Ultraproducts of L_1-predual spaces

by

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Abstract. It is shown that certain classes of Banach spaces whose duals are L_1-spaces behave well with respect to Banach ultraproducts and ultrapowers. In particular, a problem of Henson is solved: If some ultrapower of a Banach space E is a $C(K)$-space, then E itself is a $C(K)$-space. Combining the results with facts from model theory of Banach spaces, we get that some classes of L_1-preduals admit characterizations by elementary formulas.

Introduction. Ultraproducts turned out to be a useful tool in the local theory of Banach spaces. A central question arising in this framework is whether given classes of Banach spaces are closed under ultraproducts (or at least under ultrapowers). In connection with the further development of model-theoretic methods in Banach space theory, a converse question became significant, too: If an ultrapower of a Banach space belongs to a given class, does then the space itself belong to this class? It follows from results of Henson, Moore [6], Krivine [8], and Stern [19] that the class of L_p-spaces ($1 \leqslant p < \infty$) as well as the class of all L_1-preduals satisfy both conditions.

In this paper we examine the main special classes of L_1-preduals [12] from this point of view. While the (affirmative) answer to the first question is in most cases easily obtained, the converse question turns out to be more difficult and the situation looks more mixed. We prove that the converse question has a positive answer for the classes of $C(K)$-spaces (solving a problem posed by Henson [5]), $C_\Sigma(K)$-, $C_\sigma(K)$- and G-spaces. Furthermore, we give an example which shows that for $A(S)$- and $A_0(S)$-spaces the answer is negative.

Our results have some consequences of model-theoretic type such as the existence of a characterizing set of elementary formulas (i.e. norm-inequalities involving a finite number of elements) for $C(K)$-, $C_\sigma(K)$- and G-spaces. Questions like this were previously considered by Dacunha-Castelle, Krivine [1], [8], and Stern [19], who gave explicit characterizations by formulas of L_p-spaces and related structures. We also obtain that for the classes of $C(K)$-, $C_\Sigma(K)$-, $C_\sigma(K)$- and G-spaces a theorem of Löwenheim–Skolem type is valid.

Section 1 has preliminary character. We give there a brief exposition of the motivating facts from model theory of Banach spaces, due to Henson [5]. In Section 2, which contains the main results, we use geometrical methods. We carry out an analysis of the extreme point structure in ultraproducts and apply characterizations of L_1-preduals due to Lindenstrauss and Wulbert [13], [14].

1. Preliminaries. Model-theoretic aspects. Throughout the paper we consider real Banach spaces only. When speaking of a class of Banach spaces we always mean a class which is closed under isometries. The unit ball of a Banach space E is denoted by B_E, the dual space of E by E'. A Banach space E is called an *L_1-predual space* if E' is isometric to some $L_1(\mu)$. We shall consider the following classes of L_1-preduals whose definition we recall from [12]:

$C(K)$-spaces: The spaces of continuous functions on compact Hausdorff spaces K.

$C_0(K)$-spaces: The spaces of continuous functions on compact Hausdorff spaces K which vanish at a given point of K or, equivalently, the spaces of continuous functions on locally compact spaces which vanish at infinity.

$C_\sigma(K)$-spaces: The spaces of all continuous functions $x(t)$ on compact Hausdorff spaces K which satisfy $x(\sigma t) = -x(t)$ for all $t \in K$, where $\sigma\colon K \to K$ is a homeomorphism with $\sigma^2 =$ identity.

$C_\Sigma(K)$-spaces: Those $C_\sigma(K)$-spaces for which the homeomorphism is fixed point free.

M-spaces: The spaces which are isometric to closed sublattices of $C(K)$.

G-spaces: The spaces E for which there is a compact Hausdorff space K and a set of triples $(s_\alpha, t_\alpha, \lambda_\alpha)_{\alpha \in A}$ with $s_\alpha, t_\alpha \in K$, $\lambda_\alpha \in \boldsymbol{R}$, such that E is isometric to the space of all continuous functions $x(t)$ on K which satisfy $x(s_\alpha) = \lambda_\alpha x(t_\alpha)$ $(\alpha \in A)$.

$A(S)$-spaces: The spaces of affine continuous functions on compact Choquet simplexes S.

$A_0(S)$-spaces: The spaces of affine continuous functions on compact Choquet simplexes S which vanish at a given extreme point of S, or, equivalently, the simplex spaces [10].

All the function spaces are equipped with the supremum norm. A detailed treatment of these classes can be found in [12] and [14].

Let $(E_i)_{i \in I}$ be a family of Banach spaces and let $\mathfrak{U}$ be an ultrafilter on I. Let $l_\infty(I, E_i)$ be the spaces of all bounded families (x_i), $x_i \in E_i$ $(i \in I)$, equipped with the supremum norm. Then the *ultraproduct* $(E_i)_{\mathfrak{U}}$ is defined to be the quotient $l_\infty(I, E_i)/N_{\mathfrak{U}}$, where $N_{\mathfrak{U}}$ is the subspace of all those families (x_i), for which $\lim_{\mathfrak{U}} \|x_i\| = 0$. The element of $(E_i)_{\mathfrak{U}}$ which is represented by the family (x_i) is denoted by $(x_i)_{\mathfrak{U}}$. When all the spaces E_i are identical with a certain E, we speak of an *ultrapower*. For information on the structure of Banach space ultraproducts we refer to [1], [4], [20] and to the survey [3].

Model theory for Banach spaces has been developed by Krivine [8], Henson [5] and Stern [19]. We will follow the approach of [5]: The first-order language $\mathfrak{L}$ consists of a binary function symbol $+$, a unary function symbol $r\cdot$ for each rational number r, and two predicate symbols P and Q. A Banach space E can be regarded

as an $\mathfrak{L}$-structure by interpreting $+$ as the vector addition, $r\cdot$ as the multiplication by the scalar r, P as $\{x \in E\colon \|x\| \leqslant 1\}$ and Q as $\{x \in E\colon \|x\| \geqslant 1\}$. A *positive bounded sentence* is (up to equivalence with respect to the elementary theory of the class of all Banach spaces) an expression of the form

$$Q_1^B x_1 \ldots Q_n^B x_n [(\varrho_1 \wedge \ldots \wedge \varrho_{m_1}) \vee \ldots \vee (\varrho_{m_{k-1}+1} \wedge \ldots \wedge \varrho_{m_k})]\,.$$

