Normality and paracompactness of Pixley–Roy hyperspaces

by

Teodor C. Przymusński (Warszawa)


In particular, we show that the Pixley–Roy hyperspace \( \mathcal{F}[X] \) of a compact space \( X \) is normal if it is paracompact if \( X \) is scattered.

§ 1. Introduction. In this paper we study (hereditary) normality and paracompactness of Pixley–Roy hyperspaces.

In Section 2 we characterize those spaces \( X \) whose Pixley–Roy hyperspaces \( \mathcal{F}[X] \) are paracompact or hereditarily paracompact. We also show that \( \mathcal{F}[X] \) is (hereditarily) paracompact if \( \mathcal{F}[\mathcal{F}[X]] \) is (hereditarily) paracompact for all \( n < \omega \) if \( \mathcal{F}[X] \) is (hereditarily) collectionwise Hausdorff (Theorems 2.1 and 2.2).

Section 3 contains a number of applications of these results, mostly concerned with the invariance of paracompactness and hereditary paracompactness under continuous mappings and the operations of union and product of spaces. In particular, we show that if \( X \) is a \( \sigma \)-locally finite union of closed subspaces whose Pixley–Roy hyperspaces are paracompact, or if every point of \( X \) has a neighbourhood whose Pixley–Roy hyperspace is paracompact, then \( \mathcal{F}[X] \) is paracompact (Corollaries 3.4 and 3.5). We also show that if \( X \) is scattered, then \( \mathcal{F}[X] \) is paracompact and that if \( \mathcal{F}[X], \text{paracompact for } i = 1, 2, \ldots, n \), then \( \prod_{i=1}^{n} \mathcal{F}[X] \) is paracompact (Corollaries 3.6 and 3.8).

Section 4 is devoted to the investigation of Pixley–Roy hyperspaces of compact (or, more generally, locally Čech-complete) spaces. We prove that if \( X \) is locally Čech-complete, then \( \mathcal{F}[X] \) is (hereditarily) normal iff \( \mathcal{F}[X] \) is (hereditarily) paracompact if \( X \) is scattered (scattered and first countable) (Theorems 4.2 and 4.6).

In Section 5 we examine normality (= perfect normality) and metrizability (= paracompactness) of Pixley–Roy hyperspaces \( \mathcal{F}[M] \) of metrizable spaces \( M \). Theorem 5.2 asserts that \( \mathcal{F}[M] \) is metrizable iff \( M \) is \( \sigma \)-discrete. If \( \dim M = 0 \), then \( \mathcal{F}[M] \) is normal iff \( M \) is a strong \( q \)-set (Theorem 5.9). Moreover, the existence of a normal non-metrizable hyperspace \( \mathcal{F}[M] \) is equivalent to the existence of a non-\( \sigma \)-discrete strong \( q \)-set (Theorem 5.11).
Section 6 deals with Pixley-Roy hyperspaces of spaces of ordinals. Such hyperspaces $\mathcal{S}(X)$ are always paracompact (Theorem 6.1) and $\mathcal{S}(X)$ is hereditarily normal iff $\mathcal{S}(X)$ is hereditarily paracompact iff characters of all non-isolated points of $X$ coincide (Theorem 6.2). This result leads to some pathological examples of Pixley-Roy hyperspaces (Examples 6.4-6.6).

In the last Section 7 we define and examine iterated Pixley-Roy hyperspaces $\mathcal{S}^n(X)$. In particular, we prove that $\mathcal{S}^n(X)$ is paracompact for all $n \geq 2$ and, if $X$ is first countable, then $\mathcal{S}^n(X)$ is metrizable for all $n \geq 2$.

The paper contains a number of open problems.

Throughout this paper, all spaces are $T_1$, all mappings are continuous, $\mathfrak{c}$ denotes an infinite cardinal, $\kappa$ denotes a (von Neumann) ordinal and $c = 2^\kappa$.

The Pixley-Roy hyperspace of the real line was defined by Pixley and Roy in [PR] and later generalized by van Douwen in [vD]. Pixley-Roy hyperspaces were also applied and investigated in [PT], [L], [BFL1], [BFL2], [P], [R] and [B]. The Pixley-Roy hyperspace $\mathcal{S}(X)$ of a space $X$ (or, briefly, the PR-hyperspace of $X$) is the set of all non-empty finite subsets $F$ of $X$ with the topology generated by basic open sets of the form $[F, V] = \{H \in \mathcal{S}(X) : F \subseteq H \subseteq V\}$, where $F \in \mathcal{S}(X)$ and $V$ is a neighbourhood of $F$ in $X$. We shall often use the fact that $[F, V] \cap [H, W] \neq \emptyset$ if and only if $F \subseteq W$ and $H \subseteq V$. Following [BFL1] we write

$$\mathcal{S}(X)[F] = \{F \in \mathcal{S}(X) : |F| \leq m\}$$

for $m \leq \kappa$. Pixley-Roy hyperspaces are always completely regular and zero-dimensional (i.e. ind $\mathcal{S}(X) = 0$).

A completely regular space $Z$ is strongly zero-dimensional (i.e. dim $Z = 0$) if any two disjoint functionally closed subsets of $Z$ can be separated by a clopen set. A space $Z$ is collectionwise Hausdorff if every discrete collection of points of $Z$ can be separated by a disjoint collection of open sets.

A space is scattered if it contains no dense-in-itself subsets. By $X \subseteq Y$ we denote the free (i.e. disjoint) sum of the spaces $X$ and $Y$. For the undefined notions the reader is referred to [E].

**Proposition 1.1.** $\mathcal{S}(X)$ is metrizable if and only if it is first countable and paracompact.

**Proof.** $\mathcal{S}(X)$ is a Moore space iff $X$ is first countable iff $\mathcal{S}(X)$ is first countable [vD].

The proof of the following two propositions is an easy exercise.

