itself has a fixed point. Showing that openess of the mapping is essential in the result, he has constructed a fixed point free local expansion on the union of three circles ([11], p. 3 and 4). On the other hand it is easy to point some particular examples of metric continua which do not admit any local expansions onto themselves at all. Such is e.g. the unit segment of reals. Therefore it is very natural to ask about a criterion under which there exists a local expansion of a given metric continuum onto itself:

PROBLEM. Characterize metric continua X which admit a local expansion of X onto itself.

This paper does not answer the problem, however, it is a contribution to the attempt to find such a criterion for some special continua. Namely a partial answer is given by showing a necessary and sufficient condition of the existence of local expansions on linear graphs (i.e. one-dimensional connected polytopes) equipped with a convex metric.

§ 2. Definitions and preliminaries. Let ϱ be a metric on a metric space X. The statement that the mapping $f\colon X\to X$ is a local expansion means that f is continuous and that for each point $x\in X$, there is an open set U containing x and a real number $M>1$ so that if y and z belong to U, then

(1) $$\varrho(f(y),f(z))\geqslant M\varrho(y,z)$$

(see [11], p. 1). We say that a *metric space X admits a local expansion* if there exist a metric ϱ that is equivalent to the original one given on X, and a surjection $f\colon X\to X$ satisfying the conditions of the above definition.

Let a metric space X with a metric ϱ be given. Let $x, y, z\in X$. The point z is said to *lie between the points x and y* provided that $\varrho(x,y)=\varrho(x,z)+\varrho(z,y)$ (cf. [3], p. 317). The point z is said to be a *center of the pair x, y* provided that $\varrho(x,z)$

$= \varrho(y, z) = \frac{1}{2}\varrho(x, y)$. We say that the set $A \subset X$ is *linear* if there exists an isometry $\varphi\colon A \to R$ of A into the space R of all real numbers, i.e. $\varrho(x, y) = |\varphi(x) - \varphi(y)|$ for every $x, y \in A$ (cf. [6], p. 183). An arc means any subset of X that is homeomorphic to the closed unit segment $[0, 1]$ of reals. An arc contained in X is said to be a *metric segment* if it is linear.

A metric space X is said to be *convex* (in the well-known sense of Menger [7], p. 81) provided that for each two distinct points x and y of X there exists a point $z \in X$ different from x and y which lies between x and y. It was proved by Menger ([7], p. 89; see also Aronszajn [2]; cf. [3], p. 41) that in every complete convex metric space X each two points of X can be joined by a metric segment. Moreover, it is known (cf. [9], 2.3, p. 116) that a complete metric space is convex if and only if for every $x, y \in X$ and for every t, where $0 \leqslant t \leqslant 1$, there exists at least one point $z \in X$ such that

$$\varrho(x, z) = (1-t)\cdot\varrho(x, y) \quad \text{and} \quad \varrho(z, y) = t\cdot\varrho(x, y)\,.$$

If a metric space X equipped with a metric ϱ is convex, then the metric ϱ is called a *convex metric*.

Let X be a metric space and let n be a positive integer. A point $p \in X$ is said to be of *order less than or equal to n* provided that if W is any open neighborhood of p, there exists an open neighborhood U of p with $U \subset W$ and such that the boundary $\overline{U} \setminus U$ of U consists of at most n points. Clearly this amounts to saying that for every positive number ε there exists a neighborhood U of p of diameter less than ε whose boundary has at most n points. If p is of order less than or equal to n but not of order less than or equal to $n-1$, then it is said to be of order n (cf. [12], Definition on p. 48 and Note on p. 50), in writing $\operatorname{ord}_p X = n$. In particular, if $\operatorname{ord}_p X = 1$, then p is called an *end point of X*. We admit

$$\operatorname{ord} X = \max\{\operatorname{ord}_p X \mid p \in X\} \tag{2}$$

if such the maximum exists.

A connected set X is said to be *semi-locally connected* if for each its point x and for each number $\varepsilon > 0$ there exists a neighborhood U of x in X of diameter less than ε such that $X \setminus U$ has only a finite number of components ([12], p. 19).

If X is a connected space and p is a point of X such that $X \setminus \{p\}$ is not connected, then p is called a *cut point* of X ([12], p. 41). A continuum means a compact connected metric space. A simple closed curve is defined as a continuum homeomorphic to the unit circumference, i.e., to the set of all complex numbers z with $|z| = 1$.

We shall use here some notions from Whyburn's cyclic element theory (see e.g. [12], Chapter IV, p. 64–87). For the reader's convenience we recall here the basic concepts needed in the sequel.

Two points a and b of a connected set X are said to be *conjugate* provided no point separates a and b in X. If $p \in X$ is neither a cut point nor an end point of a connected set X, the set consisting of p and of all points of X conjugate to p is called a *simple link of X* (see [12], p. 64). Simple links are closed subsets of X ([12], 1.4, p. 65). In case when X is a semi-locally connected continuum, simple links coincide with non-degenerate or true cyclic elements of X (see [12], p. 66). A collection Γ of sets is said to be *coherent* if for each proper subcollection Γ' of it, an element of Γ' intersects an element of $\Gamma \setminus \Gamma'$ (see [8], p. 46). Let a semi-locally connected continuum X be given. A link of X means the union of a maximal coherent family of simple links of X (here "maximal" means that the coherent family is not a proper subfamily of another coherent family of simple links of X); compare this definition with one of an H-set in [12], p. 72. A link L of a semi-locally connected continuum X will be called an *end link* if the set $X \setminus L$ is connected.

We recall that a connected set X is a *linear graph* provided it is the union of a finite set V of points, called vertices and a finite number of arcs, called edges, so that the end points of each edge belong to V and that for every two edges their intersection is contained in V (i.e. every two edges have at most end points in common). If $v \in V$ and E is an edge such that $v \in E$, then we say that E is the *edge incident to the vertex v*. Note that the above definition of a linear graph is equivalent to one given in [12], p. 182; however the edges in [12] are defined as open free arcs in X (i.e. arcs without their end points, with the property that each such an arc without its end points is an open subset of the whole space X). But it is more convenient to us to understand each edge of a linear graph as an arc, i.e., together with its end points. It is known (see [12], p. 182) that a continuum X is a linear graph if and only if every point of X is of some finite order and almost all points of X are of order less than or equal to 2. Thus it follows that every linear graph is a regular curve, whence we conclude by Whyburn's classification of curves ([12], p. 99 and Corollary (13.21), p. 20) that every linear graph is a semi-locally connected continuum. Note further that in the graph theory the order of a point p is sometimes called the degree or the valency of p (see [1], p. 16 and [10], p. 13).

A linear graph with a fixed set of vertices is said to be *simple* if there is at most one edge joining the same pair of vertices (cf. [1], p. 20).