where ϱ_i is an atomic formula, i.e. either $P(t_i)$, $Q(t_i)$, or $t_i = 0$, t_i is a term $r_{i1}x_1 + \ldots + r_{in}x_n$ and $Q_i^B x_i$ stands for one of the restricted quantifiers $(\forall x_i)(P(x_i) \to \ldots)$ and $(\exists x_i)(P(x_i) \wedge \ldots)$. Thus, a positive bounded sentence symbolizes a system of norm estimates involving a finite number of variables whose domain is the unit ball. Given a positive bounded sentence σ as above and a rational ε with $0 < \varepsilon < 1$, the *ε-approximation* σ^ε is defined by replacing (for all i) $P(t_i)$, $Q(t_i)$ and $t_i = 0$ by $P((1-\varepsilon)t_i)$, $Q((1+\varepsilon)t_i)$ and $P((1/\varepsilon)t_i)$ respectively. A Banach space E *satisfies σ approximately* (denoted by $E \models_A \sigma$) if $E \models \sigma^\varepsilon$ for all rational $\varepsilon \in (0, 1)$. Two Banach spaces E and F are said to be *finitely equivalent* ($E \equiv_A F$) if, for each positive bounded sentence σ, $E \models_A \sigma$ if and only if $F \models_A \sigma$. The following results from [5] show the connection between model-theoretic properties of a class of Banach spaces and its behaviour with respect to ultraproducts, which we shall investigate below.

THEOREM 1.1. *Let $\mathfrak{B}$ be a class of Banach spaces. The following statements are equivalent*:

(i) *$\mathfrak{B}$ is closed under finite equivalence.*

(ii) *For each Banach space E and each ultrafilter $\mathfrak{U}$, $E \in \mathfrak{B}$ if and only if $(E)_{\mathfrak{U}} \in \mathfrak{B}$.*

THEOREM 1.2. (Löwenheim, Skolem) *Let $\mathfrak{B}$ be a class of Banach spaces which is closed under finite equivalence, let $E \in \mathfrak{B}$ and $\varkappa$ be an infinite cardinal. Then for each subspace $F \subset E$ of density character $\varkappa$ there is a subspace G of the same density character $\varkappa$ such that $F \subset G \subset E$, $G \in \mathfrak{B}$, and $G \equiv_A E$.*

THEOREM 1.3. *Let $\mathfrak{B}$ be a class of Banach spaces. The following statements are equivalent*:

(i) *$\mathfrak{B}$ is closed under ultraproducts, and for each Banach space E and each ultrafilter $\mathfrak{U}$, $(E)_{\mathfrak{U}} \in \mathfrak{B}$ implies $E \in \mathfrak{B}$.*

(ii) *There is a set of positive bounded sentences Σ which characterizes $\mathfrak{B}$ approximately within the class of all Banach spaces, i.e. a Banach space E belongs to $\mathfrak{B}$ if and only if $E \models_A \sigma$ for all $\sigma \in \Sigma$.*

In a sense, statement (ii) of Theorem 1.3 says that $\mathfrak{B}$ admits a "local" characterization. Note also that (ii) implies the existence of a set of positive bounded formulas Σ_1 which (strictly) characterizes $\mathfrak{B}$, that means $E \in \mathfrak{B}$ if and only if $E \models \sigma$ for all $\sigma \in \Sigma_1$. Obviously, $\Sigma_1 = \{\sigma^\varepsilon\colon \sigma \in \Sigma,\ \varepsilon \text{ rational},\ 0 < \varepsilon < 1\}$ is such a set.

2. The geometry of ultraproducts of L_1-preduals. We begin with a result which shows that most of the classes of L_1-preduals behave stably under ultraproducts.

PROPOSITION 2.1. *The following classes are closed under ultraproducts: $C(K)$-spaces, $C_0(K)$-spaces, $C_\sigma(K)$-spaces, M-spaces, G-spaces, $A(S)$-spaces, $A_0(S)$-spaces, L_1-predual spaces.*

Proof. Many statements are known or easy consequences of known results: It was shown in [1] that the ultraproduct of Banach lattices is, in a natural way, a Banach lattice. This implies the result for $C(K)$- and M-spaces, as observed in [1], [6], [19]. The case of L_1-preduals (contained in Th. 2.2 of [6] and Th. 7.5 of [19]) follows from the case of $C(K)$-spaces and the fact that a Banach space is an L_1-predual if and only if it is an $\mathfrak{L}_{\infty,\lambda}$-space for all $\lambda>1$ [11].

If a class $\mathfrak{B}$ is closed under ultraproducts, then the same is true for the class $\pi(\mathfrak{B})$, i.e. the class of all spaces E which are isometric to norm-one-complemented subspaces of members of $\mathfrak{B}$. We therefore get the desired result for the classes $C_\sigma(K) = \pi(C(K))$ and $G = \pi(M)$ (cf. [14]). The relation $C_0(K) = C_\sigma(K) \cap M$ ([14]) shows that the class $C_0(K)$ is closed under ultraproducts, as well.

Let now $(E_i)_{i\in I}$ be a family of simplex spaces. Then E_i can be endowed with a partial order, generated by a cone C_i, such that C_i is 1-normal, (E_i, C_i) has the decomposition property, and the open unit ball $B^0_{E_i}$ is directed upwards (cf. [10], § 19). We will show that $((E_i)_\mathfrak{U}, C)$ has the same order properties, where $C = (C_i)_\mathfrak{U} = \{(x_i)_\mathfrak{U}\colon x_i \in C_i,\ i\in I\}$. First of all, C is a cone. Obviously, $C+C\subset C$ and $\lambda C\subset C$ $(\lambda>0)$. Assume that $x\in C\cap(-C)$, and let $x = (x_i)_\mathfrak{U} = (\tilde{x}_i)_\mathfrak{U}$, $x_i\in C_i$, $\tilde{x}_i\in(-C_i)$ $(i\in I)$. Then $\tilde{x}_i - x_i \leqslant x_i \leqslant x_i-\tilde{x}_i$ and, since C_i is 1-normal, this implies $\|x_i\|\leqslant\|x_i-\tilde{x}_i\|$. Consequently, $\lim_\mathfrak{U}\|x_i\| = 0$, thus $x = 0$.

Next we prove that C is closed. Let $\{x^n\}_{n=1}^\infty$ be a sequence in C which satisfies $\|x^n-x^{n+1}\|<2^{-n}$ for all n. We choose inductively representations $x^n = (x_i^n)_\mathfrak{U}$ with $x_i^n\in C_i$ and $\|x_i^n-x_i^{n+1}\|<2^{-n}$ $(i\in I,\ n = 1,2,\ldots)$. By definition, x^1 has a representation $x^1 = (x_i^1)_\mathfrak{U}$, $x_i^1\in C_i$ $(i\in I)$. Assume that $x^n = (x_i^n)_\mathfrak{U}$ has already been found and let $x^{n+1} = (\tilde{x}_i^{n+1})_\mathfrak{U}$ with $\tilde{x}_i^{n+1}\in C_i$ $(i\in I)$. Now set $x_i^{n+1} = \tilde{x}_i^{n+1}$ if $\|x_i^n-\tilde{x}_i^{n+1}\| <2^{-n}$, $x_i^{n+1} = x_i^n$ otherwise. This yields the desired representations, and the result follows, since the C_i's are closed.