**Proposition 1.2.** If $A$ is a subspace of $X$, then $\mathcal{S}(A)$ is a closed subspace of $\mathcal{S}(X)$.

**Proposition 1.3.** $\mathcal{S}(X)$ is perfectly normal if and only if $\mathcal{S}(X)$ is normal and all points of $X$ are $G_\delta$ sets.

We do not know if the assumption of normality of $\mathcal{S}(X)$ in the following proposition is redundant.

**Proposition 1.4.** If $\mathcal{S}(X)$ is normal, then $\dim \mathcal{S}(X) = 0$.

\[ \text{Normality and paracompactness of Pixley-Roy hyperspaces} \]

\[ \text{203} \]

**Proof.** One easily proves by induction that $\dim \mathcal{S}(X) = 0$ for all $\kappa \leq \omega$. Consequently, $\dim \mathcal{S}(X) = 0$ by ([E]; Theorem 2.1).

**Proposition 1.5.** $\mathcal{S}(X \cup Y)$ is homeomorphic to $\mathcal{S}(X) \oplus \mathcal{S}(Y) \oplus (\mathcal{S}(X) \times \mathcal{S}(Y))$.

**Proof.** Clearly, $\mathcal{S}(X \cup Y)$ is a disjoint union of open subsets $A = \{F \in \mathcal{S}(X \cup Y) : F \subseteq X\}$, $B = \{F \in \mathcal{S}(X \cup Y) : F \subseteq Y\}$ and $C = \{F \in \mathcal{S}(X \cup Y) : F \cap X \neq \emptyset \text{ and } F \cap Y \neq \emptyset\}$.

Moreover, $A \cong \mathcal{S}(X)$, $B \cong \mathcal{S}(Y)$ and $C \cong \mathcal{S}(X) \times \mathcal{S}(Y)$.

Propositions 1.3 and 1.5 are also proved in [L].

\[ \text{§ 2. Characterization of paracompact and hereditarily paracompact Pixley-Roy hyperspaces} \]

In this section we shall prove the following two theorems characterizing paracompact and hereditarily paracompact $\mathcal{S}(X)$ hyperspaces. Applications of these results will be given in the next section.

**Theorem 2.1.** The following conditions are equivalent:

(i) $\mathcal{S}(X)$ is paracompact;

(ii) $\mathcal{S}(X)$ is paracompact for all $\kappa \leq \omega$;

(iii) $\mathcal{S}(X)$ is collectionwise Hausdorff;

(iv) for every non-empty finite subset $F$ of $X$ one can choose a neighbourhood $V_F$ so that the inclusions $F \subseteq V_F$ and $H \subseteq V_F$ imply $F \cap H \neq \emptyset$.

**Theorem 2.2.** The following conditions are equivalent:

(i) $\mathcal{S}(X)$ is hereditarily paracompact;

(ii) $\mathcal{S}(X)$ is hereditarily paracompact for all $\kappa \leq \omega$;

(iii) $\mathcal{S}(X)$ is hereditarily collectionwise Hausdorff;

(iv) for every non-empty finite subset $F$ of $X$ one can choose a neighbourhood $W_F$ so that the inclusions $F \subseteq W_F$ and $H \subseteq W_F$ imply $F \cap H = \emptyset$.

Rem. M. G. Tkachenko [T] introduced, for quite a different purpose, the notion of a weakly separated space: $X$ is weakly separated if for every point $x \in X$ one can choose a neighborhood $V_x$ so that if $y \in V_x$ and $x \in V_y$, then $x = y$ (cf. also [A1] and [H1]). One easily sees that $\mathcal{S}(X)$ is (hereditarily) paracompact if and only if $X$ is weakly separated. Thus conditions (iv) in Theorems 2.1 and 2.2 are natural strengthenings of this notion.

**Problem 1.** Find similar characterizations of (hereditarily) normal PR-hyperspaces.

**Problem 2.** Suppose that $\mathcal{S}(X)$ is paracompact. Is $\mathcal{S}(X) \times \mathcal{S}(X)$ paracompact?
Before proving Theorem 2.1 we shall need the following definition and two
lemmas. Let \( m \leq n \). We say that a family of open subsets of \( \mathcal{F}[X] \) is \( m \)-proper
if it covers \( \mathcal{F}[X] \) and consists of mutually disjoint sets of the form \( [F, V] \), where
\( |F| \leq m \).

**Lemma 2.3.** Every \( m \)-proper family is discrete.

**Proof.** Let \( \mathcal{A} = \{ [F_s, W_s] : s \in S \} \) be \( m \)-proper. Since \( \mathcal{A} \) is disjoint and consists of
clopen sets it suffices to show that every \( F \in \mathcal{F}[X] \) has a neighbourhood \( [F, W] \)
intersecting finitely many elements of \( \mathcal{A} \).

For every \( x \in F \) put
\[
W(x) = \bigcup \{ W_s : F_s \subseteq F, x \in W_s \text{ and } t \in S \}
\]
and
\[
W = \bigcup \{ W(x) : x \in F \}.
\]

We shall show that if \( F_s, W_s \cap [F, W] \neq \emptyset \), then \( F_s \subseteq F \), which will complete
the proof. We have \( F_s \subseteq W_s, F_s \subseteq W_t \) and \( |F_s| \leq m \). Thus there exists an \( H \subseteq F \), with
\( |H| \leq m \), such that
\[
F_s \subseteq \bigcup \{ W(x) : x \in H \}.
\]

There exists a \( t \in S \) such that \( H \cap [F_s, W_s] \), hence \( F_s \cap H \Rightarrow W_s \). We infer that
\[
\bigcup \{ W(x) : x \in H \} \supseteq W_t
\]
and therefore \( F_s \subseteq W_t \) and \( F_s \subseteq H \subseteq F \). Thus \( \mathcal{F}_s, W_s \cap [F, W] \neq \emptyset \) which implies \( s = t \) and \( F_s \subseteq F \).