Remark. Let us observe that if a linear graph is given with a convex metric and with a set of its vertices, then we can enlarge the set of vertices taking as new vertices the centers of the pair of end points for every edge, to obtain the same linear graph X (with the same convex metric) which is a simple one. Thus we can assume without loss of generality that the considered linear graphs with convex metric are simple.

An edge of a linear graph is called a *bridge* if the removal of its interior disconnects the graph (cf. [1], p. 67). In other words, an edge ab of a linear graph X is a bridge provided the set $X \setminus (ab \setminus \{a, b\})$ is not connected. In particular, if an edge E is incident to an end point of a linear graph, then E is a bridge. Let us observe that every point of a link L of a linear graph X lies in some simple closed curve contained in L. Thus every link of a linear graph is the union of a finite number of simple closed curves, and hence it can be defined as a maximal subgraph of X containing no bridge (thus no end point) of X. A linear graph containing no simple closed curve is called a tree. A (finite) sequence $C = \{E_1, E_2, \ldots, E_k\}$ of edges of a linear graph is said to be a *chain* provided $E_i \cap E_{i+1}$ is a one-point set composed of the common end-vertex of E_i and E_{i+1} for $i = 1, 2, \ldots, k-1$. If x and y are vertices such that $x \in E_1 \setminus E_2$ and $y \in E_k \setminus E_{k-1}$, then we say that the chain C joins x and y, or that C is a chain from x to y. Given a finite number r of chains $C_j = \{E_1^j, E_2^j, \ldots, E_{k_j}^j\}$ from x_j to y_j respectively, with $y_j = x_{j+1}$ for each $j = 1, 2, \ldots, r-1$, we can join them together in one chain C called the *join* of chains C_j which is defined as a sequence composed of all terms of C_j ordered linearly with respect to j:

(3) $$C = \{E_1^1, E_2^1, \ldots, E_{k_1}^1, E_1^2, E_2^2, \ldots, E_{k_2}^2, \ldots, E_{k_r}^r\}.$$

If end vertices of an edge are ordered (i.e. if one of them is distinguished as the beginning and the other as the ending of the edge), then the edge is said to be *directed* (or *oriented*). A linear graph is called *directed* if all its edges are directed (cf. [1], p. 64). By a directed path we mean a chain whose edges are oriented in such a manner that the ending of the edge E_i coincides with the beginning of E_{i+1}, where $i = 1, 2, \ldots, k-1$. Given a directed path p, i.e., a sequence of directed edges $E_1, E_2, \ldots, E_k$ with the property mentioned above, we define a sequence $\sigma(P)$ of consecutive vertices ordered along P, that is

(4) $$\sigma(P) = \{x_1, x_2, \ldots, x_{k+1}\},$$

where x_i is the beginning, and x_{i+1} is the ending of E_i, for every $i = 1, 2, \ldots, k$.

If the edges of a directed linear graph are oriented so that each vertex is accessible from each of the other vertices along a directed path, then the graph is called *strongly connected* ([1], p. 67).

The following lemma is known (see [1], Problem 18, p. 69):

LEMMA 1. *Directions can be assigned to the edges of an arbitrary linear graph without bridges to make it strongly connected.*

We shall use this lemma to prove the next one.

LEMMA 2. *If a simple linear graph X contains no end point, and if x and y are vertices of X (not necessarily distinct), then there is a chain from x to y containing all the edges of X.*

Proof. The proof runs by induction with respect to the number n of links contained in X. To show the conclusion in case $n = 1$, let us assign directions to the edges of X according to Lemma 1 and take an arbitrary finite sequence of all vertices of X such that x is the first and y is the last term of the sequence:

$$x = v_1, v_2, \ldots, v_{2j-1}, v_{2j}, \ldots, v_{2m} = y,$$

and, for each $j = 1, 2, \ldots, m$ the pair of consecutive vertices (v_{2j-1}, v_{2j}) is composed of the beginning and the ending of the directed edge E_j (here m denotes the number of all edges in X). Now it is enough to consider a sequence of $2m-1$ directed paths from v_i to v_{i+1} (where $i = 1, 2, \ldots, 2m-1$) such that

(5) the directed path from v_{2i-1} to v_{2i} is a one-term sequence, necessarily composed of the directed edge E_i.

This condition (5) guarantees that each edge of X does occur indeed in the chain to be defined. We arrange all edges of X in the sequence

(6) $$E_1, E_2, \ldots, E_s, E_{s+1}, \ldots, E_{s+j_i}, \ldots, E_t$$

relabelling them in such a way that 1) E_1 is the only (i.e. the first and the last) edge in the directed path from $x = v_1$ to v_2; 2) if E_s is determined as the last edge of the directed path from v_{i-1} to v_i, then $E_{s+1}, E_{s+2}, \ldots, E_{s+j_i}$ are sequential edges in the directed path from v_i to v_{i+1} (obviously, according to (5), we have $j_i = 1$ if index i is odd). Therefore the intersection of any two sequential edges in this sequence is not empty (it contains their common vertex, say v), and, moreover, since the first edge of these two is directed to v (i.e., v is the ending vertex of it) and the second is directed from v (i.e., it has v as the beginning), the vertex v must be the only point of the intersection. Thus sequence (6) is a chain, and therefore the proof is finished for the case $n = 1$.

Now let us assume that the conclusion of the lemma holds for every simple linear graph which contains at most n links, and consider a simple linear graph X without end points that contains $n+1$ links. Let L be an end link of X (the existence of such a link follows from [12], Theorem 8.1, p. 77 and Theorem 4.23, p. 129). Thus X can be represented in a form $X = Y \cup B \cup L$, where Y is a subgraph of X which contains n links and B denotes the union of all bridges of X between Y and L. Obviously B is an arc. Let u and w denote its end points, with $u \in Y$ and $w \in L$. Therefore by the induction hypothesis there are chains from a point of Y to any other point of Y containing all edges of Y. The same holds for the link L. To construct the chain C from x to y that contains all the edges of X, let us consider some particular cases. If both x and y are in Y, we take a chain C_1 from x to u composed of all edges in Y, a chain C_2 from u to w composed of all edges contained in B, a chain C_3 from w to w whose terms are all edges of L, a chain C_4 from w to u, again of all edges in B, and finally a chain C_5 from u to y. The chain C is defined just as the join of these five chains C_1, C_2, C_3, C_4 and C_5 in the sense of (3). If the points x and y both are in L we proceed analogously. If x is in Y and y is in L, we need only three chains: C_1 and C_2 as above, and C_3 from w to y composed of all edges of L. Then the join C of C_1, C_2 and C_3 is a chain from x to y containing all the edges of X. If x is in B and y is in L, then the chain C is defined as the join of the following chains: C_1 from x to u, with edges contained in B; C_2 from u to u covering the whole Y; C_3 from u to w covering B, and C_4 from w to y composed of all edges of L. The reader can easily construct the proper chains C for the rest cases, which are similar to ones considered above. The proof of the lemma is thereby complete.