To verify the decomposition property, let $x, y^1, y^2\in C$, $x\leqslant y^1+y^2$. Take representations $x = (x_i)_\mathfrak{U}$, $y^1 = (y_i^1)_\mathfrak{U}$, $y^2 = (y_i^2)_\mathfrak{U}$ with $x_i, y_i^1, y_i^2\in C_i$ $(i\in I)$. By the definition of C, there exist $z_i\in E_i$ with $\lim_\mathfrak{U}\|z_i\| = 0$ and $x_i\leqslant y_i^1+y_i^2+z_i$. Since $B^0_{E_i}$ is directed upwards, we may assume $z_i\geqslant 0$. Now the decomposition property of (E_i, C_i) yields the result. Finally, it is easy to see that the open unit ball of $(E_i)_\mathfrak{U}$ is directed upwards and that C is 1-normal. Thus, $(E_i)_\mathfrak{U}$ is a simplex space.

If each (E_i, C_i) has a strong order unit e_i, then $(e_i)_\mathfrak{U}$ is a strong order unit of $((E_i)_\mathfrak{U}, C)$. This implies the result for $A(S)$-spaces (cf. [10], § 2, Th. 6 and § 19, Th. 2).

Remark. Later on we prove that the class of $C_\Sigma(K)$-spaces is closed under ultrapowers, but we do not know whether this is true for ultraproducts.

We now turn to the main subject of this section, the question whether $(E)_\mathfrak{U}\in\mathfrak{B}$ implies $E\in\mathfrak{B}$. This is the case for the full class of L_1-preduals, as the following result shows.

PROPOSITION 2.2. *If an ultrapower $(E)_\mathfrak{U}$ of a Banach space E is an L_1-predual, then E is an L_1-predual.*

Proof. Using standard ultrapower-techniques (or Th. 2.2 of [6], Th. 7.5 of [19]), it can be seen that E is an $\mathfrak{L}_{\infty,\lambda}$-space for all $\lambda>1$. As already mentioned, this implies that E is an L_1-predual [11].

For special classes of L_1-preduals the question becomes more complicated since we cannot use additional structures as e.g. lattice orderings. (It is not even known whether, if $(E)_\mathfrak{U}$ is a lattice, there is any lattice structure on E, cf. [5], p. 131). We intend to apply characterizations of L_1-preduals by the extreme point structure of the spaces and their duals ([13], [14]). Yet, in general, an element of an ultrapower which is generated by a family of extreme points is not necessarily an extreme point itself. Conversely, an extreme point of an ultrapower needs not to be generated by a family of extreme points. To overcome this difficulty, we introduce an auxiliary notion which at the same time approximates and uniformizes the behaviour of extreme points (compare the definition of local uniform rotundity [2], VII, § 2).

DEFINITION 2.3. Let E be a Banach space and let $\delta>0$, $\varepsilon>0$ be reals. An element $x\in B_E$ is called a *(δ,ε)-exposed point of B_E* if, for each pair $y, z\in B_E$, $\|x-\frac{1}{2}(y+z)\|<\delta$ implies $\|y-z\|<\varepsilon$.

In what follows we undertake a study of these points, which will be crucial for the proofs of the main results. The next lemma shows that in certain spaces (δ,ε)-exposed points are close to extreme points.

LEMMA 2.4. *Let E be a $C(K)$-space or an L_1-space. Then the following hold:*

(i) *If $x\in \operatorname{ext} B_E$, then x is $(\frac{1}{2}\varepsilon,\varepsilon)$-exposed for all $\varepsilon>0$.*

(ii) *If $\varepsilon\leqslant\frac{1}{2}$ and $x\in B_E$ is (δ,ε)-exposed, then there exists an $x_0\in\operatorname{ext} B_E$ with $\|x_0-x\|<\frac{1}{2}\varepsilon$.*

Proof. Assume first that $E = C(K)$. A function $x(t)$ is an extreme point of $C(K)$ if and only if $|x(t)| = 1$ for all $t\in K$. Using this, it is elementary to verify (i). To check (ii), suppose $x = x(t)\in C(K)$ is (δ,ε)-exposed. It clearly suffices to show that $|x(t)|>1-\frac{1}{2}\varepsilon$ $(t\in K)$. Assume that this is not the case and let $t_0\in K$ be such that $|x(t_0)|<1-\frac{1}{2}\varepsilon$. Urysohn's lemma implies that there exists a function $h\in C(K)$ with $\|h\| = \frac{1}{2}\varepsilon$, $h(t_0) = \frac{1}{2}\varepsilon$ and $h(t) = 0$ for all those $t\in K$ which satisfy $(1-\frac{1}{2}\delta)|x(t)|\geqslant 1-\frac{1}{2}\varepsilon$. Setting

$$y = (1-\tfrac{1}{2}\delta)x+h,\qquad z = (1-\tfrac{1}{2}\delta)x-h,$$

we obtain a contradiction to the assumption that x is (δ,ε)-exposed.

Now let $E = L_1(\Omega,\mathfrak{A},\mu)$. The extreme points of B_E are exactly the elements of the form $x = \pm\mu(A)^{-1}\chi_A$, where A is an atom of the measure algebra (i.e. $A_1 \in \mathfrak{A}$ and $A_1 \subset A$ imply $\mu(A_1) = 0$ or $\mu(A \setminus A_1) = 0$), and χ_A denotes the characteristic function. The extreme points of the dual $E' = L_\infty(\mu)$, in turn, are the functions $f(\omega)$ with $|f(\omega)| = 1$ almost everywhere. Therefore, if $x \in \operatorname{ext} B_E$, $f \in \operatorname{ext} B_{E'}$, then $|\langle x, f\rangle| = 1$. Given $y, z \in B_E$ with $\|x - \frac{1}{2}(y+z)\| < \frac{1}{2}\varepsilon$, we get

$$|\langle \tfrac{1}{2}(y+z), f\rangle| > 1 - \tfrac{1}{2}\varepsilon$$

and, as an elementary consequence,

$$|\langle y-z, f\rangle| < \varepsilon$$

for all $f \in \operatorname{ext} B_{E'}$. Therefore $\|y-z\| < \varepsilon$, which proves (i).