**Lemma 2.4.** Let \( \mathcal{A} \) be an open covering of \( \mathcal{F}[X] \). If for all \( m \leq n \) every \( m \)-proper
family refining \( \mathcal{A} \) can be extended to an \( (m+1) \)-proper family refining \( \mathcal{A} \), then \( \mathcal{A} \)
has a disjoint refinement consisting of basic open sets.

**Proof.** Put \( \mathcal{A}_0 = \emptyset \) and let \( \mathcal{A}_{n+1} \) be an \((m+1)\)-proper extension of \( \mathcal{A}_n \)
refining \( \mathcal{A} \). The family \( \mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n \) clearly has the desired properties.

**Proof of Theorem 2.1.** The implications (ii) \( \Rightarrow \) (i) and (i) \( \Rightarrow \) (ii) are obvious.

(iii) \( \Rightarrow \) (iv). Let \( \mathcal{A} \) be an \( m \)-proper family in \( \mathcal{F}[X] \). By 2.3 \( \mathcal{A} \) is discrete and
\( \mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n \) is clopen. Let \( \mathcal{B} = \{ F \in \mathcal{F}[X] : |F| = m+1 \} \). Clearly \( \mathcal{B} \)
is a closed discrete subset of \( \mathcal{F}[X] \). Let \( \mathcal{F} = \{ [F, W] : F \in \mathcal{B} \} \) be a disjoint family of
basic open sets separating points of \( \mathcal{A} \). We may otherwise assume that
\( \mathcal{A} = \bigcup \mathcal{A} \) is clopen. The family \( \mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n \) is an \((m+1)\)-proper extension of \( \mathcal{A} \).

Put \( \mathcal{A} = \{ [F, W] : F \in \mathcal{B} \} \). For every \( F \in \mathcal{F}[X] \) there is exactly one \( s \in S \) such that \( F \in \{ F_s, W_s \} \). Put \( V_F = W_s \). One easily checks that the family \( \{ F \in \mathcal{F}[X] : F \in \mathcal{F}[X] \} \) satisfies (iv).

(iv) \( \Rightarrow \) (i). Let \( \mathcal{A} \) be an open covering of \( \mathcal{F}[X] \). By 2.4 it suffices to show that
every \( m \)-proper family \( \mathcal{A} \) refining \( \mathcal{A} \) has an \((m+1)\)-proper extension refining \( \mathcal{A} \).

Let \( \mathcal{F} = \{ [F_s, W_s] : s \in S \} \), \( \mathcal{A} = \bigcup \mathcal{A} \) and \( \mathcal{B} = \{ [F, W] : |F| = m+1 \} \) and \( F \in \mathcal{B} \). Since by 2.3 \( \mathcal{A} \) is clopen it suffices to show that points of \( \mathcal{B} \) can be
separated by disjoint basic open sets (we can always require that those sets refine \( \mathcal{A} \)
and are disjoint from \( \mathcal{A} \)).
inductive assumption, implies that either $K = H$ or $H \subset K$. But $|K| = m$ and $|H| \leq m$, thus $H \subset K \subset F$, which completes the inductive construction.

(iv) $\Rightarrow$ (i). Let $U$ be an open subset of $\mathcal{P}(X)$ and let $\mathcal{G}$ be an open covering of $U$. We can clearly assume that for every $F \in \mathcal{G}$ there exists a $G \in \mathcal{G}$ such that $[F, W_F] \subset G$.

Let us put

$$\mathcal{V}_{m+1} = \{ [F, W_F] : F \in U \} \cup \mathcal{V}_m,$$

and

$$\mathcal{V} = \bigcup_{m \leq \infty} \mathcal{V}_m.$$  

Clearly $\mathcal{V}$ is a covering of $U$, refining $\mathcal{G}$. In order to show that $\mathcal{V}$ is open and locally finite in $U$ it suffices to verify that for all $m \leq \infty$:

(\ast) the family $\mathcal{V}_m$ is locally finite in $U$ and consists of clopen subsets of $U$.

Clearly (\ast) is true. If (\ast) holds for all $i \leq m$, then clearly the elements of $\mathcal{V}_m$ are clopen in $U$. It is enough to check that $\mathcal{V}_m$ is locally finite in $U$. Let $K \in U$. If $K \in \mathcal{V}_m$, then $W_F$ is a neighbourhood of $K$ intersecting no elements of $\mathcal{V}_{i+1}$, where $i \leq m$.

If $K \not\in \mathcal{V}_m$, then $W_K$ is a neighbourhood of $K$ intersecting no elements of $\mathcal{V}_{i+1}$, where $i \leq m$.

Hence $F \subset K$. Thus $[K, W_K]$ intersects only finitely many elements of $\mathcal{V}_{i+1}$, which proves (\ast) and completes the proof of the equivalence of (i), (ii), and (iv).

(iv) $\Rightarrow$ (ii). As in the proof of the implication (iv) $\Rightarrow$ (i) in Theorem 2.1, one easily sees that it suffices to show that if $X$ satisfies (iv) then so does $Z = \bigcup_{i=1}^n X_i$, where $X_i = X$, for $i = 1, 2, ..., n$. This, in turn, can be reduced by induction to the case of $Z = X_1 \cup X_2$, where $X_1 = X_2 = X$.

For a subset $A$ of $X$ by $A_1$ and $A_2$ we shall denote the corresponding subsets of $X_1$ and $X_2$, respectively. Let $K \subset X$ and $F = K \cup L$ be an arbitrary element of $\mathcal{P}(Z)$. We shall assign to $F$ a neighbourhood $W_F$ in $Z = X_1 \cup X_2$ so that (iv) is satisfied.

Let us put

$$W_F = (W_{K \cup L} \setminus (L \setminus K)) \cup (W_{K \cup L} \setminus (K \setminus L)).$$

Suppose that $K^* \subset L^* \subset X$, $P^* = K^* \cup L^*$ and that $P \subset W_F$ and $P^* \subset W_F$.