Given a directed path P from a vertex a to a vertex b in a linear graph X, we denote by P^{-1} the directed path from b to a oriented in the opposite manner as P is, i.e., composed of the same edges as P but taken in the inverse order and with the inverse direction. Given an arc A joining in X a vertex a with a vertex b, we denote by $P(A; a, b)$ the directed path from a to b such that the union of all its edges is just the arc A. Similarly, given a simple closed curve S in X and a point $p \in S$, we denote by $P(S; p)$ one of the two possible directed paths from p to p such that the union of all its edges is equal to S.

We shall need also the following

LEMMA 3. *Let a simple linear graph X be given which is not a simple closed curve. Let $c \in X$ be a point of the maximal order in X, i.e., $\mathrm{ord}_c X = \mathrm{ord} X$, such that for every component of $X \setminus \{c\}$ its closure contains a simple closed curve. Then for any two edges ca and cb (not necessarily different) there exists a directed path P from c to c in X such that ca (directed from c to a) is the first, and bc (directed from b to c) is the last edge of P. Moreover, the number of edges of P can be arbitrarily large.*

Proof. Let ν be an arbitrary natural number. First, assume that a and b both lie in the same component K of $X \setminus \{c\}$. We consider the following two cases: (i) $a \neq b$, and (ii) $a = b$.

In case (i) there exists a nondegenerate arc $ab \subset K$. Denote by P_0 the join of $P(ca; c, a)$, $P(ab; a, b)$ and $P(bc; b, c)$, and define P as the join of the directed paths $P_1 P_2, \ldots, P_\nu$, where $P_i = P_0$ for $i = 1, 2, \ldots, \nu$.

In case (ii) we discuss the three possibilities:

1^0 $\mathrm{ord}_c \overline{K} = 1$; 2^0 $\mathrm{ord}_c \overline{K} = 2$; 3^0 $\mathrm{ord}_c \overline{K} \geqslant 3$.

If 1^0 holds, take a simple closed curve $S \subset \overline{K}$ and an arc cd such that $ca \subset cd$ and $cd \cap S = \{d\}$. Put $P_0 = P(cd; c, d)$ and $P_i = P(S; d)$ for $i = 1, 2, \ldots, \nu$, and define P as the join of $P_0, P_1, P_2, \ldots, P_\nu, P_0^{-1}$.

If 2^0 holds, let ca and cx be the two different edges in $\overline{K}$ incident to the vertex c. Then $a \neq x$ and, as in case (i) above, there is a directed path P_0 from c to c having ca as its first and xc as its last edges. Since X is not a simple closed curve, we have $\mathrm{ord}_c X > 2$, and by $\mathrm{ord}_c \overline{K} = 2$ we see that there is another component, say K', of $X \setminus \{c\}$, and by hypothesis there is a simple closed curve S' in K'. Let cd' be an arc in K' joining c with S', i.e., such that $cd' \cap S' = \{d'\}$ (this arc can be degenerate if $c \in S'$). Let P' be the join of $P(cd'; c, d')$, $P(S'; d')$ and $P(cd'; d', c)$. Define P as the join of the directed paths $P_1, P_2, \ldots, P_\nu, P', P_0^{-1}$, where $P_i = P_0$ for $i = 1, 2, \ldots, \nu$.

If 3^0 holds, then there are three different edges ca, cx and cy in $\overline{K}$. Thus the vertices a, x and y are distinct and, as in case (i) above, there are a directed path P_0 from c to c having ca (directed from c to a) as its first and xc (directed from x to c) as its last edge, and a directed path P' from c to c having cy (directed from c to y) and ac (directed from a to c) as its first and last edges correspondingly. Putting $P_i = P_0$ for $i = 1, 2, \ldots, \nu$ we define P as the join of $P_1, P_2, \ldots, P_\nu, P'$.

Second, assume that a and b lie in two different components K and K' of $X \setminus \{c\}$. Using the same argumentation as previously one can find two directed paths P_0 and P', both from c to c, such that ca (oriented from c to a) is the first edge of P_0 and all edges of P_0 are contained in $\overline{K}$, and that bc (oriented from b to c) is the last edge of P', and all edges of P' are in $\overline{K}'$. Let P_i for $i = 1, 2, \ldots, \nu$ be the join of P_0 and P'. Then we define P as the join of $P_1, P_2, \ldots, P_\nu$. The proof of the lemma is finished.

§ 3. **Standard mappings.** It is known that every linear graph X can be remetrized by a convex metric: it was shown by Borsuk [4], Section 6 and 7, p. 329–332, that for every polytope P of dimension less than or equal to 2 there exists a metric ϱ such that (P, ϱ) is a convex (and locally strongly convex) metric space.

Let B_1 and B_2 denote two closed intervals of reals. A surjection $g\colon B_1 \to B_2$ is called *affine* if there are reals $\alpha \neq 0$ and β such that $g(x) = \alpha x + \beta$ for every $x \in B_1$. For $k = 1$ and 2 let X_k be a convex metric space and let A_k be a metric segment contained in X_k. Thus there is an isometry $\varphi_k\colon A_k \to B_k$, where B_k is a closed interval of reals. We say that a surjection $f\colon A_1 \to A_2$ is *linear* if there exists an affine surjection $g\colon B_1 \to B_2$ such that $f = \varphi_2^{-1} g \varphi_1$. We say that a mapping $f\colon X_1 \to X_2$ of X_1 into X_2 is piecewise linear if there is a partition of the domain X_1 into a finite number n of metric segments A_1^j, i.e., $X_1 = \bigcup \{A_1^j \mid j = 1, 2, \ldots, n\}$ such that $f(A_1^j)$ is a metric segment in X_2 and that the restrictions $f|A_1^j$ are linear for every $j = 1, 2, \ldots, n$. Obviously each piecewise linear mapping is continuous.