Assume that $x \in B_E$ is (δ, ε)-exposed. Let x_A denote the restriction of the function x to a set $A \in \mathfrak{A}$, that means $x_A(\omega) = \chi_A(\omega)x(\omega)$ $(\omega \in \Omega)$. We first show that for all $A \in \mathfrak{A}$,

(1) $\quad$ if $\|x_A\| \geq \|x_{\Omega\setminus A}\|$, then $\big\|x - x_A/\|x_A\|\big\| < \frac{1}{2}\varepsilon$.

Indeed, we set $y = x_A/\|x_A\|$, $z = 2x - y$. Then

$$\|z\| = \big\|2x - x_A/\|x_A\|\big\| = \big|2\|x_A\| - 1\big| + 2\|x_{\Omega\setminus A}\| \leq 1$$

and (1) follows from the fact that x is (δ, ε)-exposed. We get from (1) that

$$\tfrac{1}{2}\varepsilon > \big\|x - x_A/\|x_A\|\big\| = 1 - \|x_A\| + \|x_{\Omega\setminus A}\| \geq 2\|x_{\Omega\setminus A}\|.$$

This shows that also the following holds:

(2) $\quad$ if $\|x_A\| \geq \|x_{\Omega\setminus A}\|$, then $\|x_A\| \geq \|x\| - \frac{1}{4}\varepsilon$.

Now define

$$\mathfrak{A}_0 = \{A \in \mathfrak{A}:\ \|x_A\| \geq \|x\| - \tfrac{1}{4}\varepsilon\}.$$

Given $A_1, A_2 \in \mathfrak{A}_0$, we have

$$\|x_{A_1 \cap A_2}\| = \|x_{A_1}\| + \|x_{A_2}\| - \|x_{A_1 \cup A_2}\| \geq \|x\| - \tfrac{1}{2}\varepsilon.$$

Since $\varepsilon \leq \frac{1}{2}$ and (obviously) $\|x\| > 1 - \frac{1}{2}\varepsilon$, this implies

$$\|x_{A_1 \cap A_2}\| > 1 - \varepsilon \geq \tfrac{1}{2} > \|x_{\Omega\setminus(A_1 \cap A_2)}\|.$$

Applying (2), we get $A_1 \cap A_2 \in \mathfrak{A}_0$. This allows to find a sequence $A_1 \supseteq A_2 \supseteq \ldots$ with $A_n \in \mathfrak{A}_0$ $(n = 1, 2, \ldots)$ and

$$\lim_{n\to\infty} \|x_{A_n}\| = \inf\{\|x_A\|:\ A \in \mathfrak{A}_0\}.$$

We define $B = (\bigcap A_n) \setminus \Omega_0$, where $\Omega_0 = \{\omega \in \Omega:\ x(\omega) = 0\}$. By continuity, $B \in \mathfrak{A}_0$, and it is easy to see that B is an atom. Finally, we have $\|x_B\| \geq \|x_{\Omega\setminus B}\|$. Therefore, setting $x_0 = x_B/\|x_B\|$, it follows from (1) that $\|x - x_0\| < \frac{1}{2}\varepsilon$. This concludes the proof.

Remark. Lemma 2.4 (i) holds for arbitrary (real) L_1-preduals. In this case E can be embedded isometrically into some $C(K)$ in such a way, that a given extreme point of B_E is mapped onto the function identically equal to 1 (cf. [13], the proof of Th. 6.1). Part (ii) fails for general L_1-preduals, as shows Example 2.14 below.

The following lemma clarifies the behaviour of (δ, ε)-exposed points under ultraproducts.

LEMMA 2.5. *Let $(E_i)_{i\in I}$ be a family of Banach spaces, let x be an element of the unit ball of $(E_i)_\mathfrak{U}$, and let $x = (x_i)_\mathfrak{U}$ be a representation with $x_i \in B_{E_i}$ $(i \in I)$. Then the following hold:*

(i) *If each x_i $(i \in I)$ is a (δ, ε)-exposed point of B_{E_i}, then x is a $(\delta, \varepsilon+\eta)$-exposed point of $B_{(E_i)_\mathfrak{U}}$ for each $\eta > 0$.*

(ii) *If x is a (δ, ε)-exposed point of $B_{(E_i)_\mathfrak{U}}$, then for each $\eta > 0$ there exists a set $I_0 \in \mathfrak{U}$ such that, for $i \in I_0$, x_i is a $(\delta-\eta, \varepsilon)$-exposed point of B_{E_i}.*

Proof. (i) Let $y, z \in B_{(E_i)_\mathfrak{U}}$ and assume that $\|x - \frac{1}{2}(y+z)\| < \delta$. Choose representations $y = (y_i)_\mathfrak{U}$ and $z = (z_i)_\mathfrak{U}$ with $y_i, z_i \in B_{E_i}$ $(i \in I)$. Then there is a set $I_0 \in \mathfrak{U}$ such that $\|x_i - \frac{1}{2}(y_i+z_i)\| < \delta$ for all $i \in I_0$. The hypothesis implies $\|y_i - z_i\| < \varepsilon$ $(i \in I_0)$, from which the result follows.

(ii) Assume that the statement is false for a certain $\eta > 0$. Then the set

$$I_1 = \{i:\ x_i \text{ is not } (\delta-\eta, \varepsilon)\text{-exposed}\}$$

belongs to $\mathfrak{U}$. For each $i \in I_1$ there exist elements $y_i, z_i \in B_{E_i}$ with

$$\|x_i - \tfrac{1}{2}(y_i+z_i)\| < \delta - \eta, \qquad \|y_i - z_i\| \geq \varepsilon.$$

Setting $y = (y_i)_\mathfrak{U}$, $z = (z_i)_\mathfrak{U}$, we obtain

$$\|x - \tfrac{1}{2}(y+z)\| < \delta, \qquad \|y - z\| \geq \varepsilon,$$

which is a contradiction.

The above two lemmas enable us to give a partial description of the extreme points in the dual $(E_i)'_\mathfrak{U}$ when the E_i are L_1-preduals. Recall that the product of the duals $(E'_i)_\mathfrak{U}$ can be identified isometrically with a subspace of $(E_i)'_\mathfrak{U}$, though not necessarily (and in the case of L_1-preduals E_i definitely not) with the whole space (cf. [3], Section 7).

