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CHELSEA COLLEGE, UNIVERSITY OF LONDON

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A characterization of expandability of models for ZF to models for KM

by

Zygmunt Ratajczyk (Warszawa)

Abstract. In this paper we characterize KM-expandable and KM-non-β-expandable models by means of certain games. Also another characterization is given. It is proved that KM-expandability and KM-non-β-expandability are equivalent in a wide class of models for ZF.

§ 0. **Introduction.** The primary aim of our paper is the characterization, with the aid of a certain closed game, of those KM-expandable models for ZF whose height has a cofinality character equal to ω.

We characterize KM-expandable models in a way similar to that of Bieliński [3] in the case of countable models. We do this by means of approximations for recursive closed game formulas considered by Barwise in [1].

The investigation of properties of KM-expandable models was initiated by Marek and Mostowski in [6]. The authors focus their attention on KM-β-expandable models and give their full characterization. Among other things, they show that KM-expandability is not an elementary property in a 1-st order language. In that paper a characterization of KM-expandable standard models whose height has a cofinality character $>\omega$ is given. In fact, it is shown that any model for KM possessing such a set universe is automatically a β-model, hence KM-expandability can be reduced to KM-β-expandability in that case.

Let K be a language. By $K_{\infty\omega}$ we shall denote the class of all infinitary formulas of the language K. Let L_α: $\alpha \in \mathrm{Ord}$ be the hierarchy of constructible sets. By K_α we shall denote $K_{\infty\omega} \cap L_\alpha$ and by $\mathrm{ZF}^{\mathrm{KM}}_\infty$ the class of all formulas φ from language $(\mathscr{L}_{\mathrm{ZF}})_{\infty\omega}$ such that their relativization φ^V is a theorem in KM. By $\mathrm{ZF}^{\mathrm{KM}}_\alpha$ we shall understand the intersection of $\mathrm{ZF}^{\mathrm{KM}}_\infty$ and L_α. Note that for admissible $\alpha>\omega$,

$$\mathrm{ZF}^{\mathrm{KM}}_\alpha = \{\varphi \in (\mathscr{L}_{\mathrm{ZF}})_\alpha \colon L_\alpha \models (\mathrm{KM} \vdash \varphi^V)\}.$$

K. Bieliński, in [3], shows that although KM-expandability is not an elementary property (in language $(\mathscr{L}_{\mathrm{ZF}})_{\omega\omega}$), nevertheless it can be characterized in a uniform manner in the class of countable models $\underline{M}$ by a theory which is Σ_1 in $\mathrm{HYP}_{\underline{M}}$.

Namely:

THEOREM. *If $\underline{M}$ is countable model for* ZFC, $\alpha = o(\mathrm{HYP}_{\underline{M}})$, *then $\underline{M}$ is* KM-*expandable* $\Leftrightarrow$ $\underline{M} \models \mathrm{ZF}_{\alpha}^{\mathrm{KM}}$.

The above theorem is a special case of a more general theorem proved by Bieliński and giving a characterization of a very general problem of T-expandability, where T is an arbitrary, recursively enumerable theory. An investigation of expandability to theories other than KM is also presented by Barwise and Schlipf in [2], where a characterization of models for arithmetic expandable to models of weaker fragments of analysis is given.

A related problem of characterization of models of arithmetic expandable to models of ZF is studied in Wilmers [11].

Let us mention that the first known paper devoted to the characterization of countable models expandable to KM by properties possible to describe in HYP_M is a paper by Marek and Srebrny [7]. The authors show, among other things, that the KM-extendability for countable models can be described by Π_1 sentence in HYP_M. This is a weaker result than that obtained by Bieliński and can be obtained directly through the application of the completeness theorem to HYP_M.

If K is a sublanguage of the language of a theory T, then $(K)_{\alpha}^{T} \underset{\mathrm{df}}{=} \{\varphi \in K_{\alpha}: T \vdash \varphi\}$. The theorem of Bieliński can be formulated in the following way:

If $\underline{M}$ is a countable K-structure and $\alpha \geqslant o(\mathrm{HYP}_M)$ is an admissible ordinal, then $\underline{M}$ is T expandable iff $\underline{M} \models (K)_{\alpha}^{T}$.

The above theorem does not generalize to uncountable models provided $o(\mathrm{HYP}_M) > \omega$. If $o(\mathrm{HYP}_M) = \omega$ we can replace the countability assumption by that of resplendency, cf. [3], Theorem 2.7.

In the present paper it is shown that if we restrict our attention to theories such as KM or KM_n, (fragments of KM), we obtain a similar characterization for certain classes of uncountable models. The statement of results and the proofs are given only for KM; in the case of KM_n the proofs must be modified by using the results of Ratajczyk [10].

Here, among other things, we show the following:

THEOREM. *If $\underline{M} \models$ ZF and the height of $\underline{M}$ has cofinality character ω, then $\underline{M}$ is* KM-*expandable iff $\underline{M} \models \mathrm{ZF}_{\alpha}^{\mathrm{KM}}$, where $\alpha = o(\mathrm{HYP}_{\underline{M}})$.*

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§ 1. Preliminaries. We use the standard set and model-theoretical notation. In particular, by KM we mean set theory in a two-sorted language with classes and full comprehension schema, but without any form of the axiom of choice. By $\mathrm{r.a}(T,.)$ we shall denote the formula describing the ramified analysis hierarchy in KM and

by $\mathrm{RA}_{\alpha}^{\underline{M}}$ we shall denote the ramified analysis hierarchy over model $\underline{M}$ in the world. The formula $\mathrm{r.a}^{U}(T,.)$ determines the relativized hierarchy, starting from $\{U\}$.