Let Y be a complete convex metric space with a metric ϱ_2. Consider a finite sequence of $n+1$ not necessarily distinct points

$$b_0, b_1, \ldots, b_n \tag{7}$$

in Y, and for every $j = 0, 1, \ldots, n-1$ take an arc $b_j b_{j+1}$ which is a metric segment. Further, let a metric segment pq be given in a complete convex metric space X with a metric ϱ_1, and put

$$\mu = \Big[\sum_{j=0}^{n-1} \varrho_2(b_j, b_{j+1})\Big] / \varrho_1(p, q). \tag{8}$$

We define $n+1$ distinct points $p = a_0, a_1, \ldots, a_n = q$ in the metric segment pq putting for every $k = 1, 2, \ldots, n$

$$\mu \cdot \varrho_1(p, a_k) = \sum_{j=0}^{k-1} \varrho_2(b_j, b_{j+1}). \tag{9}$$

Thus we have $\mu \varrho_1(a_j, a_{j+1}) = \varrho_2(b_j, b_{j+1})$ for every $j = 0, 1, \ldots, n-1$. The mapping

$$f\colon pq \to \bigcup \{b_j b_{j+1} \mid j = 0, 1, \ldots, n-1\} \tag{10}$$

defined in such a way that the partial mapping

$$f|a_j a_{j+1}\colon a_j a_{j+1} \to b_j b_{j+1} \tag{11}$$

is linear, with

$$f(a_j) = b_j \quad \text{for every } j = 0, 1, \ldots, n-1, \tag{12}$$

will be called the *standard mapping* associated with sequence (7). In particular we see that for the standard mapping f just defined we have

$$f(p) = b_0 \quad \text{and} \quad f(q) = b_n. \tag{13}$$

The number μ defined by (8) will be called the *coefficient of the standard mapping* f, and will be denoted by $\tau(f)$.

It is manifestly evident that the standard mapping f is piecewise linear, and thereby continuous. Further, it follows from the linearity of the partial mapping (11) that, for every $j = 0, 1, \ldots, n-1$,

(14) if $y, z \in a_j a_{j+1}$, then $\varrho_2(f(y), f(z)) = \mu\varrho_1(y, z)$.

Hence we have

STATEMENT 1. *Let X and Y be simple linear graphs with convex metrics ϱ_1 and ϱ_2 respectively and let an edge pq be given in X. Let $f\colon pq \to Y$ be a standard mapping associated with sequence (7) of distinct points b_j such that for every $j = 1, 2, \ldots, n-1$ the point b_j lies between b_{j-1} and b_{j+1}, and let $\mu = \tau(f)$. Then for every point $x \in pq \setminus \{p, q\}$ there exists a neighborhood U of x such that if $y, z \in U$, then*

$$\varrho_2(f(y), f(z)) = \mu\varrho_1(y, z). \tag{15}$$

Indeed, if x is an interior point of a segment $a_j a_{j+1}$ for some $j = 0, 1, \ldots, n-1$, where the points a_j are defined by equality (9), then taking $U = a_j a_{j+1} \setminus \{a_j, a_{j+1}\}$ we see that U is a neighborhood of x and that the conclusion follows from (14). If $x = a_j$ for some $j = 1, 2, \ldots, n-1$, then take $U = a_{j-1}a_{j+1} \setminus \{a_{j-1}, a_{j+1}\}$. If both y and z are in $a_{j-1}a_j$ or in $a_j a_{j+1}$, the argumentation for (15) is exactly the same as previously. If $y \in a_{j-1}a_j$ and $z \in a_j a_{j+1}$, then $f(y) \in b_{j-1}b_j$ and $f(z) \in b_j b_{j+1}$. Since b_j lies between b_{j-1} and b_{j+1} and since $b_{j-1} \neq b_{j+1}$ by assumption, we conclude from [9], 2.2, p. 116 that $b_{j-1}b_j \cup b_j b_{j+1}$ is a metric segment. Hence we have by (12) and (14) that

$$\begin{aligned}\varrho_2(f(y), f(z)) &= \varrho_2(f(y), b_j) + \varrho_2(b_j, f(z)) = \varrho_2(f(y), f(a_j)) + \varrho_2(f(a_j), f(z)) \\ &= \mu\varrho_1(y, a_j) + \mu\varrho_1(a_j, z) = \mu\varrho_1(y, z),\end{aligned}$$

and (15) follows.

STATEMENT 2. *Let X and Y be simple linear graphs with convex metrics ϱ_1 and ϱ_2 respectively. Given a vertex $p \in X$ of a finite order n, let $pq_1, pq_2, \ldots, pq_n$ be edges in X incident to the point p. For every $i = 1, 2, \ldots, n$ let $f_i\colon pq_i \to Y$ be a standard mapping with the coefficient $\tau(f_i) = \mu_i > 1$ and such that there exists a point $r \in Y$ with $f_i(p) = r$. Assume further that for every $i = 1, 2, \ldots, n$ there is an edge rr_i in Y such that $rr_i \subset f_i(pq_i)$ and that rr_i are distinct for distinct indices i. Let a mapping $f\colon \bigcup\{pq_i \mid i = 1, 2, \ldots, n\} \to Y$ be defined by*

$$f | pq_i = f_i. \tag{16}$$

Then there is an open neighborhood U of p and a real number $M > 1$ such that if y and z are in U, then

$$\varrho_2(f(y), f(z)) \geqslant M \cdot \varrho_1(y, z). \tag{17}$$

Proof. For $i = 1, 2, \ldots, n$ let a_i be the nearest to p point of the edge pq_i such that $f_i(a_i) = r_i$. Let U be the component of the open set $X \setminus \{a_i \mid i = 1, 2, \ldots, n\}$ which contains the point p. In other words, we put $U = \bigcup\{pa_i \setminus \{a_i\} \mid i = 1, 2, \ldots, n\}$. Hence U is a neighborhood of p in X. Further put $M = \min\{\mu_i \mid i = 1, 2, \ldots, n\}$. Since $\mu_i > 1$ for every i, we have $M > 1$. Take two arbitrary points y and z of U and consider two cases. Firstly, let y and z be in the same arc pa_i for some $i = 1, 2, \ldots, n$. Then by (14) we have that

$$\varrho_2(f(y), f(z)) = \varrho_2(f_i(y), f_i(z)) = \mu_i\varrho_1(y, z) \geqslant M\varrho_1(y, z),$$

and (17) follows. Secondly, let $y \in pa_i$ and $z \in pa_j$, where $i \neq j$. Since rr_i and rr_j are distinct metric segments by hypothesis, we see that 2.2 of [9], p. 116 can be applied, and thus $rr_i \cup rr_j$ is a metric segment. It follows from assumptions that $f(y) = f_i(y) \in f_i(pa_i) = rr_i$ and $f(z) = f_j(z) \in f_j(pa_j) = rr_j$, and since $f(p) = r$, we have by (14) and (16) that

$$\begin{aligned}\varrho_2(f(y), f(z)) &= \varrho_2(f(y), r) + \varrho_2(r, f(z)) \\ &= \varrho_2(f(y), f(p)) + \varrho_2(f(p), f(z)) \\ &= \varrho_2(f_i(y), f_i(p)) + \varrho_2(f_j(p), f_j(z)) \\ &= \mu_i\varrho_1(y, p) + \mu_j\varrho_1(p, z) \\ &\geqslant M(\varrho_1(y, p) + \varrho_1(p, z)) = M\varrho_1(y, z),\end{aligned}$$

and thereby (17) follows in this case, too. Thus the proof is complete.