PROPOSITION 2.6. *Let $(E_i)_{i\in I}$ be a family of L_1-preduals, and let $f \in (E'_i)_\mathfrak{U} \subset (E_i)'_\mathfrak{U}$. Then f is an extreme point of the unit ball of $(E_i)'_\mathfrak{U}$ if and only if f possesses a representation $f = (f_i)_\mathfrak{U}$ with $f_i \in \operatorname{ext} B_{E'_i}$ $(i \in I)$.*

Proof. Assume that f has a representation $f = (f_i)_\mathfrak{U}$, $f_i \in \operatorname{ext} B_{E'_i}$ $(i \in I)$. Then, by Lemma 2.4, each f_i is $(\frac{1}{2}\varepsilon, \varepsilon)$-exposed for all $\varepsilon > 0$. Lemma 2.5, in turn, shows that f is an $(\frac{1}{2}\varepsilon, 2\varepsilon)$-exposed point of the unit ball of $(E'_i)_\mathfrak{U}$. We will prove that f is also exposed with respect to the ball of the whole dual space $(E_i)'_\mathfrak{U}$. By the local duality of ultraproducts (cf. [20], [9], or Th. 7.6 of [3]) there is an ultrafilter $\mathfrak{B}$ (on a certain index set) and an isometry from $(E_i)'_\mathfrak{U}$ into $((E'_i)_\mathfrak{U})_\mathfrak{B}$ such that the restriction of the isometry to $(E_i')_{\mathfrak{U}}$ coincides with the canonical embedding of $(E_i')_{\mathfrak{U}}$ into its ultrapower $((E_i')_{\mathfrak{U}})_{\mathfrak{V}}$. Thus, f is mapped onto $(f)_{\mathfrak{V}}$, which is, by the above and Lemma 2.5, an $(\frac{1}{2}\varepsilon, 3\varepsilon)$-exposed point of the ball of $((E_i')_{\mathfrak{U}})_{\mathfrak{V}}$, and therefore also of the ball of $(E_i)'_{\mathfrak{U}}$. Obviously, this implies that f is an extreme point.

To prove the converse direction, assume that f is an extreme point of $B_{(E_i)'_{\mathfrak{U}}}$. Then f is also an extreme point of the ball of $(E_i')_{\mathfrak{U}}$. The latter space is an L_1-space. Therefore Lemma 2.4 yields that f is $(\frac{1}{2}\varepsilon, \varepsilon)$-exposed for all $\varepsilon>0$. Let $f = (\tilde{f}_i)_{\mathfrak{U}}$ be a representation with $\tilde{f}_i \in B_{E_i'}$. For each $i \in I$ define

$$\varepsilon_i = \inf\{\varepsilon>0\colon \tilde{f}_i \text{ is } (\tfrac{1}{2}\varepsilon, \varepsilon)\text{-exposed}\}.$$

By Lemma 2.5, we have

$$\lim_{\mathfrak{U}} \varepsilon_i = 0.$$

Now we define the desired representation $f = (f_i)_{\mathfrak{U}}$. If $0<\varepsilon_i\leqslant\frac{1}{2}$, then we can use Lemma 2.4 to find an extreme point $f_i \in B_{E_i'}$ such that $\|f_i-\tilde{f}_i\|<\varepsilon_i$. If $\varepsilon_i = 0$, then we set $f_i = \tilde{f}_i$, since in this case $\tilde{f}_i$ is an extreme point itself. For $\varepsilon_i>\frac{1}{2}$ (which happens only outside a certain set $I_0 \in \mathfrak{U}$) we take f_i to be an arbitrary extreme point of $B_{E_i'}$. This concludes the proof.

We are now ready to pass on to the main results of the paper.

THEOREM 2.7. *Let E be a Banach space and let $\mathfrak{U}$ be an ultrafilter. If the ultrapower $(E)_{\mathfrak{U}}$ is a G-space, then E itself is a G-space.*

Proof. We shall show that, given $x, y \in E$, there exists a $z \in E$ with

$$\langle z,f\rangle = \max\{\langle x,f\rangle, \langle y,f\rangle, 0\}+\min\{\langle x,f\rangle, \langle y,f\rangle, 0\}$$

for all $f \in \operatorname{ext} B_{E'}$. By Theorem 2 of [14], this property characterizes the G-spaces. By assumption, $(E)_{\mathfrak{U}}$ is a G-space, therefore we can find an element $(z_i)_{\mathfrak{U}} \in (E)_{\mathfrak{U}}$ with

$$(1)\quad \langle (z_i)_{\mathfrak{U}}, (f_i)_{\mathfrak{U}}\rangle = \max\{\langle (x)_{\mathfrak{U}}, (f_i)_{\mathfrak{U}}\rangle, \langle (y)_{\mathfrak{U}}, (f_i)_{\mathfrak{U}}\rangle, 0\}+ +\min\{\langle (x)_{\mathfrak{U}}, (f_i)_{\mathfrak{U}}\rangle, \langle (y)_{\mathfrak{U}}, (f_i)_{\mathfrak{U}}\rangle, 0\}$$

for each family (f_i) with $f_i \in \operatorname{ext} B_{E'}$ (by Lemma 2.6 such a family generates an extreme point of the ball of $(E)'_{\mathfrak{U}}$). This yields, in particular,

$$(2)\quad \lim_{\mathfrak{U}} \langle z_i,f\rangle = \max\{\langle x,f\rangle, \langle y,f\rangle, 0\}+\min\{\langle x,f\rangle, \langle y,f\rangle, 0\}$$

for all $f \in \operatorname{ext} B_{E'}$. We show next that the convergence in (2) is uniform on the set of all $f \in \operatorname{ext} B_{E'}$. Indeed, if this were not the case, we could find an $\varepsilon>0$ such that the set

$$\{i\colon \text{there exists an } f_i \in \operatorname{ext} B_{E'} \text{ with}$$
$$|\langle z_i,f_i\rangle-\max\{\langle x,f_i\rangle, \langle y,f_i\rangle, 0\}-\min\{\langle x,f_i\rangle, \langle y,f_i\rangle, 0\}|\geqslant\varepsilon\}$$

belongs to $\mathfrak{U}$. But this clearly contradicts (1). The uniform convergence in (2) implies that for each $\varepsilon>0$ there is a set $I_\varepsilon \in \mathfrak{U}$ such that

$$\|z_i-z_j\|<\varepsilon \quad (i,j \in I_\varepsilon).$$

This means that the filter of subset of E which is generated by the family $(\{z_j\colon j \in J\})_{J\in\mathfrak{U}}$ is a Cauchy filter with respect to the norm topology. Since E is complete, this filter converges. In other words, there is a $z \in E$ with

$$\lim_{\mathfrak{U}} \|z_i-z\| = 0.$$

Clearly, (2) implies that z is the element we were looking for.