Following Barwise, by $\mathrm{HYP}_{\underline{M}}$ we shall denote the least admissible set with urelements $A_{\underline{M}}$ such, that $\underline{M} \in A_{\underline{M}}$, where by $A_{\underline{M}}$ we denote the structure $(\underline{M}; A, \in, \ldots)$, with M as the set of urelements.

If $\underline{M} = \langle M, \in \rangle$ is a model for ZF, $\mathscr{F} \subseteq P(M)$, then by $\langle \mathscr{F}, M, E^* \rangle$ we denote a structure for a two-sorted language with relation E^* defined by the expresion $E^* = E \cup \{\langle x, A \rangle: A \in \mathscr{F} \,\&\, x \in A\}$. Note that any model for KM whose set universe is $\langle M, E \rangle$ is isomorphic to a model of the form $\langle \mathscr{F}, M, E^* \rangle$. $\mathscr{F}$ is the set of proper classes of this model. Hence, we shall say that a model $\langle M, E \rangle$ is KM-*expandable* if there exists a family $\mathscr{F} \subseteq P(M)$ such that $\langle \mathscr{F}, M, E^* \rangle \models$ KM.

We say that $\mathscr{F}_0 \subseteq \mathscr{F}$ is *codable in* $\mathscr{F}$ iff there exists a class $X \in \mathscr{F}$ such that $\mathscr{F}_0 = \{X^{(a)}: a \in M\}$, where $X^{(a)} = \{b: \langle a, b \rangle^{\underline{M}} \in X\}$. If the class X can be chosen in such a way that $\underline{\mathscr{F}} \models (\mathrm{Dom}\, X \in V)$, then we say that $\mathscr{F}_0$ is *codable by a class with a set domain.*

The height of an admissible set A_M is denoted by $o(A_{\underline{M}})$. If $\underline{M} = \langle M, E \rangle \models$ KP then by $\mathrm{sp}(\underline{M})$ we shall denote the standard part of the model and by $\mathrm{osp}(\underline{M}) = \mathrm{sp}(\underline{M}) \cap \mathrm{Ord}$, the height of the standard part. Similarly if $\underline{\mathscr{F}} = \langle \mathscr{F}, M, E^* \rangle \models \mathrm{GB} + \Sigma_1^1$-comprehension then by $\mathrm{osp}(\underline{\mathscr{F}})$ (or $\mathrm{osp}(\mathscr{F})$) we denote the upper bound of those ordinals α which are representable in $\mathscr{F}$, i.e. such that there exists a $T \in \mathscr{F}$ which is a wellordering in $\underline{\mathscr{F}}$ ($\langle \mathscr{F}, M, E^* \rangle \models \mathrm{W.O}(T)$) and that the ordinal type of the linear ordering relation $T_E = \{\langle x, y \rangle: \langle x, y^{\underline{M}} E^* T\}$ is equal to α.

We shall also say that $\underline{\mathscr{F}}$ is not a β-model if in the model there exists such a wellordering T, in the sense of $\underline{\mathscr{F}}$, such that T_E is not a wellordering in the world.

By the content of element $x \in M$, where $\underline{M} = \langle M, E \rangle \models$ KP, we shall understand the set $x_E = \{y \in M: yEx\}$. This notion will be used rather freely. For example, if $h \in M$ is a function in $\underline{M}$, then by the contents of h we shall understand the function $\{\langle x, y \rangle: \langle x, y \rangle^{\underline{M}} Eh\}$, whenever this does not lead to confusion.

Any notion relativised to $\underline{M}$ is denoted by the superscript $\underline{M}$. Hence, $On^{\underline{M}} = \{x \in M: \underline{M} \models \mathrm{Ord}(x)\}$, $R_{\alpha}^{\underline{M}}$ is an element a belonging to M and such that $\underline{M} \models$ "$a = R_{\alpha}$" provided $\alpha \in On^{\underline{M}}$.

We shall also adopt the following convention: the notation $\langle M, R_1, \ldots, R_k \rangle$ does not imply that R_i is a relation in M, but denotes the structure

$$\langle M, R_1 \upharpoonright M, \ldots, R_k \upharpoonright M \rangle .$$

For any linear ordering T the notion of the index of cofinality of T will be useful. We define it, as for ordinals, as the least ordinal which can be cofinally embedded in the ordering T.

If $\underline{M} = \langle M, E \rangle \models$ ZF, then we assume cf $On^{\underline{M}} = \mathrm{cf}(On^{\underline{M}}, E \rangle)$.

For any language K, $K_{\omega\omega}$ denotes the set of all finite formulas, and $K_{\infty\omega}$ the set of all proper and infinite ones, i.e. the least class of formulas possesing a finite number of variables and closed with respect to finite operations and infinite alternative — $\bigvee\bigvee$ and conjunction — $\bigwedge\bigwedge$.

The language $K_{\infty G}$, similar to that introduced by Moschovakis in [9], is obtained by the use of all the above rules of building the formulas together with the following:

If $\varphi_0(x_0)$, $\varphi_1(x_0, x_1)$, $\varphi_2(x_0, x_1, x_2)$, ... is such a sequence of formulas that for every formula $\varphi_i(x_0, x_1, ..., x_i)$, its free variables other than $x_0, ..., x_i$ belong to a fixed finite set (independent of i), and if $Q_0, Q_1, Q_2, ...$ is an infinite sequence of quantifiers, then

$$\{(Q_0 x_0)(Q_1 x_1)(Q_2 x_2) ...\} \bigwedge_{i\in\omega}\bigwedge \varphi_i(x_0, ..., x_i),$$

$$\{(Q_0 x_0)(Q_1 x_1)(Q_2 x_2) ...\} \bigvee_{i\in\omega}\bigvee \varphi_i(x_0, ..., x_i)$$

are also formulas.