§ 4. Existence of local expansions-sufficiency. Now let a linear graph X endowed with a convex metric ϱ be given. We can assume without loss of generality that (i) every edge of X is a metric segment of X, and that (ii) every such edge is uniquely determined by two its end points (i.e. that there is at most one edge between the two vertices). Indeed, both (i) and (ii) can be realized simply by enlarging of the number of vertices in a proper way, e.g. by taking as new vertices the centers of the pair of the end points of every edge. In other words condition (ii) says that the linear graph X is simple (compare the remark in § 2). Since now, in the present paragraph, X always will mean a convex metric space being a simple linear graph that satisfies condition (i).

Now we are going to prove some sufficient conditions for the existence of a local expansion on such an X. The existence depends heavily on the structure of the linear graph. Namely if the graph contains a point of the maximal order which does not disconnect the graph, then a local expansion does exist. If every point of the maximal order in the graph disconnects the graph, but if there is one such that the closure of every component of its complement contains a simple closed curve, then also the graph admits a local expansion.

Invertedly, it will be shown in sixth paragraph that if there exists a local expansion on a linear graph metrized by a convex metric, then either there is a point of the maximal order which does not disconnect the graph, or there is one which disconnects the graph in such a way that the closure of no component of its complementary is a tree. Therefore a characterization of linear graphs that admit local expansions will be obtained.

THEOREM 1. *Let a simple linear graph X metrized by a convex metric be given. If there is a point $c \in X$ of the maximal order in X, i.e.,*

$$\operatorname{ord}_c X = \operatorname{ord} X \tag{18}$$

such that for every component of $X \setminus \{c\}$ its closure contains a simple closed curve, then there exists a local expansion $f\colon X \to X$ of X onto itself.

Proof ([1]). Since the unit circle $\{z\colon |z| = 1\}$ admits a local expansion, e.g. $z \to z^2$, we may assume that X is not a simple closed curve. To describe the local expansion f mentioned in the conclusion of the theorem we distinguish some subsets of the graph X. Every of them will be the union of some edges of X.

The set X_1 is defined as the union of all end edges of X: an edge E of X is contained in X_1 if and only if there is an end point of X which belongs to E. Note that we can assume without loss of generality that

(19) all edges of X contained in X_1 are disjoint.

In fact, take two edges $p_1 q_1$ and $p_2 q_2$ contained in X_1, where p_1 and p_2 are end points of X. If $q_1 = q_2$, i.e., if these edges have a common vertex q, then we take centers r_1 and r_2 of the pairs (p_1, q) and (p_2, q) respectively as some new vertices of X. Such an improvement does not violate conditions (i) and (ii) assumed on X, and it let us to consider only the new edges $p_1 r_1$ and $p_2 r_2$ as contained in X_1. Obviously they are disjoint, and therefore we may assume that the set X_1 satisfies (19) indeed.

Observe that if the graph X has no end point, then the set X_1 is empty by definition.

Further, define X_2 as the union of all edges E of X such that E is contained either in some link of X or in some arc whose end points are in two different links of X. Thus X_2 is a subgraph of X which contains no end point of itself. Note that X_2 can never be empty, because X contains a simple closed curve by assumption, and every edge of X which lies in a simple closed curve is contained in X_2. Apart from this we have $c \in X_2$. Finally put $X_3 = \overline{X \setminus (X_1 \cup X_2)}$. Thus

$$X = X_1 \cup X_2 \cup X_3 . \tag{20}$$

Now we are going to define $f\colon X \to X$. To describe $f|X_1\colon X_1 \to X$ we define it separately on each end edge of X the set X_1 is composed of. Let pq be such an edge, where p denotes an end point of X. We denote by T the closure of the component containing the point p of the set $X \setminus X_2$, and by b the only point of the intersection $T \cap X_2$. Let pc be an arbitrary but fixed arc from p to c. Obviously we have

([1]) The authors thank Krzysztof Omiljanowski for valuable suggestions concerning a part of this proof.

$pq \subset pb \subset pc$ and $pb \cap X_2 = \{b\}$. Observe that it can happen $q = b$ or $b = c$ but never $q = c$ because otherwise we would have a component $pc \setminus \{c\}$ of $X \setminus \{c\}$ such that its closure $pc = pq$ would contain no simple closed curve, contrary to the hypothesis. Thus

(21) pq is a proper subset of pc.

Take the sequence of all vertices of X lying in the arc pc and ordered from p to c:

$$p, q, \ldots, b, \ldots, c, \tag{22}$$

and define

$$f|pq\colon pq \to pc \tag{23}$$

as the standard mapping associated with sequence (22). It follows from (13) by the definition of a standard mapping that

$$f(p) = p \quad \text{and} \quad f(q) = c . \tag{24}$$

Further, since $f|pq$ maps pq onto pc by its definition, we conclude from (21) that

$$\tau(f|pq) > 1 \quad \text{for every edge } pq \subset X_1 . \tag{25}$$

Since T is a tree by definition, with p and b as its end points, hence each point of T lies in some arc $p'b$ for some end point $p' \neq b$ of T, whence we conclude

$$T \subset f(X_1 \cap T) . \tag{26}$$

Taking the union over all end points p of X we infer from (26) that

$$X_1 \cup X_3 \subset f(X_1) . \tag{27}$$

To describe the mapping $f|X_2 \cup X_3$ we need some preliminary constructions. First, we arrange all vertices of X which are not end points of X in a (finite) sequence

$$c = v_1, v_2, \ldots, v_i, \ldots, v_m \tag{28}$$

such that $i_1 \neq i_2$ implies $v_{i_1} \neq v_{i_2}$. Second, we set up all edges of X contained in X_2 or in X_3 in a sequence

$$E_1, E_2, \ldots, E_j, \ldots, E_n \tag{29}$$

such that $j_1 \neq j_2$ implies $E_{j_1} \neq E_{j_2}$ and that E_1 is an edge incident to the vertex $c = v_1$. Now we are going to assign to each edge E_j of (29) a directed path P_j from c to c in X_2. This is done by the finite induction. Then we define $f|E_j\colon E_j \to X$ as the standard mapping associated with the sequence $\sigma(P_j)$. This is realized in such a way that not only the auxiliary partial mappings $f|E_j$ are local expansions, but also the resulting mapping $f\colon X \to X$ is. For this purpose we use Statements 1 and 2. To guarantee that all assumptions of Statement 2 are fulfilled, we define, again by the finite induction with respect to j and simultaneously with P_j, a sequence of families M_i^j of some edges incident to the vertex c (here $i = 1, \ldots, m$ and $j = 0, 1, 2, \ldots, n$), and we choose the directed paths in such a way that either the first or the last edge of the path just being defined is taken from out of the corresponding family M_i^j in a proper manner.