THEOREM 2.8. *Let E be a Banach space and let $\mathfrak{U}$ be an ultrafilter. If $(E)_{\mathfrak{U}}$ is a $C_\sigma(K)$-space, then E is a $C_\sigma(K)$-space. If $(E)_{\mathfrak{U}}$ is a $C_\Sigma(K)$-space, then E is a $C_\Sigma(K)$-space.*

Proof. Assume that $(E)_{\mathfrak{U}}$ is a $C_\Sigma(K)$-space. Then, by Proposition 2.2, E is a predual of L_1. The set of extreme points of the ball of $(E)'_{\mathfrak{U}}$ is w^*-compact, and it suffices to show that the same holds for $\operatorname{ext} B_{E'}$ ([14], Th. 1). Let f_0 be an element of the w^*-closure of $\operatorname{ext} B_{E'}$ and set $f^0 = (f_0)_{\mathfrak{U}}$. Let $x^1, \ldots, x^n$ be elements of $(E)_{\mathfrak{U}}$ with representations $x^k = (x_i^k)_{\mathfrak{U}}$ $(k = 1, \ldots, n)$, and let $\eta>0$. Then we can find for each $i \in I$ an $f_i \in \operatorname{ext} B_{E'}$ with

$$|\langle x_i^k,f_0\rangle-\langle x_i^k,f_i\rangle|\leqslant\eta \quad (k = 1, \ldots, n).$$

Setting $f = (f_i)_{\mathfrak{U}}$, we get

$$|\langle x^k,f^0\rangle-\langle x^k,f\rangle|\leqslant\eta \quad (k = 1, \ldots, n).$$

Since, by Proposition 2.6, f is an extreme point of $B_{(E)'_{\mathfrak{U}}}$, it follows that f^0 belongs to the w^*-closure of $\operatorname{ext} B_{(E)'_{\mathfrak{U}}}$. Thus, $f^0 = (f_0)_{\mathfrak{U}}$ is an extreme point of $B_{(E)'_{\mathfrak{U}}}$. Lemmas 2.4 and 2.5 show that f_0 is $(\frac{1}{3}\varepsilon, \varepsilon)$-exposed for each $\varepsilon>0$, consequently $f_0 \in \operatorname{ext} B_{E'}$.

If $(E)_{\mathfrak{U}}$ is a $C_\sigma(K)$-space, then the w^*-closure of $\operatorname{ext} B_{(E)'_{\mathfrak{U}}}$ is contained in $\operatorname{ext} B_{(E)'_{\mathfrak{U}}} \cup \{0\}$. The considerations presented above yield that the corresponding statement holds for $\operatorname{ext} B_{E'}$, and the result follows from the corollary on p. 218 of [10].

Using a similar argument, we now complement the results of Proposition 2.1.

PROPOSITION 2.9. *The class of $C_\Sigma(K)$-spaces is closed under ultrapowers.*

Proof. If E is a $C_\Sigma(K)$-space, then, by Proposition 2.1, $(E)_{\mathfrak{U}}$ is a $C_\sigma(K)$-space. Now assume that zero belongs to the w^*-closure of $\operatorname{ext} B_{(E)'_{\mathfrak{U}}}$. The set of all elements $(f_i)_{\mathfrak{U}}$ with $f_i \in \operatorname{ext} B_{E'}$ $(i \in I)$ is norming for $(E)_{\mathfrak{U}}$ and therefore w^*-dense in $\operatorname{ext} B_{(E)'_{\mathfrak{U}}}$ (cf. [2], V § 1). It follows that, given $x_1, \ldots, x_n \in E$, $\eta>0$, we can find $f_i \in \operatorname{ext} B_{E'}$ $(i \in I)$ such that

$$|\langle (x_k)_{\mathfrak{U}}, (f_i)_{\mathfrak{U}}\rangle|<\eta \quad (k = 1, \ldots, n).$$

Consequently, there exists a set $I_0 \in \mathfrak{U}$ with

$$|\langle x_k, f_i\rangle| < \eta \quad (k = 1, \ldots, n,\ i \in I_0).$$

This shows that the element 0 belongs to the w^*-closure of $\operatorname{ext} B_{E'}$, contradicting the assumption that E is a $C_\Sigma(K)$-space.

Our next result answers a question posed by Henson ([5], p. 130, question (1)).

THEOREM 2.10. *Let E be a Banach space and $\mathfrak{U}$ be an ultrafilter. If $(E)_\mathfrak{U}$ is a $C(K)$-space, then E is a $C(K)$-space.*

Proof. If $(E)_\mathfrak{U}$ is a $C(K)$-space, then E is a predual of L_1 and the set $\operatorname{ext} B_{E'}$ is w^*-closed (Theorem 2.8). By Theorem 6.6 of [13] it remains to show that B_E has an extreme point. We construct inductively a sequence $\{x_n\}_{n=1}^{\infty} \subset B_E$ with the following properties: For each n,

(1) x_n is a $(2^{-(k+3)}, 2^{-(k+1)})$-exposed point of B_E $(k = 1, 2, \ldots, n)$,

(2) $\|x_n - x_{n+1}\| < 2^{-(n+1)}$.

The construction starts as follows: $B_{(E)_\mathfrak{U}}$ has an extreme point, say $z^1 = (z_i^1)_\mathfrak{U}$, $z_i^1 \in B_E$ $(i \in I)$. By Lemma 2.5, there exists an index $i \in I$ such that z_i^1 is $(\frac{1}{16}, \frac{1}{4})$-exposed. We define $x_1 = z_i^1$. Now assume that $x_1, \ldots, x_n$ have already been found. Lemma 2.5 shows that the element $x^n = (x_n)_\mathfrak{U}$ is a $(2^{-(n+3)}, 2^{-n})$-exposed point of $B_{(E)_\mathfrak{U}}$. Using Lemma 2.4, we can find an extreme point z^{n+1} of $B_{(E)_\mathfrak{U}}$ with

$$\|x^n - z^{n+1}\| < 2^{-(n+1)}.$$

Take a representation $z^{n+1} = (z_i^{n+1})_\mathfrak{U}$, $z_i^{n+1} \in B_E$. It follows that there is a set $I_0 \in \mathfrak{U}$ with

$$\|x_n - z_i^{n+1}\| < 2^{-(n+1)} \quad (i \in I_0).$$

On the other hand, Lemma 2.5 implies the existence of sets $I_1, I_2, \ldots, I_{n+1} \in \mathfrak{U}$ such that z_i^{n+1} is $(2^{-(k+3)}, 2^{-(k+1)})$-exposed for each $i \in I_k$. Now pick an index $i \in \bigcap_{k=0}^{n+1} I_k$ and set $x_{n+1} = z_i^{n+1}$. This completes the induction.