For the formulas of this language, as in $K_{\omega\omega}$, satisfaction has a recursive character. For the formulas with an infinite sequence of quantifiers at the beginning satisfaction is defined with the aid of the game connected with such a sequence. Now we shall distinguish formulas of the form

$$(\exists x_1)(\forall x_2)(\exists x_3) ... \bigwedge_{i\in\omega}\bigwedge \varphi_i(x_1, ..., x_{p_i})$$

such that $\{\varphi_i(x_1, ..., x_{p_i}): i\in\omega\}$ is a recursive subset of $K_{\omega\omega}$, which, following Barwise [1], we shall call recursive closed games formulas.

We shall denote by $\mathscr{G}\vec{x}$ the sequence of quantifiers $(\exists x_1)(\forall x_2)(\exists x_3)$ ... A simple interpretation of formula $\mathscr{G}\vec{x}\varphi(\vec{x})$ can be formulated with the aid of the Skolem function:

$$\langle M, R_1, ..., R_k\rangle \models \mathscr{G}\vec{x}\varphi(\vec{x}) \quad \text{iff} \quad \{(\exists f_0)(\exists f_1) ...\},$$
$$\langle M, R_1, ..., R_k\rangle \models \{(\forall x_2)(\forall x_4) ...\}\varphi(f_0, x_2, f_1(x_2), x_4, ...).$$

This should be sufficient for an accurate presentation of our results.

§ 2. Approximations of models for KM. In this section we shall consider some model-theoretic consequences of the reflection principle related to the problem of expandability. As is well known, the theory KM does not have the property of reflection. Note, however, that the class of KM-expandable models coincides with the class of models expandable to the theory KM+$(\forall X)$r.a(X) — a theory possessing the reflection property.

Before, formulating the theorem characterizing KM-expandable models, we shall recall two theorems from the paper of Marek and Mostowski [6] concerning reflection:

THEOREM 2.1 (The reflection principle in KM+$(\forall X)$r.a(X)). *If $\Phi(X)$ is a formula of language $\mathscr{L}_{\mathrm{KM}}$, with one free variable X, then*

$$\mathrm{KM}+(\forall X)\mathrm{r.a}(X) \vdash (\forall T)(\exists T_1)[\mathrm{W.O}(T) \Rightarrow \mathrm{W.O}(T_1) \,\&$$
$$\& \, T_1 > T \,\&\, (\forall X)\big(\mathrm{r.a}(T_1, X) \Rightarrow (\Phi^{\mathrm{r.a}(T_1,.)}(X) \Leftrightarrow \Phi(X))\big)].$$

Substituting for Φ the conjunction of the formula universal for Σ_n^1 formulas and the formula which is a basis for this universal formula, we can represent the above theorem in the following, model-theoretical form:

If $\underline{\mathscr{F}} = \langle \mathscr{F}, M, E^*\rangle \models \mathrm{KM}+(\forall X)\mathrm{r.a}(X)$, $n\in\omega$, $X\in\mathscr{F}$, then there exists a subset $\mathscr{F}_0$ of $\mathscr{F}$ codable in $\underline{\mathscr{F}}$ such, that

$$\langle \mathscr{F}_0, M, E^*\rangle \prec_n^1 \langle \mathscr{F}, M, E^*\rangle \quad \text{and} \quad X\in\mathscr{F}_0.$$

Let KM_n denote the fragment of KM with the comprehension scheme restricted to Σ_n^1 formulas. KM_n is finitely axiomatizable, hence we can additionally assume that $\langle \mathscr{F}_0, M, E^*\rangle \models \mathrm{KM}_n$ in the above remark.

The following theorem we present directly in the model-theoretic version:

THEOREM 2.2. *If $\underline{\mathscr{F}} = \langle \mathscr{F}, M, E^*\rangle \models \mathrm{KM}$, $\mathscr{F}_0$ is a subset of $\mathscr{F}$ codable in $\underline{\mathscr{F}}$, $\mathscr{F}_1 \subseteq \mathscr{F}_0$ and $\mathscr{F}_1$ is codable by a class with a set domain, then there exists $\alpha\in On^{\underline{M}}$, $F\in M$ and a mapping H codable in $\underline{\mathscr{F}}$ which is an elementary embedding of $\langle F_E, (R_\alpha^{\underline{M}})_E, E^*\rangle$ in $\langle \mathscr{F}_0, M, E^*\rangle$. H is constant on elements of the set $(R_{\overline{\alpha}}^{'})_E$ and it is such that $\mathscr{F}_1 \subseteq H''F_E$.*

Theorems 2.1 and 2.2 hold also for the relativized ramified analysis hierarchy.

LEMMA 2.3. *If $\underline{M} = \langle M, E\rangle \models \mathrm{ZF}$, cf. $On^{\underline{M}} = \omega$, then*

(a) *$\underline{M}$ is* KM*-expandable iff there exists a direct system:*

$$g_i\colon \langle \mathscr{F}_i, R_i, E^*\rangle \prec_i^1 \langle \mathscr{F}_{i+1}, R_{i+1}, E^*\rangle, ...; \quad i\in\omega,$$

such that for all $i\in\omega$, $\langle \mathscr{F}_i, R_i, E^*\rangle \models \mathrm{KM}_i$, where $\mathscr{F}_i$, R_i, g_i are the respective contents of elements F_i, $R_{\alpha_i}^{\underline{M}}$, h_i belonging to M, $\alpha_i\in On^{\underline{M}}$, $g_i \restriction R_i = \mathrm{id}_{R_i}$ and the sequence $\alpha_1, \alpha_2, ...$ is cofinal with the height of $\underline{M}$.