To begin with, for every $i = 1, 2, \ldots, m$ we define M_i^0 as follows. If a vertex v_i from sequence (28) belongs to an end edge pv_i (i.e. if $v_i \in X_1 \cap (X_2 \cup X_3)$) we put $M_i^0 = \{cx\}$, where x is the last point in (22) that is different from c. Otherwise we put $M_i^0 = \emptyset$. Now recall that X_2 is a subgraph of X which contains no end point of itself, whence by Lemma 2 there is a chain C from c to c containing all edges of X_2. Choose an orientation on C to get a fixed directed path P from c to c. Consider the first edge E_1 in (29) and let $E_1 = cv_r$. If the last edge of P is not a member of M_r^0, then we define $P_1 = P$. In the opposite case we take an arbitrary edge cy incident to c and distinct from E_1, we construct a directed path P' having cy (directed from c to y) as the first edge and yc (directed from y to c) as the last one (such a path does exist in X_2 by Lemma 3), and we define P_1 as the join of P and P'. Next we define $f|E_1 = f|v_1v_r\colon v_1v_r \to X$ as the standard mapping associated with the sequence $\sigma(P_1)$. Since the directed path P runs over all edges of X_2 by construction, the same holds for P_1, and it follows from the definition of $f|E_1$ that

$$f(E_1) = X_2\,, \tag{30}$$

and since E_1 is a proper subset of X_2, we conclude from (30) that

$$\tau(f|E_1) > 1\,. \tag{31}$$

Further, if A and B denote the first and the last edge of the directed path P_1 respectively, we put

$$M_1^1 = M_1^0 \cup \{A\}, \quad M_r^1 = M_1^0 \cup \{B\} \quad \text{and} \quad M_i^1 = M_i^0 \tag{32}$$

for every $i \in \{1, 2, \ldots, m\} \setminus \{1, r\}$.

Now let us fix some $k \in \{1, 2, \ldots, n-1\}$ and assume that for all $j \in \{1, 2, \ldots, k\}$ there are defined directed paths P_j from c to c and standard mappings $f|E_j\colon E_j \to X$ associated with the sequences $\sigma(P_j)$ and such that

$$\tau(f|E_j) > 1\,. \tag{33}$$

Furthermore, we assume that, for every pair of standard mappings defined on two edges incident to the same vertex, the assumptions of Statement 2 are satisfied, i.e., the following assertion is assumed:

(34) For every vertex v_i, where $i = 1, 2, \ldots, m$, if two distinct edges E_{j_1} and E_{j_2} with $1 \leqslant j_1 < j_2 \leqslant k$ are both incident to v_i and considered as directed from v_i (it means that the vertex v_i is the common beginning of E_{j_1} and E_{j_2}), then the first directed edges of directed paths P_{j_1} and P_{j_2} are distinct, too. Further, if pv_1 is an edge contained in X_1, then the first directed edge of P_{j_1} and the element of M_1^0 also are distinct.

Moreover, we assume that

(35) for every $i = 1, 2, \ldots, m$ the family M_i^k consists of all edges incident to c which are either the first edges of a directed path P_j for some $j \leqslant k$ if E_j is considered as directed from v_i or the last edges of P_j, if the corresponding edge E_j is considered as directed to v_i, or — finally — belong to M_i^0.

Now we are ready to go to the next step, i.e. to define P_{k+1}. Suppose $E_{k+1} = v_sv_t$ where v_s and v_t are some vertices taken from sequence (28). Note that, by the induction hypotheses (34) and (35), we have $\operatorname{card} M_s^k \leqslant \operatorname{ord}_{v_s} X - 1 < \operatorname{ord}_c X = \operatorname{ord} X$, and similarly $\operatorname{card} M_t^k < \operatorname{ord}_c X$. Therefore we can find an edge ca which is not in M_s^k and an edge cb not in M_t^k. By Lemma 3 there is a directed path P_{k+1} from c to c such that ca (directed from c to a) is the first, and bc (directed from b to c) is the last edge of P_{k+1}, and that the number of edges of this path can be arbitrarily large. This condition guarantees that for the mapping $f|E_{k+1} = f|v_sv_t\colon v_sv_t \to X$ defined as the standard mapping associated with the sequence $\sigma(P_{k+1})$ we have

$$\tau(f|E_{k+1}) > 1\,. \tag{36}$$

Finally, put

$$M_s^{k+1} = M_s^k \cup \{ca\}, \quad M_t^{k+1} = M_t^k \cup \{cb\} \quad \text{and} \quad M_i^{k+1} = M_i^k \tag{37}$$
for $i = \{1, 2, \ldots, m\} \setminus \{s, t\}$.

Therefore the inductive procedure is finished, both for the families M_i^j (see (32), (35) and (37)) and for the directed paths P_j, and so the mapping f is defined on each edge E of $X_2 \cup X_3$ with the property $\tau(f|E) > 1$ (see (31), (33) and (36)), from which we see by (20) and (25) that

$$\tau(f|E) > 1 \quad \text{for each edge } E \text{ of } X\,. \tag{38}$$

Let us recall that $f|X_1$ has already been defined, separately for each edge of X_1 (which are disjoint — see (19)), in such a way that each end point of X is a fixed point and each vertex of X which is not an end point is mapped onto the point c (see (24)). Similarly every vertex in $X_2 \cup X_3$ is sent to c under $f|X_2 \cup X_3$. Thus mappings $f|X_1$ and $f|X_2 \cup X_3$ do agree on the common part $X_1 \cap (X_2 \cup X_3)$ which is a subset of the set of vertices of X, and it follows from (20) that f is well-defined on the whole X. The continuity of f follows from continuity of every partial mapping $f|E$, where E is an edge of X (see [5], Theorem 9.4, p. 83). It follows from (20), (27) and (30) that f is surjective.

To see that f is a local expansion it is enough to apply Statements 1 and 2. In fact, let us observe that the inequality $\tau(f|E) > 1$ holds for all edges E of X. If a point $x \in X$ is not a vertex of X, then a suitable neighborhood U mentioned in the definition of a local expansion (see the very beginning of § 2) exists by Statement 1.

If x is a vertex of X, then U is constructed in Statement 2, and the existence of the constant $M>1$ follows from (38) in both the cases. Thus the proof is complete.

§ 5. Some properties of local expansions on linear graphs. To show that the condition mentioned in Theorem 1 is not only sufficient but also necessary for a linear graph to admit a local expansion, we shall use some properties of these mappings. We will prove them now.