A simple geometric argument shows that the limit of a sequence of (δ, ε)-exposed points is $(\delta, \varepsilon+\eta)$-exposed for each $\eta \geq 0$. Therefore $x = \lim_{n\to\infty} x_n$ is a $(2^{-(n+3)}, 2^{-n})$-exposed point of B_E for all $n \geq 1$. Hence x is an extreme point.

Remark. Theorem 2.10 remains valid for complex $C(K)$-spaces. In this case one has to use the analogous characterization for complex $C(K)$-spaces, due to Hirsberg and Lazar ([7], Cor. 3.4). It is easy to see that Lemma 2.4 holds for complex spaces if we replace in part (i) the phrase "$(\frac{1}{2}\varepsilon, \varepsilon)$-exposed" by "$(1-(1-\frac{1}{4}\varepsilon^2)^{1/2}, \varepsilon)$-exposed". Now those considerations which were necessary for the proof of 2.10 can be carried over to the complex case without difficulty.

We are now in the position to apply the model-theoretic results of Section 1.

COROLLARY 2.11. *The classes of $C(K)$-, $C_\Sigma(K)$-, $C_\sigma(K)$- and G-spaces are closed under finite equivalence. Consequently, they satisfy the Löwenheim–Skolem theorem (1.2).*

For the class of $C(K)$-spaces this result was obtained by Henson ([5], Th. 3.9) using lattice arguments. A construction closely related to ultrapowers is the *nonstandard hull* $\hat{E}$ of a Banach space E. For the definition we refer to [17], [15], or [5]. (As in [5] we shall assume that the nonstandard hull is constructed with respect to an $\aleph_1$-saturated extension $\mathfrak{M}^*$ of a set-theoretical structure $\mathfrak{M}$.)

COROLLARY 2.12. *If the nonstandard hull $\hat{E}$ of a Banach space E is a $C(K)$-space ($C_\Sigma(K)$-, $C_\sigma(K)$-, G-space, respectively), then E is a $C(K)$-space ($C_\Sigma(K)$-, $C_\sigma(K)$-, G-space, resp.).*

Proof. By Corollary 2.11 and the fact that $\hat{E}$ is finitely equivalent to E ([5], Cor. 1.11).

Finally, we obtain the desired existence of characterizing sets of sentences:

COROLLARY 2.13. *The classes of $C(K)$-, $C_\sigma(K)$- and G-spaces can be characterized approximately (and also strictly) by sets of positive bounded sentences.*

Remark. Let $\mathfrak{B}$ be one of the classes $C(K)$, $C_\sigma(K)$ or G, and let Σ be the set of all positive bounded sentences which are satisfied by each member of $\mathfrak{B}$. Then it follows from Corollary 2.13 that Σ characterizes $\mathfrak{B}$. We do not know a characterizing set of sentences for $\mathfrak{B}$ which admits a more explicit description. For the class of all L_1-predual spaces such a set was given by Stern ([19], Th. 7.5) involving sentences of a slightly different type.

We conclude by giving a counterexample simultaneously for the classes of $A_0(S)$- and $A(S)$-spaces.

EXAMPLE 2.14. *There exists a separable L_1-predual E with the following two properties:*

(i) *For each $\varepsilon > 0$ there is a $(\frac{1}{5}\varepsilon, \varepsilon)$-exposed point in B_E.*

(ii) *B_E has no extreme points.*

Consequently, E is not an $A_0(S)$-space, but has an ultrapower $(E)_\mathfrak{U}$ which is an $A(S)$-space.

Proof. We use the method of matrix representations (cf. [16], [12]). Define a triangular matrix $A = \{a_{i,n}\}_{n=1,2,\ldots}^{1 \leq i \leq n}$ by

$$a_{i,n} = \begin{cases} 0 & \text{if } n \text{ is odd and } i = n, \\ (-1)^{i+1} 2^{-[(i+3)/2]} & \text{otherwise} \end{cases}$$

where $[\gamma]$ denotes the largest integer not exceeding γ. Thus, the matrix A looks like this:

1.	2.	3.	4.	...	(2n−1)-th	2nth	... column
0	1/4	1/4	1/4	...	1/4	1/4	...
	−1/4	−1/4	−1/4	...	−1/4	−1/4	...
		0	1/8	...	1/8	1/8	...
			−1/8	...	−1/8	−1/8	...
					...	...	...
					0	$1/2^{n+1}$	...
						$-1/2^{n+1}$	...
							...

The L_1-predual space E defined by this matrix is the completion of $\bigcup_{n=1}^{\infty} E_n$, where $E_1 \subset E_2 \subset \ldots$, each E_n is isometric to l_∞^n, and the embedding J_n of E_n into E_{n+1} is defined by

$$J_n e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1} \quad (1 \leqslant i \leqslant n)$$

with $\{e_{i,n}\}_{1 \leqslant i \leqslant n}$ being the canonical basis of E_n. Given $x \in E_n$, let $x(i)$ denote the ith coordinate of x with respect to the basis $\{e_{i,n}\}$. Obviously, $x(i)$ does not depend on the choice of n.

We first show that E satisfies condition (i). Fix $n \geqslant 1$ and choose $x \in E_{2n}$ with $x(i) = (-1)^{i+1}$ $(1 \leqslant i \leqslant 2n)$. It can be read immediately from the definition of A that for $i > 2n$, the sequence of coordinates $x(i)$ stabilizes, namely

$$x(i) = 1 - 2^{-n} \quad (i > 2n).$$

Thus, x is $(2^{-n}, 4 \cdot 2^{-n})$-exposed in each E_m, and therefore $(2^{-n}, 5 \cdot 2^{-n})$-exposed in E. It is elementary to show that a (δ, ε)-exposed point is also $(\lambda\delta, \lambda\varepsilon)$-exposed for each $\lambda > 1$, which proves (i).

The idea in the proof of (ii) is to show that the signs of the coordinates $x(i)$ of a point $x \in \bigcup E_n$ must be alternating on an arbitrary given interval, provided only that x is well enough exposed. On the other hand, for each $x \in \bigcup E_n$, the sequence of coordinates stabilizes. Therefore there cannot be an "accumulation" of well-exposed points in $\bigcup E_n$ like it would necessarily be in a neighbourhood of an extreme point of B_E. We now give the details.