Then

(1) $K^* \subset W_{K \cup L} \setminus (L \setminus K)$;

(2) $L^* \subset W_{K \cup L} \setminus (K \setminus L).$

This implies that $K^* \subset L^* \subset W_{K \cup L}$ and $K \cup L \subset W_{K \cup L}$, thus, by (iv), either $K \cup L \subset K^* \subset L^* \subset K \cup L$ or $K^* \cup L^* \subset K \cup L$. Suppose e.g. $K \cup L \subset K^* \subset L^*$ and assume that $P \subset W_F$. Then either $K^* \neq \emptyset$ or $L^* \neq \emptyset$. Suppose e.g. $K \subset \emptyset$, $L^* \neq \emptyset$. This implies that $\emptyset \neq K \subset L^*$, hence $K \cap (L^* \setminus K^*) \neq \emptyset$. But $K \cap (L^* \setminus K^*) = \emptyset$ by (2); contradiction. All other cases are dealt with similarly. This completes the proof.

Remark. From the above proof it follows that each of the conditions (i) and (iv) in Theorem 2.2 is equivalent to the following condition:

(iv') for every $m \leq \infty$ and every $m$-element subset $F$ of $X$ one can choose a neighbourhood $V_F$ so that inclusions $F \subset V_F$ and $F \subset V_F$ imply $F = F$.

§ 3. Applications. This section contains a number of applications of Theorems 2.1 and 2.2 mostly concerned with the invariance of paracompactness and hereditary paracompactness of PR-hyperspaces under continuous mappings and the operations of union and product of spaces.

Proposition 3.1. Let $(S, <)$ be a partially ordered set and suppose that $X = \bigcup \{ X_i : s \in S \}$, where $X_s \cap X_t = \emptyset$ for $s \neq t$ and $\bigcup \{ X_i : i \in I \}$ is open in $X$ for every $s \in S$.

If $\mathcal{P}(X)$ is paracompact for every $s \in S$, then $\mathcal{P}(X)$ is paracompact.

Proof. By Theorem 2.1 for every $s \in S$ and every non-empty finite subset $F$ of $X_s$ one can assign a neighbourhood $V_F$ of $F$ in $X_s$ so that if $F \subset V_F$ and $F \subset V_F$, then $F \cap H \neq \emptyset$.

For every subset $F$ of $X$ put $F_s = F \cap X_s$ and define $V_F = \bigcup \{ F_s : s \in S \}$, $F \neq \emptyset$ for every non-empty finite subset $F$ of $X$. One easily checks that $V_F$ is a neighbourhood of $F$ in $X$. Suppose that $F \subset V_F$ and $F \subset V_F$.

Let $A = \{ s \in S : F_s \neq \emptyset \}$. The set $A$ is finite and thus $A$ contains a maximal element $s_0$. Suppose e.g. $F_{s_0} \neq \emptyset$. Since $F_{s_0} \subset V_{F_{s_0}}$, there exists an $s$ such that $H_s \neq \emptyset$ and $F_{s_0} \cap (V_{F_{s_0}} \setminus X_s) \neq \emptyset$. From the maximality of $s_0$ and the inequality $s_0 < s$, we infer that $s = s_0$ and that $F_{s_0} \subset V_{F_{s_0}}$. Similarly, we prove that $H_s \subset V_{F_{s_0}}$. Therefore, $F \subset V_{F_{s_0}} \cap H_{s_0} \neq \emptyset$.

Corollary 3.2 [BFL]. Suppose that $X$ can be partially ordered by $<$ in such a way that $(y, y < x)$ is open in $X$ for every $x \in X$. Then $\mathcal{P}(X)$ is paracompact.

Corollary 3.3. Suppose that $X = \bigcup A_s$ and the sets $K_s = \bigcup \{ A_t : \beta < \alpha \}$ are open (or closed) for every $\alpha < x$.

If $\mathcal{P}(A_s)$ is paracompact for every $\alpha < x$, then $\mathcal{P}(X)$ is paracompact.

Proof. Putting $A = A_s \cup A_s$ and using Proposition 1.2 we can always make the sets $A_s$ disjoint. If the sets $K_s$ are open then also the sets $\bigcup \{ A_t : \beta < \alpha \} = K_{s+1}$
are open. If the sets $X_\beta$ are closed, then the sets $\bigcup \{X_\beta : \beta \geq \alpha\}$ are open. Now, it suffices to apply Proposition 3.1.

**Corollary 3.4.** If $X$ is a $\alpha$-locally finite union of closed subspaces whose PR-hyperspaces are paracompact, then $\mathcal{F}[X]$ is paracompact. ■

**Corollary 3.5.** If every point of $X$ has a neighborhood whose PR-hyperspace is paracompact, then $\mathcal{F}[X]$ is paracompact. ■

**Corollary 3.6.** If $X$ is scattered, then $\mathcal{F}[X]$ is paracompact.

Proof. Clearly, for some $x$, $X \setminus (x^{(0)} \cup x^{(a+1)}(t))$ and the sets $X \setminus (x^{(0)} \cup x^{(a+1)}(t))$ are open for every $a < x$. Our assertion follows now from Corollary 3.3. ■

The following corollary generalizes one of the results obtained by M. E. Rudin [R] for $X$ being a subspace of a Suslin line.

**Corollary 3.7.** Suppose that $X = \bigcup A_\alpha$, where sets $A_\alpha$ are countable and the set $S = \{\alpha : \bigcup A_\alpha \text{ is not closed}\}$ is not stationary. Then $\mathcal{F}[X]$ is paracompact.

Proof. Let $C$ be a closed unbounded subset of $\omega_1$ disjoint from $S$ and put $B_\beta = \bigcup A_\alpha$, for $\alpha < \beta \in C$. Then sets $B_\beta$ are countable, $X = \bigcup B_\beta$ and $B_\beta$ is closed for every $\beta \in C$. By 3.3 $\mathcal{F}[X]$ is paracompact. ■

Our next corollary concerns products of PR-hyperspaces.