(b) *$\underline{M}$ is expandable to a model for* KM *which is not a β-model iff there exists a direct system:*

$$g_i\colon \langle \mathscr{F}_i, R_i, T_i, E^*\rangle \prec_i^1 \langle \mathscr{F}_{i+1}, R_{i+1}, T_{i+1}, E^*\rangle;$$

$i\in\omega$ such that for all $i\in\omega$, $\langle \mathscr{F}_i, R_i, E^*\rangle \models \mathrm{KM}_i$, $\langle \mathscr{F}_i, R_i, T_i, E^*\rangle \models \mathrm{W.O}(T_i)$, $\mathscr{F}_i$, R_i, g_i, T_i are the respective contents of the elements F_i, $R_{\alpha_i}^{\underline{M}}$, h_i, t_i belonging to $\underline{M}$, $\alpha_i\in On^{\underline{M}}$, $g_i\restriction R_i = \mathrm{id}_{R_i}$, the sequence $\alpha_1, \alpha_2, ...$ is cofinal with the height of $\underline{M}$ and there exists a sequence of different elements $a_1, a_2, ...$ belonging to M such that for every $i\in\omega$.

$$\langle \mathscr{F}_i, R_i, T_i, E^*\rangle \models a_i E^* \mathrm{Dom}\, T_i$$

and

$$\langle \mathscr{F}_{i+1}, R_{i+1}, T_{i+1}, E^*\rangle \models \langle a_{i+1}, a_i\rangle E^* T_{i+1}.$$

Proof. Assume that $\underline{M} = \langle M, E\rangle$ is KM-expandable and possesses all the properties mentioned in the assumption.

Let $\underline{\mathscr{F}} = \langle \mathscr{F}, M, E^*\rangle \models \mathrm{KM}+(\forall X)\mathrm{r.a}(X)$ and let $\overline{\mathscr{F}}_1 \subseteq \overline{\mathscr{F}}_2$ ... be a sequence of codable subsets of $\mathscr{F}$, such that $\langle \overline{\mathscr{F}}_i, M, E^*\rangle \prec_i^1 \langle \mathscr{F}, M, E^*\rangle$: $i\in\omega$. The existence of such a system follows from the above mentioned Theorem 1.

Let $\beta_1, \beta_2, \ldots$ be a sequence of elements $On^{\underline{M}}$ cofinal with the height of $\underline{M}$. Applying Theorem 2.2 mentioned at the beginning, we conclude that there exist sequences: a sequence $\alpha_1, \alpha_2, \ldots$ of elements $On^{\underline{M}}$; $F_1, F_2, \ldots$ — a sequence of elements of M and $H_1, H_2, \ldots$, and a sequence of mappings codable in $\mathscr{F}$, having the following properties: for every i: $\alpha_i > \beta_i$, $\alpha_{i+1} > \alpha_i$, H_i is an elementary embedding $\langle (F_i)_E, (R^M_{\alpha_i})_E, E^* \rangle$ in $\langle \mathscr{F}_i, M, E^* \rangle$, H_i is an identity on the elements of set $(R^M_{\alpha_i})_E$ and $H_i''(F_i)_E \subseteq H_{i+1}''(F_{i+1})_E$.

Let us denote by g_i the mapping $H_{i+1}^{-1} \circ H_i$. Obviously g_i is a mapping from $\langle (F_i)_E, (R^M_{\alpha_i}), E^* \rangle$ into $\langle (F_{i+1})_E, (R^M_{\alpha_{i+1}}), E^* \rangle$ which is constant on the elements of $(R^M_{\alpha_i})_E$. It remains to show that g_i is a Σ^1_i elementary monomorphism. In order to do this, note that g_i can be represented as a composition of three mappings:

$$H_{i+1}^{-1} \restriction H_i''(F_i)_E, \ \mathrm{id}_{\mathscr{F}_i}: \langle \mathscr{F}_i, M, E^* \rangle \prec^1_i \langle \mathscr{F}_{i+1}, M, E^* \rangle$$

and H_i. Hence, from the fact that H_i and H_{i+1} are elementary we infer that g_i is a Σ^1_i-elementary monomorphism.

Next, since g_i is a composition of mappings codable in $\mathscr{F}$ it is $\mathscr{F}$-codable, hence, as a subset of a content of some element of M it is a content of a mapping $h_i \in M$. If we now put $\mathscr{F}_i = (F_i)_E$, $R_i = (R^M_{\alpha_i})_E$, this, together with g_i is the desired direct system.

To prove the converse implication let us assume that

$$\langle \mathscr{F}, R, E_1^* \rangle = \mathrm{Lim} \langle \mathscr{F}_i, R, E^* \rangle$$

is a direct limit of our system supplied with the missing monomorphisms g_{ij} defined as follows: $g_{ij} = g_{j-1} \circ \ldots \circ g_i$ for $i<j$. By $g_{i\infty}$ let us denote the natural monomorphism from $\langle \mathscr{F}_i, R_i, E^* \rangle$ into $\langle \mathscr{F}, R, E_1^* \rangle$.

From the accepted model-theoretical facts we infer that $g_{i\infty}$ is a Σ^1_i-elementary monomorphism.