Assume that B_E has an extreme point y. By the remark following Lemma 2.4, y is $(\frac{1}{2}\varepsilon, \varepsilon)$-exposed for all $\varepsilon > 0$. Let $\{x_n\}$ be a sequence with $x_n \in B_{E_n}$ and $\lim_{n\to\infty} \|y - x_n\| = 0$. Choose $n \geqslant 1$ in such a way that

(1) $$\|x_{2n} - x_m\| < \tfrac{1}{3} \quad (m > 2n).$$

Since $\{x_n\}$ converges to y, it follows immediately that there is an index $m > 2n+2$ such that x_m is a $(2^{-(n+4)}, 2^{-(n+2)})$-exposed point of B_E. Fix this m. Lemma 2.4 yields

(2) $$|x_m(i)| > 1 - 2^{-(n+3)} \quad (i = 1, 2, \ldots).$$

Taking, if necessary, $-y$ instead of y, we may assume $x_m(1) > 0$. We now prove that

(3) $$\operatorname{sign} x_m(i) = \operatorname{sign} a_{i,2n+2} \quad (1 \leqslant i \leqslant 2n+2).$$

Assume that there is an index j, $1 \leqslant j \leqslant 2n+2$, with $\operatorname{sign} x_m(j) \neq \operatorname{sign} a_{j,2n+2}$. By definition,

$$x_m(m+1) = \sum_{i=1}^{m} a_{i,m} x_m(i)$$
$$= \sum_{\operatorname{sign} x_m(i) = \operatorname{sign} a_{i,m}} |a_{i,m}|\,|x_m(i)| - \sum_{\operatorname{sign} x_m(i) \neq \operatorname{sign} a_{i,m}} |a_{i,m}|\,|x_m(i)|.$$

Therefore

$$x_m(m+1) \leqslant 1 - |a_{j,m}|\,|x_m(j)| = 1 - 2^{-[\frac{1}{2}(j+3)]}|x_m(j)| \leqslant 1 - 2^{-(n+3)}.$$

On the other hand, since $\operatorname{sign} x_m(1) = \operatorname{sign} a_{1,m}$,

$$x_m(m+1) \geqslant |a_{1,m}|\,|x_m(1)| - 1 \geqslant \tfrac{1}{8} - 1 = -\tfrac{7}{8}.$$

Summarizing both inequalities, we get

$$|x_m(m+1)| \leqslant 1 - 2^{-(n+3)}$$

which contradicts (2). This proves (3). Inequalities (1)–(3) yield

$$|x_{2n}(i)| > 0, \quad \operatorname{sign} x_{2n}(i) = \operatorname{sign} a_{i,2n+2} \quad (1 \leqslant i \leqslant 2n+2).$$

But a look at A shows that

$$x_{2n}(2n+1) = x_{2n}(2n+2).$$

We have reached a contradiction which proves (ii).

Next we verify that a space E which satisfies (i) and (ii) cannot be an $A_0(S)$-space. Assume the contrary. Then, by Theorem 5.3 of [12], E has a representing matrix $(b_{i,n})$ with $b_{i,n} \geqslant 0$ $(1 \leqslant i \leqslant n, n = 1, 2, \ldots)$. Let $E = \overline{\bigcup_{n=1}^{\infty} \tilde{E}_n}$ be the corresponding representation. For each n, let $x_n \in \tilde{E}_n$ be such that $x_n(i) = 1$ $(1 \leqslant i \leqslant n)$. Since, for each $y \in B_{\tilde{E}_n}$, $|y(i)| \leqslant x_n(i)$ $(i = 1, 2, \ldots)$ we derive from condition (i) that

$$\lim_{n\to\infty} (\inf_{i \in N} x_n(i)) = 1.$$

This shows that the sequence $\{x_n\}$ converges to an extreme point $x \in B_E$, a contradiction to (ii).

To conclude the proof of 2.14, let $\mathfrak{U}$ be a non-trivial ultrafilter on the set of natural numbers, and let x_n be a $(2^{-n}, 5 \cdot 2^{-n})$-exposed point of B_E. Then x_n is also $(2^{-k}, 5 \cdot 2^{-k})$-exposed for $1 \leqslant k \leqslant n$. Finally, the family $(x_1, x_2, \ldots)$ generates an extreme point $(x_n)_{\mathfrak{U}}$ of $(E)_{\mathfrak{U}}$, by Lemma 2.5. Therefore $(E)_{\mathfrak{U}}$ is an $A(S)$-space (cf. [18]).

Remark. We do not know whether $(E)_{\mathfrak{U}} \in \mathscr{B}$ implies $E \in \mathscr{B}$ when $\mathscr{B}$ is the class of M-spaces or the class of $C_0(K)$-spaces.

The following table summarizes the results of this paper as well as the open questions:

	$C(K)$	$C_\Sigma(K)$	$C_\sigma(K)$	$C_0(K)$	M	G	$A(S)$	$A_0(S)$	$E'=L_1$
Closed under ultrapowers	+	+	+	+	+	+	+	+	+
Closed under ultraproducts	+	?	+	+	+	+	+	+	+
If $(E)_{\mathfrak{U}} \in \mathfrak{B}$, then $E \in \mathfrak{B}$	+	+	+	?	?	+	−	−	+

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Local expansions on graphs

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Abstract. A necessary and sufficient condition is proved under which a linear graph admits a local expansion.

§ 1. Introduction. This paper is motivated by a short note of Rosenholtz [11] who studied local expansions on metric continua and proved that every open local expansion on a metric continuum onto itself has a fixed point. Showing that openess of the mapping is essential in the result, he has constructed a fixed point free local expansion on the union of three circles ([11], p. 3 and 4). On the other hand it is easy to point some particular examples of metric continua which do not admit any local expansions onto themselves at all. Such is e.g. the unit segment of reals. Therefore it is very natural to ask about a criterion under which there exists a local expansion of a given metric continuum onto itself:

PROBLEM. Characterize metric continua X which admit a local expansion of X onto itself.

This paper does not answer the problem, however, it is a contribution to the attempt to find such a criterion for some special continua. Namely a partial answer is given by showing a necessary and sufficient condition of the existence of local expansions on linear graphs (i.e. one-dimensional connected polytopes) equipped with a convex metric.

§ 2. Definitions and preliminaries. Let ϱ be a metric on a metric space X. The statement that the mapping $f\colon X\to X$ is a local expansion means that f is continuous and that for each point $x\in X$, there is an open set U containing x and a real number $M>1$ so that if y and z belong to U, then

$$\varrho(f(y),f(z))\geqslant M\varrho(y,z) \tag{1}$$

(see [11], p. 1). We say that a *metric space X admits a local expansion* if there exist a metric ϱ that is equivalent to the original one given on X, and a surjection $f\colon X\to X$ satisfying the conditions of the above definition.

Let a metric space X with a metric ϱ be given. Let $x, y, z\in X$. The point z is said to *lie between the points x and y* provided that $\varrho(x,y)=\varrho(x,z)+\varrho(z,y)$ (cf. [3], p. 317). The point z is said to be a *center of the pair x, y* provided that $\varrho(x,z)$