PROPOSITION 1. *Let $f\colon X\to X$ be a local expansion on a metric space X. Then f is locally one-to-one: for every point $x\in X$ and for the open neighborhood U of x that exists by the definition of the local expansion, the restricted mapping $f|U\colon U\to f(U)$ is one-to-one.*

Indeed, let U be as in the definition of the local expansion (see § 2). Then, if y and z are distinct points of U, we have $\varrho(y,z)>0$, whence $\varrho(f(y),f(z))>0$ by (1), and the conclusion follows.

PROPOSITION 2. *Let $f\colon X\to X$ be a local expansion on a metric space X. Then for every point $x\in X$ and for the open neighborhood U of x that exists by the definition of the local expansion, every arc $ab\subset U$ is mapped onto an arc $f(a)f(b)$ homeomorphically under f.*

In fact, it follows from the previous proposition that if $ab\subset U$, then $f|ab\colon ab\to f(ab)$ is a homeomorphism.

PROPOSITION 3. *Let $f\colon X\to X$ be a local expansion on a metric space X. Then for each simple closed curve S contained in X its image $f(S)$ does not contain end points of itself.*

To see this, let $p\in f(S)$ and let a point $x\in S$ be such that $p=f(x)$. Take the open neighborhood U of x that exists by the definition of the local expansion, and choose two points x_1, x_2 in $S\cap U$ both different from x and such that x lies in the arc $x_1x_2\subset S\cap U$. Then the arc x_1x_2 is mapped homeomorphically under f (see Proposition 2) onto the arc $f(x_1)f(x_2)$ which contains $p=f(x)$ with $f(x_1)\neq p\neq f(x_2)$, whence p cannot be an end point of $f(S)$.

PROPOSITION 4. *Let $f\colon X\to X$ be a local expansion on a metric space X. Then for each arc ab contained in X no point of $ab\setminus\{a,b\}$ can be mapped on an end point of $f(ab)$.*

Indeed, as in the proof of the previous proposition, we take

$$p\in f(ab)\setminus\{f(a),f(b)\}$$

and $x\in ab\setminus\{a,b\}$ such that $p=f(x)$; in the open neighborhood U of x as in the definition of the local expansion we choose two points x_1, x_2 of $ab\setminus\{a,b\}$ such that x lies between them. The rest of the argumentation is exactly the same as for the proof of Proposition 3.

PROPOSITION 5. *Let $f\colon X\to X$ be a local expansion on a linear graph X. Then for every point $x\in X$ we have*

$$\operatorname{ord}_x X\leqslant \operatorname{ord}_{f(x)}X.$$

To see this, let us consider the points x and $f(x)$ as vertices of the graph X. Let $n=\operatorname{ord}_x X$ and $m=\operatorname{ord}_{f(x)}X$ be the numbers of edges of X incident to x and to $f(x)$ respectively. Further, let U be the open neighborhood of x as in the definition of the local expansion f. For every $i=1,2,\ldots,n$ choose exactly one point $a_i\in U\cap E_i$, where E_i is an edge of X incident to x. Thus, according to Propositions 1 and 2, the union $\bigcup\{xa_i|\ i=1,2,\ldots,n\}$ is mapped homeomorphically under f onto the union $\bigcup\{f(x)f(a_i)|\ i=1,2,\ldots,n\}$, whence we conclude that the number m of edges incident to $f(x)$ must be greater than or equal to n.

PROPOSITION 6. *Let $f\colon X\to X$ be a local expansion on a linear graph X. If a point $e\in X$ is an end point of X, then also $f(e)$ is an end point of X.*

Indeed, recall that the set F of all end points of X is finite, and put $k=\operatorname{card}F$. Suppose, on the contrary, that there is a point $e\in F$ with $f(e)\in X\setminus F$. Since the mapping f is a surjection, there is a set of k points of X, say $x_1,x_2,\ldots,x_k$, with the property that $f(x_1),f(x_2),\ldots,f(x_k)$ are all k end points of X. Thus at least one of $x_1,x_2,\ldots,x_k$ is not an end point of X; call it x. This means that X is of order greater than 1 at x and of order 1 at $f(x)$, contrary to Proposition 5.

Let X be a simple linear graph with a convex metric ϱ, and let an arc $A=ab\subset X$ be given. Consider the sequence of all vertices of X lying in A, ordered from a to b:

$$a\leqslant v_1<v_2<\ldots<v_n\leqslant b$$

and put

$$\lambda(A)=\varrho(a,v_1)+\sum_{i=1}^{n-1}\varrho(v_i,v_{i+1})+\varrho(v_n,b).$$

It is evident that if we take, instead of $\{v_i\}$, some other finite sequence of points, say $\{p_i|\ i=0,1,\ldots,k\}$, lying in A, ordered from a to b:

$$a=p_0<p_1<p_2<\ldots<p_k=b$$

and such that the consecutive points p_i, p_{i+1} lie closely enough, e.g.,

$$\varrho(p_i,p_{i+1})<\tfrac{1}{2}\min[\varrho(a,v_1),\varrho(v_1,v_2),\ldots,\varrho(v_{n-1},v_n),\varrho(v_n,b)] \tag{39}$$

for every $i=0,1,\ldots,k-1$, then we have

$$\lambda(A)=\sum_{i=0}^{k-1}\varrho(p_i,p_{i+1}). \tag{40}$$

PROPOSITION 7. *Let $f\colon X\to X$ be a local expansion of a simple linear graph X (with a convex metric) onto itself. If an arc $A\subset X$ is mapped onto an arc $f(A)$ under f, then*

$$\lambda(A)<\lambda(f(A)). \tag{41}$$

Proof. Every point $x\in A$ has an open and connected neighborhood U_x with the property as in the definition of the local expansion f. Furthermore, we can take neighborhoods U_x sufficiently small, e.g. such that the diameter of each U_x is less than the minimum of the lengths of edges of the graph X. The family $\{U_x \mid x \in A\}$ is an open covering of A. Take a finite subcovering of A such that no its element is contained in another, and let $U_{x_1}, U_{x_2}, \ldots, U_{x_k}$ be a sequence of all elements of the subcovering ordered in such a way that if a and b are end points of A, then $a \in U_{x_1}$, $b \in U_{x_k}$ and $U_{x_i} \cap U_{x_{i+1}} \neq \emptyset$ for every $i = 1, 2, \ldots, k-1$. Let $M_i > 1$, where $i = 1, 2, \ldots, k$, be the constants mentioned in the definition of the local expansion, i.e. such that if $y, z \in U_{x_i}$, then (see (1))