Hence, because of the fact that $(\forall_i)(\langle \mathscr{F}_i, R_i, E^* \rangle \models \mathrm{KM}_i)$, $\langle \mathscr{F}, R, E_1^* \rangle \models \mathrm{KM}$. Thus, if suffices to show, that $\langle R, E_1 \rangle \simeq \langle M, E \rangle$. Assuming that $g(x) = g_{i\infty}(x)$ for $x \in M$, where i is the least natural number such, that $x \in (R^M_{\alpha_i})_E$, we conclude that g is the desired isomorphism. Indeed, if $x \neq y$ or $xEy, g(x) = g_{i\infty}(x)$, $g(y) = g_{j\infty}(y)$, $i<j$, then $g(x) = g_{i\infty}(x) = g_{j\infty}(g_{ij}(x)) = g_{j\infty}(x)$. Hence $g(x) = g_{j\infty}(x) \neq g_{j\infty}(y) = g(y)$. Similarly $g(x) = g_{j\infty}(x) E_1 g_{j\infty}(y) = g(y)$. Finally if $z \in R$, then $z = g_{j\infty}(x)$ for some $x \in R_j$. Thus $z = g(x)$.

The proof of (b) is similar to the proof of (a). The main differences in the proof of the implication $\Rightarrow$ are as follows:

Having a model $\langle \mathscr{F}', M, E^* \rangle$ for KM which is not a β-model, we choose $T \in \mathscr{F}'$ such that $\langle \mathscr{F}', M, E^* \rangle \models \mathrm{W.O}(T)$, T_E is not a wellordering in the world. Next we "cut out" $\mathscr{F} \subseteq \mathscr{F}'$ in such a way that the structure $\langle \mathscr{F}, M, T, E^* \rangle$ is a model for $\mathrm{KM} + (\forall X)\mathrm{r.a}^T(X)$. Finally, we choose a sequence $a_1, a_2, \ldots$ of different elements of M which shows that T_E is not well-founded. The construction of an adequate "tower" is analogous, with one difference, namely that in the ith step we extend the set universe in such a way that in the next step it contains the element a_{i+1}.

To prove the converse implication, assume that $\langle \mathscr{F}, R, T, E_1^* \rangle$ is a direct limit of the system $\{\langle \mathscr{F}_i, R_i, T_i, E^* \rangle, \ldots\}$ with the required properties. As in the proof of (a), we conclude that $\langle R, E_1 \rangle \simeq \langle M, E \rangle$ and that $\langle \mathscr{F}, R, T, E_1^* \rangle \models \mathrm{W.O}(T)$. Let $b_i = g_{i\infty}(a_i)$ for $i \in \omega$. Since $b_i = g_{i\infty}(a_i) = g_{i+1\infty}(a_i)$ and $\langle a_{i+1}, a_i \rangle^{\underline{R_{i+1}}} E^* T_{i+1}$, we have

$$\langle b_{i+1}, b_i \rangle^{\underline{R}} = g_{i+1\infty}(\langle a_{i+1}, a_i \rangle^{\underline{R_{i+1}}}) E_1^* g_{i+1\infty}(T_{i+1}) = T.$$

This proves, that $\langle \mathscr{F}, R, E_1^* \rangle$ is not a β-model.

§ 3. Characterization of KM-expandability by a game formula. Now we shall prove that the existence of a direct system with elements codable in the set universe and "approximating" the model for KM can be described by a recursive game.

THEOREM 3.1. *There exists a recursive and closed game, formula* $\mathscr{G}\vec{x}\varphi(\vec{x})$, *belonging to language* $(\mathscr{L}_{\mathrm{ZF}})_{\infty G}$ *such that*

(1) $\underline{M}$ *is* KM-*expandable* $\Rightarrow$ $\underline{M} \models \mathscr{G}\vec{x}\varphi(\vec{x})$,

(2) $(\underline{M} \models \mathrm{ZF} \ \& \ \mathrm{cf}\, On^{\underline{M}} = \omega \ \& \ \underline{M} \models \mathscr{G}\vec{x}\varphi(\vec{x})) \Rightarrow \underline{M}$ *is* KM-*expandable.*

Proof. We define

$$\begin{aligned} \mathscr{G}\vec{x}\varphi(\vec{x}) = \{&(\exists \alpha_1, F_1)(\forall \beta_1)(\exists \alpha_2, F_2, h_1)(\forall \beta_2) \ldots \\ &\ldots (\exists \alpha_{i+1}, F_{i+1}, h_i) \ldots (\forall \beta_{i+1}) \ldots\} \bigwedge_{i \in \omega}\bigwedge [\alpha_{i+1} > \beta_i \ \& \\ &\& \ (h_i: \langle F_i R_{\alpha_i}, \in \rangle \prec^1_i \langle F_{i+1}, R_{\alpha_{i+1}}, \in \rangle) \ \& \\ &\& \ (\langle F_i, R_{\alpha_i}, \in \rangle \models \mathrm{KM}_i) \ \& \ h_i \restriction R_{\alpha_i} = \mathrm{id}_{R_{\alpha_i}}]. \end{aligned}$$

Now let assume that $\underline{M}$ is KM-expandable; $\langle \mathscr{F}, M, E^* \rangle \models \mathrm{KM}$. To simplify the description of the winning strategy, we assume that $(\exists)$ chooses two additional elements in each move. Assume that player $(\forall)$ has chosen element $\beta_n \in On^{\underline{M}}$ and that $(\alpha_1, F_1, \beta_1, \alpha_2, F_2, h_1, \ldots, \alpha_n, F_n, h_{n-1}, \beta_n)$ is a fragment of the already completed part of the play. Let $\mathscr{F}_1, H_1, \ldots, \mathscr{F}_n, H_n$ be the sequence of the additional elements already chosen by $(\exists)$, satisfying the following conditions:

1) $(\forall i)_{\leqslant n}(\langle \mathscr{F}_i, M, E^* \rangle \prec^1_i \langle \mathscr{F}, M, E^* \rangle)$,

2) $(\forall_i)_{<n}(\mathscr{F}_i \subseteq \mathscr{F}_{i+1})$,

3) $(\forall i)_{\leqslant n} H_i: \langle (F_i)_E, (R^M_{\alpha_i})_E, E^* \rangle \prec \langle \mathscr{F}_i, M, E^* \rangle$.