**Corollary 3.8.** If $\mathcal{F}[X]$ is paracompact for $i = 1, 2, \ldots, n$, then $\prod_{i=1}^n \mathcal{F}[X_i]$ is paracompact.

Proof. By Proposition 1.5 the space $\prod_{i=1}^n \mathcal{F}[X_i]$ is a closed subspace of $\mathcal{F}[Z]$, where $Z = \bigoplus_{i=1}^n X_i$. By 3.5, $\mathcal{F}[Z]$ is paracompact. ■

Remark. No one of the above results 3.1–3.8 is valid for hereditarily paracompactness. There exist spaces $X$ and $Y$ such that $\mathcal{F}[X]$ and $\mathcal{F}[Y]$ are hereditarily paracompact, but $\mathcal{F}[X] \otimes \mathcal{F}[Y]$ and $\mathcal{F}[X] \times \mathcal{F}[Y]$ are not (see Example 6.6). ■

With regard to the (inverse) invariance of paracompactness under continuous mappings we have the following corollaries to Theorems 2.1 and 2.2.

**Corollary 3.9.** Let $f: X \rightarrow Y$ be a continuous bijection. If $\mathcal{F}[Y]$ is (hereditarily) paracompact, then $\mathcal{F}[X]$ is (hereditarily) paracompact.

Proof. Use (iv) of Theorems 2.1 and 2.2, respectively. ■

**Corollary 3.10.** Let $f: X \rightarrow Y$ be a closed finite-to-one mapping of $X$ onto $Y$. If $\mathcal{F}[X]$ is (hereditarily) paracompact, then $\mathcal{F}[Y]$ is (hereditarily) paracompact.

Proof. Suppose that $X$ is paracompact. By Theorem 2.1 for every finite subset $K$ of $X$ one can choose an open neighborhood $V_K$ so that (iv) is satisfied. Let us put

$$V_f = \bigcap_{\beta \in \mathbb{N}} (X \setminus F \setminus V_{\beta})$$

for every non-empty finite subset $F$ of $Y$. Clearly the sets $V_f$ are open neighborhoods of $F$ in $Y$. Let $F = V_f$ and $H \subseteq V_f$, then $f^{-1}(F) \subseteq V_{f^{-1}(F)}$ and $f^{-1}(H) \subseteq V_{f^{-1}(H)}$, hence $f^{-1}(F) \cap f^{-1}(H) \neq \emptyset$. Therefore $F \cap H \neq \emptyset$.

The proof for hereditary paracompactness is analogous. ■

**Problem 3.** Let $f: X \rightarrow Y$ be a perfect mapping of $X$ onto $Y$. Is $\mathcal{F}[Y]$ (hereditarily) paracompact if $\mathcal{F}[X]$ is such?

This problem has an affirmative answer if $X$ is either locally Čech-complete (see Theorems 4.2 and 4.6) or metrizable (see Theorem 5.2 and also Problem 9). As it follows from the next result, hereditary paracompactness of $\mathcal{F}[X]$ in many cases implies that $\mathcal{F}[X]$ is perfectly paracompact.

**Corollary 3.11.** If $X$ contains a countable non-discrete subset and if $\mathcal{F}[X]$ is hereditarily paracompact, then $\mathcal{F}[X]$ is perfectly paracompact.

Proof. By 2.2 $\mathcal{F}[X]^2$ is hereditarily paracompact. It is easy to see that $\mathcal{F}[X]$ also contains a countable non-discrete subset. Now, it suffices to apply Katetov's theorem (see [E]; Problem 2.7.15). ■

Also the relation between normality and paracompactness of PR-hyperspaces is delicate. Assuming MA + CH there exists a separable metric space $M$ such that $\mathcal{F}[M]$ is normal but not paracompact [PT]. On the other hand, the following corollary is an immediate consequence of Theorems 2.1, 2.2 and a theorem of Fleissner [F].

**Corollary 3.12.** ($\forall = \mathbb{L}$). If $\mathcal{F}[X]$ is (hereditarily) normal and its character is $\leq \mathbb{L}$, then $\mathcal{F}[X]$ is (hereditarily) paracompact. ■

**Problem 4.** Give a "real" example of a normal non-paracompact Pixley–Roy hyperspace $\mathcal{F}[X]$.

It follows from Theorem 4.2 and Corollary 3.12 that $X$ can be neither locally Čech-complete nor first countable.

§ 4. Pixley–Roy hyperspaces of compact spaces. In this section we investigate (hereditary) normality and (hereditary) paracompactness of PR-hyperspaces of locally Čech-complete (in particular, compact) spaces.

**Lemma 4.1.** If $X$ has a closed irreducible mapping onto the Cantor set $C$, then $\mathcal{F}[X]$ is not normal.

Proof. Let $f: X \rightarrow C$ be such a mapping and suppose that $\mathcal{F}[X]$ is normal. Let $C = A \cup B$, where $A \cap B = \emptyset$ and both $A$ and $B$ are of the second category in every non-empty open subset of $C$ (see e.g. [E]; Problem 5.5.4). Put $K = f^{-1}(A)$ and $L = f^{-1}(B)$. Since $K$ and $L$ are disjoint subsets of $X$, there exist disjoint open subsets $U$ and $V$ of $\mathcal{F}[X]$ such that $\{(x) : x \in K\} \subseteq U$ and $\{(x) : x \in L\} \subseteq V$. For

(*) $X^{(0)}$ denotes the $0$th derivative of $X$; sets $X^{(0)}, x^{(a+1)}(t)$ are discrete.
every \( x \in X \) choose a neighbourhood \( V_x \) such that \([x], V_x\) is either contained in \( U \) or in \( V \).