(42) $$\varrho(f(y), f(z)) \geqslant M_i \cdot \varrho(y, z).$$

Put $M = \min(M_1, M_2, \ldots, M_k)$. Thus $M > 1$. Choose points $p_i \in U_{x_i} \cap U_{x_{i+1}}$ for $i = 1, 2, \ldots, k-1$ and define $p_0 = a$ and $p_k = b$. The sets U_{x_i} being connected by construction, we have $p_i p_{i+1} \subset U_{x_{i+1}}$ for every $i = 0, 1, \ldots, k-1$, and, by (42)

(43) $$\varrho(f(p_i), f(p_{i+1})) \geqslant M_{i+1} \cdot \varrho(p_i, p_{i+1}).$$

Further, we see that points $f(p_i)$ lie in the arc $f(A)$ in the same order as points p_i lie in A. The neighborhoods U_{x_i} are small enough so that condition (39) holds and thus (40) can be applied. Therefore by (43) we have

$$\lambda(A) = \sum_{i=0}^{k-1} \varrho(p_i, p_{i+1}) < M \cdot \sum_{i=0}^{k-1} \varrho(p_i, p_{i+1}) \leqslant \sum_{i=0}^{k-1} M_{i+1} \cdot \varrho(p_i, p_{i+1})$$
$$\leqslant \sum_{i=0}^{k-1} \varrho(f(p_i), f(p_{i+1})) = \lambda(f(A)),$$

the last equality being an easy consequence of the uniform continuity of f so that formula (40) is applicable to the image $f(A)$. Thus (41) follows.

§ 6. Existence of local expansions-necessity. We are ready now to prove our second main result.

THEOREM 2. *Let a simple linear graph X metrized by a convex metric be given. If there exists a local expansion f: $X \to X$ of X onto itself, then there is a point $c \in X$ of the maximal order in X, i.e., satisfying* (18), *such that for every component of $X \setminus \{c\}$ its closure contains a simple closed curve.*

Proof. Suppose the contrary. This means that for every point c of the maximal order in X there is a component of $X \setminus \{c\}$ such that its closure $T(c)$ contains no simple closed curve (in other words, $T(c)$ is a tree). Let P denote the set of all points c in X which are of the maximal order $n = \operatorname{ord} X$ in X. Thus $\operatorname{ord}_c X = n$ for every $c \in P$. It follows from Proposition 5 that $f(P) \subset P$. For each point $c \in P$ take the open neighborhood U of c as in the definition of the local expansion f and, for each index $i = 1, 2, \ldots, n$ choose exactly one point $a_i \in U \cap E_i$, where E_i is an edge of X incident to c. We distinguish two kinds of arcs $ca_i \subset E_i$. Namely, an arc ca_i is said to be of the first kind if it is contained (together with the edge E_i) in a tree $T(c)$; it is said to be of the second kind if it is contained (also together with the edge E_i containing ca_i) in the closure $K(c)$ of a component of $X \setminus \{c\}$ such that $K(c)$ contains a simple closed curve. Given a point $c \in P$, let $n_1(c)$ or $n_2(c)$ denote the number of arcs ca_i which are of the first or of the second kind respectively. Thus,

(44) $$n_1(c) + n_2(c) = n \quad \text{for every point } c \in P.$$

Let us recall that, by Proposition 2, every arc ca_i is mapped homeomorphically onto an arc $f(c)f(a_i)$. We claim that

(45) if ca_i is of the second kind, then its image $f(ca_i)$ is of the second kind, too.

Indeed, if not, then consider two cases. If ca_i lies on a simple closed curve S, then, since $f(ca_i)$ is contained in some tree $T(f(c))$, the image $f(S)$ has a nondegenerate intersection with $T(f(c))$. Thus $f(S) \cap T(f(c))$ is a tree as a subcontinuum of $T(f(c))$, and hence $f(S)$ contains an end point of itself, contrary to Proposition 3. If ca_i is contained in no simple closed curve, then, since it is of the second kind, there is an arc cb such that

$$ca_i \subset E_i \subset cb \subset cb \cup S' \subset K(c)$$

and $cb \cap S' = \{b\}$, where $K(c)$ is the closure of the component of $X \setminus \{c\}$ containing ca_i, and S' is a simple closed curve contained in $K(c)$. Since no point of $cb \setminus \{c, b\}$ is mapped to an end point of $f(cb)$ by Proposition 4, and since $f(cb)$ is a subcontinuum of a tree $T(f(c))$, hence $f(cb)$ is an arc with $f(b) \neq f(c)$. Thus $f(S')$ has a nondegenerate intersection with $T(f(c))$, which implies, as in the previous case, a contradiction with Proposition 3. So claim (45) is proved.

The restricted mapping $f \mid U$ being a homeomorphism by Proposition 1, we conclude from (45) that

(46) $$n_2(c) \leqslant n_2(f(c)).$$

Let us take a subset Q of P composed of all points $c \in P$ for which the number $n_2(c)$ is maximal. Since $\operatorname{ord}_c X = \operatorname{ord}_{f(c)} X = n$, both (44) and (46) imply that $n_2(c) = n_2(f(c))$ for every point $c \in Q$, whence we conclude at once that

(47) if $c \in Q$ and if an arc ca_i is of the first kind, then its image $f(ca_i)$ is of the first kind, too.

Finally take a subset R of Q composed of all points $c \in Q \subset P$ with the property that for all end points e of all trees $T(c)$ the distance $\varrho(c, e)$ is maximal. In other words, a point $c \in Q$ is in the set R if and only if there is a tree $T(c)$ and an end point $e \in T(c)$ such that

(48) $\varrho(c', e') \leqslant \varrho(c, e)$ for all points $c' \in Q$, all trees $T(c')$ and all end points e' of $T(c')$.

Consider an arc ce with $c \in R$, with an end point $e \in T(c)$ and with the maximal distance $\varrho(c, e)$. Thus the arc $ca_i \subset ce$ is of the first kind, and we see by (47) that its image $f(ca_i)$ is also of the first kind. Since $f(e)$ is an end point of X by Proposition 6, and since the image $f(ce)$ does not contain end points of itself except of $f(c)$ and

$f(e)$ by Proposition 4, we see that $f(ce)$ is an arc contained in a tree $T(f(c))$ being the closure of some component of $X\setminus\{c\}$. So the arcs ce and $f(ce)$ are contained in the trees $T(c)$ and $T(f(c))$ respectively, and thereby we conclude from the convexity of the metric ϱ that

$$\varrho(c,e) = \lambda(ce) \quad \text{and} \quad \varrho(f(c),f(e)) = \lambda(f(ce)). \tag{49}$$

Further, $f(c)$ is a point c' of Q and $f(e)$ is an end point e' of $T(c')$, and therefore we have $\lambda(f(ce)) \leqslant \lambda(ce)$ by (49) and (48), contrary to Proposition 7. The contradiction completes the proof.

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