Then $(\exists)$ chooses an additional $\mathscr{F}$-codable element $\mathscr{F}_{n+1}$ such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1} \subseteq \mathscr{F}$ and $\langle \mathscr{F}_n, M, E^* \rangle \prec^1_{n+1} \langle \mathscr{F}, M, E^* \rangle$, and chooses elements $\alpha_{n+1}, F_{n+1}, h_n \in M$ and one additional element H_{n+1} such that $\alpha_{n+1} > \max(\alpha_n, \beta_n)$,

$$H_{n+1}: \langle (F_{n+1})_E, (R^M_{\alpha_{n+1}})_E, E^* \rangle \prec \langle \mathscr{F}_{n+1}, M, E^* \rangle,$$

H_{n+1} is codable in $\mathscr{F}$, $H_{n+1} \restriction (R^M_{\alpha_{n+1}})_E = \mathrm{id}$, $H_{n+1}^{-1} \circ H_n$ is the content of h_n. From the proof of the lemma it follows that player $(\exists)$ can always choose the elements described above. This leads $(\exists)$ to winning the game. We must add that the property

"$\langle (F_n)_E, (R^{M}_{\alpha_n})_E, E^*\rangle \models \mathrm{KM}_n$" is absolute with respect to $\underline{M}$ because of the finite axiomatizability of KM_n and can be expressed in the following form:

$$\underline{M} \models \text{"}\langle F_n, R_{\alpha_n}, \in\rangle \models \mathrm{KM}_n\text{"} .$$

For similar reasons, property $\prec^1_n$ can also be expressed in $\underline{M}$ in the required form.

Now, let $\underline{M} \models \mathrm{ZF}$, cf. $On^{\underline{M}} = \omega$ and let $\underline{M}$ satisfy the recursive closed game formula defined above. Let $\gamma_1, \gamma_2, \ldots$ be a fixed sequence of elements of $On^{\underline{M}}$ cofinal with the height of $\underline{M}$.

Let player $(\forall)$ play as follows: $\beta_n = \gamma_n$ for all $n \in \omega$, and let player $(\exists)$ use his winning strategy. Hence, as a result of the described game, we obtain a sequence $\alpha_1, F_1, \gamma_1, \alpha_2, F_2, h_1, \gamma_2, \ldots$ such that for all $i \in \omega$:

$$\underline{M} \models (h_i\colon \langle F_i, R_{\alpha_i}, \in\rangle \prec^1_i \langle F_{i+1}, R_{\alpha_{i+1}}, \in\rangle),$$
$$\underline{M} \models (\langle F_i, R_{\alpha_i}, \in\rangle \models \mathrm{KM}_i) \mathbin{\&} (h_i \upharpoonright R_{\alpha_i} = \mathrm{id}_{R_{\alpha_i}}),$$
$$\gamma_i E \alpha_{i+1} .$$

This means that the sequence $\alpha_1, \alpha_2, \ldots$ is cofinal with the height of $\underline{M}$ and in consequence,

$$\langle (F_i)_E, (R_{\alpha_i})_E, E^*\rangle, (h_i)_E\colon i \in \omega$$

is a direct system described in the lemma. Hence, from this lemma we infer that $\underline{M}$ is KM-expandable.

Referring to Lemma 2.3 (b) we can, by a similar reasoning, obtain the following theorem.

THEOREM 3.2. *There exists a recursive closed game formula $\mathcal{G}\vec{x}\varphi_1(\vec{x})$ belonging to language $(\mathcal{L}_{\mathrm{ZF}})_{\infty G}$ such that*

(1) *If $\underline{M}$ has an extension $\mathcal{F}$ such that $\langle \mathcal{F}, M, E^*\rangle$ is not a β-model, then $\underline{M} \models \mathcal{G}\vec{x}\varphi_1(\vec{x})$.*

(2) *If $\underline{M} \models \mathrm{ZF}$, cf. $On^{\underline{M}} = \omega$, $\underline{M} \models \mathcal{G}\vec{x}\varphi_1(\vec{x})$, then $\underline{M}$ has an extension to a model of KM which is not a β-model.*

The following theorem, proved by Barwise [1], is the key to further considerations.

THEOREM (Barwise). *If $\mathcal{G}\vec{x}\varphi(\vec{x})$ is a recursive closed game formula belonging to $K_{\infty G}$, then there exists a $\Delta_1^{\mathrm{KP+Inf}}$-operation $\sigma\colon On \to K_{\infty\omega}$ such that*

(1) *for every $n \in \omega$, $\sigma_n \in K_{\omega\omega}$, and if M is a structure for K, then*

(2) $(M \models \mathcal{G}\vec{x}\varphi(\vec{x})) \Rightarrow M \models \bigwedge_{\alpha}\!\!\bigwedge \sigma_\alpha$,

(3) $(\alpha = o(\mathrm{HYP}_M) \mathbin{\&} M \models \bigwedge_{\beta<\alpha}\!\!\bigwedge \sigma_\beta) \Rightarrow M \models \mathcal{G}\vec{x}\varphi(\vec{x})$.