Let us notice that sets \( V_x \) have the following property:

(i) if \( x \in K \) and \( z \in L \) then either \( x \not\in V_z \) or \( z \not\in V_x \).

Indeed, if \( x \in K, z \in L, x \in V_z \) and \( z \in V_x \), then \([x], V_x\) \(\cap\) \([z], V_z\) \(\subset\) \(\overline{U \cup V} = \emptyset\), which is impossible.

For every \( y \in C \) find a neighbourhood \( U_y \) of \( y \) such that

\[
f^{-1}(U_y) = \{ x \in f^{-1}(y) : f^{-1}(y) \subset W \}.
\]

For every open subset \( W \) of \( X \) let \( \mathcal{W} = \{ f^{-1}(y) : f^{-1}(y) \subset W \} \). Since \( f \) is closed \( f(\mathcal{W}) \) is open in \( C \) and \( W \) is closed \( f(\mathcal{W}) \). Since \( f \) is irreducible, for every non-empty \( W \), we have \( \mathcal{W} \not\subset \emptyset \).

Put \( W_y = U_y \cap \{ f^{-1}(y) : f^{-1}(y) \subset W \} \). The set \( W_y \) is open and contained in \( U_y \). Let us observe that

(3) \( W_y \) is dense in \( U_y \).

Indeed, let \( G \) be non-empty and open in \( U_y \). Then, by (2),

\[
f^{-1}(G) = \{ x \in f^{-1}(y) : f^{-1}(y) \subset G \}.
\]

There exists an \( x \) such that \( W = f^{-1}(G) \cap V_x \subset \emptyset \). We have \( \emptyset \not\subset f(\mathcal{W}) \subset f(G) \subset \emptyset \).

Let us put \( A_x = \{ a \in A : B(a, 1/n) \subset U_x \} \), where \( B(a, 1/n) \) denotes a ball in \( C \) of radius \( 1/n \) and center at \( a \). Since \( A \) is of the second category, there exists an \( \delta \) such that \( A_x \) is not nowhere dense. Find an open subset \( G \) of \( C \) such that

\[
\emptyset \neq G \subset A_x
\]

and let \( Q \) be a countable dense subset of \( A \). Hence \( Q \subset A_x = A_x \). By (3) the set

\[
T = \bigcup \{ U_y \cap W_y : a \in Q \}
\]

is of the first category. Therefore there exists a \( b \in (B \cap T) \subset G \).

We have \( b \in B(b, 1/n) \cap G \subset U_y \not\subset \emptyset \), hence by (3), \( B(b, 1/n) \cap G \subset W_y \not\subset \emptyset \).

Since \( Q \) is dense in \( C \), there exists an \( a \in Q \) \(\subset\) \( B(b, 1/n) \cap G \subset W_y \). We have \( a \in A_x \), hence \( b \in U_y \), but \( b \not\in T \subset U_y \cap W_y \), hence \( b \not\in W_y \).

Finally, we obtain \( a \in W_y \) and \( b \in W_y \). Therefore there exist \( x \in f^{-1}(a) \) and \( z \in f^{-1}(b) \) such that \( a \in f^{-1}(y) \) and \( b \in f^{-1}(y) \). We have \( x \in f^{-1}(y) \subset V_x \) and \( x \in f^{-1}(y) \subset \emptyset \), which is impossible by (1).

**Theorem 4.4.** For a locally \( \mathfrak{C} \)-complete space \( X \) the following conditions are equivalent:

(i) \( \mathcal{F}[X] \) is normal,

(ii) \( \mathcal{F}[X] \) is paracompact,

(iii) \( X \) is scattered.

Proof. Implication (iii) \(\rightarrow\) (ii) follows from Corollary 3.6 and implication (ii) \(\rightarrow\) (i) is obvious.

(i) \(\rightarrow\) (iii) (cf. [A]). Suppose that \( X \) is not scattered. Then \( X \) contains a dense-in-itself, closed and \( \mathfrak{C} \)-complete subspace, which implies that there exists a perfect mapping \( f : Y \rightarrow C \) of some (closed) subspace \( Y \) onto the Cantor set. We can assume that \( f \) is irreducible (cf. [E]; Exercise 3.1.C). By Proposition 1.2 \( \mathcal{F}[Y] \) is normal, which contradicts Lemma 4.1. ■

**Lemma 4.3.** Let \( X \) be an uncountable compact space with exactly one non-isolated point \( x_0 \). Then \( \mathcal{F}[X] \) is not hereditarily normal.

Proof. Let \( A \) be an arbitrary countably infinite subset of \( X \) not containing \( x_0 \). We claim that disjoint closed subsets \( K = \{ (x_0, a) : a \in A \} \) and \( L = \{ (x_0, b) : b \in B : X = X \cup \{ x_0 \} \} \) of \( \mathcal{F}[X] \) are not separable by disjoint open subsets \( U \) and \( V \) of \( \mathcal{F}[X] \). Suppose the contrary. For every \( a \in A \cup B \) find a neighbourhood \( V(a) \) of \( x_0 \) such that the set \([x_0, a], V(x) \) \(\cup\) \([x_0, b], V(b) \) \(\neq \emptyset \).

Clearly the complement of every \( V(a) \) is finite, thus there exists a \( b \in B \). \(\subset\) \( Y \) \(\neq \emptyset \). Since \( |X \setminus V(b)| < \aleph_0 \), there exists an \( \alpha \in A \) such that \( a \not\in V(b) \). We have \( b \not\in V(a) \) and \( a \not\in V(b) \), which is a contradiction. ■

**Remark.** There exists an uncountable Lindelöf space \( X \) with exactly one non-isolated point such that \( \mathcal{F}[X] \) is hereditarily paracompact (see Example 6.5).

**Theorem 4.4.** For a compact space \( X \) the following conditions are equivalent:

(i) \( \mathcal{F}[X] \) is hereditarily normal,

(ii) \( \mathcal{F}[X] \) is hereditarily paracompact,

(iii) \( \mathcal{F}[X] \) is metrizable,

(iv) \( X \) is countable.