THEOREM 3.3. *If $\underline{M} \models \mathrm{ZF}$, cf. $On^{\underline{M}} = \omega$, $\alpha = o(\mathrm{HYP}_{\underline{M}})$, then $\underline{M}$ is KM-expandable iff $\underline{M} \models \mathrm{ZF}^{\mathrm{KM}}_\alpha$.*

Proof. Let $\mathcal{G}\vec{x}\varphi(\vec{x})$ be the formula from Theorem 3.1 describing the expandability to KM for models of cofinality character ω. Let σ_α be the approximations $\mathcal{G}\vec{x}\varphi(\vec{x})$ mentioned above. Obviously, for every α, $\sigma_\alpha \in (\mathcal{L}_{\mathrm{ZF}})_{\infty\omega}$. We shall show that

for countable α, $\mathrm{KM} \models \sigma^V_\alpha$. Indeed, if $\langle \mathcal{F}, N, E^*\rangle \models \mathrm{KM}$, then $\langle N, E\rangle \models \mathcal{G}\vec{x}\varphi(\vec{x})$ hence $\langle N, E\rangle \models \bigwedge_{\alpha<\omega_1}\!\!\bigwedge \sigma_\alpha$. This proves that $\mathrm{KM} \models \sigma^V_\alpha$ for $\alpha < \omega_1$. Applying the completeness theorem to formulas from language $K_{\omega_1\omega}$, we conclude that $\mathrm{KM} \vdash \sigma^V_\alpha$ for countable $\alpha \in On$.

Since the relation $\mathrm{KM} \vdash \varphi$ for $\varphi \in K_{\infty\omega}$ is of class Δ_1 (see Barwise [1]), employing the Lévy principle we obtain $(\forall\alpha)(\mathrm{KM} \vdash \sigma^V_\alpha)$.

Now, let us assume that $\alpha = o(\mathrm{HYP}_{\underline{M}}) > \omega$, cf $On^{\underline{M}} = \omega$, $\underline{M} \models \mathrm{ZF}^{\mathrm{KM}}_\alpha$. Since σ is a $\Delta_1^{\mathrm{KP+Inf}}$-operation, $\alpha > \omega$ is an admissible ordinal, we have $(\forall\beta)_{<\alpha}(\sigma_\beta \in L_\alpha)$. Hence $(\forall\beta)_{<\alpha}(\mathrm{ZF}^{\mathrm{KM}}_\alpha \vdash \sigma_\beta)$, and in consequence $\underline{M} \models \bigwedge_{\beta<\alpha}\!\!\bigwedge \sigma_\beta$. Thus, by the quoted Barwise Theorem, $\underline{M} \models \mathcal{G}\vec{x}\varphi(\vec{x})$. This proves that $\underline{M}$ is KM-expandable.

If $o(\mathrm{HYP}_{\underline{M}}) = \omega$, *cf.* $On^{\underline{M}} = \omega$, $\underline{M} \models \mathrm{ZF}^{\mathrm{KM}}_\omega$, then, from the fact that $(\forall n)_\omega(\sigma_n \in (\mathcal{L}_{\mathrm{ZF}})_{\omega\omega})$ we infer that $\underline{M} \models \bigwedge_{n\in\omega}\!\!\bigwedge \sigma_n$. Hence we find that $\underline{M}$ is KM-expandable.

THEOREM 3.4. *If $\underline{M} \models \mathrm{ZF}$, cf $On^{\underline{M}} = \omega$, $\underline{M}$ is KM-expandable, then it admits an expansion to KM which is not a β-model.*

An example of a model for ZF with an uncountable height expandable to a non-β-model of KM was given by Marek and Nyberg in [5]. They proved that, for the least α such that R_α is KM-expandable, R_α is not KM-β-expandable. Let us note here that the conclusion of Theorem 3.4 can be strenghtened, since as is shown in the present author's thesis, under the assumptions of Theorem 3.4 the intersection of all KM-expansions is included in $\mathrm{HYP}_{\underline{M}}$ (this is a direct generalization of the Gandy Kreisel Tait result for the case of KM-expansions). This result will be published elsewhere.

Theorem 3.4 is also a generalization of the known result that every countable and KM-extendable model for ZFC has an extension which is not a β-model (see [8]).

In the proof of 3.4 we shall employ the fact, that this theorem holds for countable models. We can not, however, refer directly to the quoted result, since our assumptions are weaker. Therefore we shall first prove the following auxiliary lemma:

LEMMA 3.5. *If $\underline{M} = \langle M, E\rangle \models \mathrm{ZF}$, $\underline{M}$ is countable and KM-expandable, then M possesses an extension to KM which is not a β-model and whose standard part is of a height equal to the height of $\mathrm{HYP}_{\underline{M}}$.*

Proof. If $\underline{M}$ is a nonstandard model, then, because of the expandability to KM (compare Barwise, Schlipf [2], Bieliński [3]), $\mathrm{osp}(\underline{M}) = o(\mathrm{HYP}_{\underline{M}})$. Also, for every expansion $\mathcal{F}$ such that $\langle \mathcal{F}, M, E\rangle \models \mathrm{KM}$, $\mathrm{osp}(\mathcal{F}) = \mathrm{osp}(\underline{M}) = o(\mathrm{HYP}_{\underline{M}})$. Assume that $\underline{M}$ is standard and assume that the lower limit of $\mathrm{osp}(\mathcal{F})$ for $\mathcal{F}$ such that $\langle \mathcal{F}, M, E^*\rangle \models \mathrm{KM}$ is $\geqslant \alpha > o(\mathrm{HYP}_{\underline{M}})$. It follows from the absoluteness of $\mathrm{R.A}^{\underline{M}}_\beta$ for ordinals β representable in a model for KM, that

$$(\forall \mathcal{F})(\langle \mathcal{F}, M, E^*\rangle \models \mathrm{KM} \Rightarrow \mathrm{R.A}^{\underline{M}}_\alpha \subseteq \mathcal{F}) .$$

Hence the required contradiction is a consequence of the following theorems:

1. Grillot [4]: $\langle M, E\rangle$ *is* KM-*expandable* $\Rightarrow \bigcap\{\mathscr{F}: \langle \mathscr{F}, M, E^*\rangle \models \mathrm{KM}\} \subseteq \mathrm{HYP}_{\underline{M}} \cap P(M)$.

2. Moschovakis [9]: *If* M *is acceptable*, $\varkappa = \mathrm{o}(\mathrm{HYP}_{\underline{M}})$, *then* $\mathrm{HYP}_{\underline{M}} \cap P(M) = \mathrm{R.A}^{\underline{M}}_{\varkappa}$.

We shall not recall the definition of acceptable models. Roughly speaking, they are models possessing a definable copy of ω and a definable function coding finite sequences. For our purposes it suffices to note that standard models for ZF are acceptable.

3. Marek and Mostowski [6]: *If* $\mathrm{R.A}^{\underline{M}}_{\gamma} = \mathrm{R.A}^{\underline{M}}_{\gamma+1}$, $\underline{M}$ *is* $\mathrm{R.A}^{\underline{M}}_{\gamma}$-*amenable* (*i.e. the image of a set by a class in* $\mathrm{R.A}^{\underline{M}}_{\gamma}$ *is also a set*), *then* $\langle \mathrm{R.A}^{\underline{M}}_{\gamma}, M, E^*\rangle \models \mathrm{KM}$.

And the known fact that

4. *If* $\varkappa = \mathrm{o}(\mathrm{HYP}_{\underline{M}})$ *then* $\langle \mathrm{R.A}^{\underline{M}}_{\varkappa}, M, E^*\rangle$ *not* $\models \mathrm{KM}$.

Thus we have shown that there exists a model $\langle \mathscr{F}, M, E^*\rangle \models \mathrm{KM}$ such that $\mathrm{osp}(\mathscr{F}) = \mathrm{o}(\mathrm{HYP}_{\underline{M}})$ which evidently is not a β-model.

Proof of Theorem 3.4: Assume, that $\underline{M} = \langle M, E\rangle$ is KM-expandable and that cf. $On^{\underline{M}} = \omega$. Let σ'_β: $\beta \in On$ be the class of formulas from Theorem 3.2 approximating the formula $\mathscr{G}\vec{x}\varphi_1(\vec{x})$ characterizing the expandability to a model for KM which is not a β-model. From the preceding lemma we conclude that, for every countable M, if $\langle \mathscr{F}, M, E^*\rangle \models \mathrm{KM}$, then $\langle M, E\rangle \models \mathscr{G}\vec{x}\varphi_1(\vec{x})$. This, together with the completeness theorem for formulas from language $(\mathscr{L}_{\mathrm{ZF}})_{\omega_1\omega}$ gives $(\forall \beta)_{<\omega_1}(\mathrm{KM} \vdash (\sigma'_\beta)^V)$. Applying the Lévy principle, we obtain $(\forall \beta)(\mathrm{KM} \vdash (\sigma'_\beta)^V)$. Thus $\langle M, E\rangle \models \mathscr{G}\vec{x}\varphi_1(\vec{x})$, and hence we infer that $\langle M, E\rangle \models \mathscr{G}\vec{x}\varphi_1(\vec{x})$. This consequently, means that $\langle M, E\rangle$ has an extension to KM which is not a β-model.

Note that an alternative proof is also possible. Namely, the part of the result 3.5 employed above can be expressed by a single formula from $(\mathscr{L}_{\mathrm{ZF}})_{\infty G}$ in the following form:

$$(\forall \underline{M})\{\underline{M} \text{ is countable} \Rightarrow \underline{M} \models [\textstyle\bigwedge\bigwedge \mathrm{ZF} \Rightarrow (\mathscr{G}\vec{x}\varphi(\vec{x}) \Rightarrow \mathscr{G}\vec{x}\varphi_1(\vec{x}))]\}.$$

Applying the Skolem–Löwenheim theorem for language $K_{\infty G}$ (proved by Moschovakis in [9]) in a version similar to the Lévy principle, we infer that

$$(\forall \underline{M})(\underline{M} \models [\textstyle\bigwedge\bigwedge \mathrm{ZF} \Rightarrow (\mathscr{G}\vec{x}\varphi(\vec{x}) \Rightarrow \mathscr{G}\vec{x}\varphi_1(\vec{x}))]).$$

Hence, we immediately obtain the required result.

The technique presented in this part of the paper may be useful in generalizations of theorems on countable models to the uncountable case. Moschovakis [9] used a similar idea when he proved the existence of countable structures with some property if there exists an uncountable model with that property, provided that the property can be expressed in the language $K_{\infty G}$ by a hereditary countable set of formulas.

The following theorems, proved orginally for the countable models, can be generalized to the case $\mathrm{cf}(M) = \omega$ by means of the above technique:

THEOREM (Felgner). *If* $\langle M, E\rangle \models \mathrm{ZFC}$, $\overline{\overline{M}} \leqslant \aleph_0$, *then there exists an* $A \in P(M)$ *such that* $\langle M, A, E\rangle \models \mathrm{ZFC}(A)$+"*A well orders* V".

THEOREM (Hutchinson): *If* $\langle M, E\rangle \models \mathrm{ZFC}$ *is* Δ^1_1-ZFC *expandable* (Δ^1_1-ZFC *is* GB+Δ^1_1-*comprehension schema*), *then* $\underline{M} = \langle M, E\rangle$ *has an elementary end extension with a first new ordinal "equal" to* $On^{\underline{M}}$.

The following theorem can also be proved.

THEOREM 3.6. *If* $\underline{M} = \langle M, E\rangle$ *is* KM-*expandable, cf.* $On^{\underline{M}} = \omega$, *then* $\underline{M}$ *has an expansion whose ordinal standard part is of a height equal to the height of* $\mathrm{HYP}_{\underline{M}}$.

We mentioned this result when discussing our Theorem 3.4.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW

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