Proof. The implications (iii) \(\rightarrow\) (ii) and (ii) \(\rightarrow\) (i) are obvious. The implication (iv) \(\rightarrow\) (iii) follows from Proposition 1.1, because \( \mathcal{F}[X] \) is then countable (hence Lindelöf) and \( X \) is first countable.

(i) \(\rightarrow\) (iv). Since \( X \) is compact, it suffices to show that \( X \) is locally countable. Let \( x \) be the first ordinal such that there exists an \( x \in X \times X^{\omega(x)} \) with no countable neighbourhood (Theorem 4.2 implies that \( X \) is scattered). Find a neighbourhood \( V \) of \( x \) such that \( \forall V \times X^{\omega(x)} = \{ x \} \). For every neighbourhood \( W \) of \( x \) in \( X \) the set \( V \setminus W \) is compact and contained in \( X \times X^{\omega(x)} \). From our assumption on \( x \) we infer that \( V \setminus W \) is countable for any such \( W \). If \( V \setminus W \) is finite for all neighbourhoods \( W \) of \( x \), then, applying Lemma 4.3, we infer that \( \mathcal{F}[X] \) is not hereditarily normal, which is a contradiction.

If there exists a \( W \) such that \( V \setminus W \) is infinite, then \( V \setminus W \) is compact, metrizable and infinite, hence it contains a non-trivial convergent sequence. Using the fact that
a complement of every neighbourhood of \( x \) in \( V \) is countable, one shows, as in the proof of the implication (i) \( \rightarrow \) (iii) in Theorem 6.2, that \( S[V] \) is not hereditarily normal, which again is a contradiction. ■

Remark. In fact, Theorem 4.4 is valid for Lindelöf Čech-complete spaces. ■

Proposition 4.5. Let \( X \) be a space of point-countable type (e.g. a \( p \)-space). If \( S[X] \) is hereditarily normal, then \( X \) is first countable and hence \( S[X] \) is perfectly normal.

Proof. By the definition of a space of point-countable type for every point \( x \in X \) there exists a compact set \( K \times x \) of countable character in \( X \). Since \( S[K] \) is hereditarily normal, \( K \) is countable by Theorem 4.4. Thus the point \( x \) has a countable character in \( X \) (cf. [E]; Exercise 3.1.E). Hence \( S[X] \) is perfectly normal by Proposition 1.3. ■

Theorem 4.6. For a locally Čech-complete space \( X \) the following conditions are equivalent:

(i) \( S[X] \) is hereditarily normal,
(ii) \( S[X] \) is hereditarily paracompact,
(iii) \( S[X] \) is metrizable,
(iv) \( X \) is first countable and scattered.

Proof. The implications (iii) \( \rightarrow \) (ii) and (ii) \( \rightarrow \) (i) are obvious. The implication (i) \( \rightarrow \) (iv) follows from Proposition 4.5 and Theorem 4.2. The implication (iv) \( \rightarrow \) (iii) follows from Proposition 1.1 and Theorem 4.2. ■

Remarks. 1. There exist first countable, paracompact, scattered and Čech-complete spaces which are not locally countable, e.g. the real line with all non-zero points isolated.

2. Since normality and paracompactness in PR-hyperspaces of metric spaces do not, in general, coincide [PT], Theorems 4.2 and 4.6 are not valid for (locally) \( p \)-spaces. ■

§ 5. Pixley–Roy hyperspaces of metrizable spaces. In this section we investigate the normality (= perfect normality) and metrizability (= paracompactness) of PR-hyperspaces of metrizable spaces. Throughout this section \( M \) always denotes a metrizable space.

Proposition 5.1. If \( F[M] \) is collectionwise Hausdorff, then \( M \) is \( \sigma \)-discrete.

Proof. For every \( x \in M \) find an \( n(x) < \omega \) such that the family \([\{x\}, B(x, 1/n(x)) \cap F[M)]_{x \in M}\)
is disjoint. One easily checks that sets \( A_n = \{ x \in M : n(x) = n \} \) are closed and discrete. ■

Metrizability of the hyperspaces \( F[M] \) is fully characterized by the following theorem, which is a consequence of Proposition 5.1 and Corollary 3.4.

Theorem 5.2. \( F[M] \) is metrizable if and only if \( M \) is \( \sigma \)-discrete. ■

Theorem 5.2 as well as Proposition 5.1 are valid for semi-metric spaces. As for the normality of hyperspaces \( F[M] \) the following result is known (the "if" part proved in [PT] and the "only if" part in [R]).

Theorem 5.3. If \( M \) is separable, then \( F[M] \) is normal if and only if \( M \) is a strong \( Q \)-set. ■

Recall that \( M \) is a \( Q \)-set if \( M \) is separable and all subsets of \( M \) are \( G_\delta \)'s and that \( M \) is a strong \( Q \)-set if all finite powers of \( M \) are \( Q \)-sets.

Theorem 5.4. The following conditions are equivalent:

(i) there exists and uncountable \( Q \)-set;
(ii) there exists an uncountable strong \( Q \)-set;
(iii) there exists a separable \( M \) such that \( F[M] \) is normal but not metrizable.

Proof. By [P], the existence of an uncountable \( Q \)-set is equivalent to the existence of an uncountable strong \( Q \)-set. Thus it suffices to apply Theorems 5.2 and 5.3. ■

In an attempt to generalize Theorems 5.3 and 5.4 for the non-separable case it is natural to introduce non-separable \( Q \)- and strong \( Q \)-sets. However, as we shall see, the generalization will not be complete. We say that a metric space \( M \) is a \( q \)-set if all subsets of \( M \) are \( G_\delta \)'s and that \( M \) is a strong \( q \)-set if all finite powers of \( M \) are \( Q \)-sets. Thus, \( Q \)-sets are separable \( q \)-sets.

Proposition 5.5. \( F[M] \) is normal if and only if \( M \) is a \( q \)-set. ■

Corollary 5.6. \( \{ x \} \) (\( V = 1 \)). Every \( q \)-set is \( \sigma \)-discrete.

Proof. Let \( M \) be a \( q \)-set. By 5.6 \( F[M] \) is normal and hence, by [F], \( F[M] \) is collectionwise Hausdorff. Thus 5.1 implies that \( M \) is \( \sigma \)-discrete. ■

Lemma 5.7. If \( M \) is a strong \( q \)-set and \( \dim M = 0 \), then \( F[M] \) is normal.

Proof. By a result of Herrlich [H], the space \( M \) is linearly ordered. Now the proof on page 294 of [PT] applies with obvious modifications. ■

Lemma 5.8. If \( F[M] \) is normal and if \( f : X \rightarrow M \) is a one-to-one continuous mapping of a metric space \( X \) with \( \dim X = 0 \) onto \( M \), then \( F[X] \) is normal and \( X \) is a strong \( q \)-set.

Lemma 5.8 will be proved at the end of this section.

Theorem 5.9. If \( \dim M = 0 \), then \( F[M] \) is normal if and only if \( M \) is a strong \( q \)-set.

Proof. This is an immediate consequence of Lemmas 5.7 and 5.8. ■

Problem 5. Can the assumption that \( \dim M = 0 \) in Theorem 5.9 be omitted?

Problem 6. Is every \( q \)-set strongly zerodimensional?

Remark. From Proposition 5.5 it follows that a positive answer to Problem 6 implies a positive answer to Problem 5. Since every \( Q \)-set is obviously strongly zerodimensional, Theorem 5.9 in fact generalizes Theorem 5.3. ■
Lemma 5.10. For every non $\sigma$-discrete metric space $M$ there exists a non-$\sigma$-discrete metric space $X$ with $\dim X = 0$ and a one-to-one continuous mapping $f: X \to M$ of $X$ onto $M$. Moreover, if $M$ is a (strong) $q$-set, then also $X$ is such.

Proof. By (E), Exercise 4.4.1 there exists a strongly zero-dimensional metric space $Z$ and a perfect mapping $g: Z \to M$ of $Z$ onto $M$. Let $X$ be a subspace of $Z$ such that $f = g: X \to M$ is one-to-one and onto. Suppose that $X = \bigcup A_n$ is a union of countably many discrete subspaces $A_n$. For every $n$, $A_n$ is an open subspace of $C_\infty$, hence $A_n = \bigcup F_{x_n}$ is a union of countably many closed discrete subspaces of $Z$. Thus $M = \bigcup g(F_{x_n})$ is $\sigma$-discrete, which is impossible. The last assertion is obvious. □

The following theorem (partially) generalizes Theorem 5.4.

Theorem 5.11. The following two conditions are equivalent:
   (i) there exists a non-$\sigma$-discrete strong $q$-set;
   (ii) there exists an $M$ such that $\mathcal{F}[M]$ is normal but not metrizable.

Proof. If (i) holds, then by 5.10 there exists a strongly zero-dimensional non-$\sigma$-discrete strong $q$-set $M$. By Theorems 5.9 and 5.2, $\mathcal{F}[M]$ is normal but not metrizable.

If (ii) holds, then by 5.2 $M$ is not $\sigma$-discrete and by 5.10 there exists a strongly zero-dimensional non-$\sigma$-discrete metric space $X$ and a continuous one-to-one mapping $f: X \to M$ of $X$ onto $M$. By 5.8 $X$ is a strong $q$-set. □

Problem 7. Is the existence of a non-$\sigma$-discrete $q$-set equivalent to the existence of a non-$\sigma$-discrete strong $q$-set?

Problem 8. Is the existence of a non-$\sigma$-discrete $q$-set equivalent to the existence of an uncountable $Q$-set?

A comparison of Theorems 4.2 and 5.2 suggests the following problem (see also Problem 3).

Problem 9. Let $X$ be a paracompact $\rho$-space. Is $\mathcal{F}[X]$ paracompact if and only if $X$ is $\sigma$-scattered?

Proof of Lemma 5.8. By Lemma 5.7 it is enough to show that $X$ is a strong $q$-set. Take $n < \infty$ and assume that we have already proved that $X^n$ is a $q$-set. Let $\leq$ be a linear order on $X$ generating the topology of $X$ [H]. Let

$$Z = \{(x_1, ..., x_{n+1}) \in X^{n+1} : x_1 < ... < x_{n+1}\}.$$ 

One easily sees that $X^{n+1}$ is a finite union of its $G_\delta$ subspaces which are either homeomorphic to $Z$ or to some $X^k$, for $k < n$. Thus in order to show that $X^{n+1}$ is a $q$-set it suffices to show that $Z$ is a $q$-set. Since $f$ is one-to-one we can identify the points of $X$ and $M$. Let $A \subset Z, B = Z \setminus A$ and let $g: Z \to \mathcal{F}[M]$ be defined by $g(x_1,...,x_{n+1}) = \{x_1,...,x_{n+1}\}$.

Clearly the sets $g(A)$ and $g(B)$ are disjoint closed subsets of $\mathcal{F}[M]$, hence there exist open and disjoint subsets $U$ and $V$ of $\mathcal{F}[M]$ such that $U \cap g(A)$ and $V \cap g(B)$. Define $A_n = \{x \in A : g(x), B(g(x), 1/m) \subset U\}$, where $B(p, 1/m)$ is a ball in $M$ of radius $1/m$.

Since $A = \bigcup A_n$, it suffices to prove that $C_\infty A_n \subset A$. Suppose the contrary and let $z^* \in C_{\infty} A_n \cap B$. There exists a $k > m$ such that $g(z^*), B(g(z^*), 1/k) \subset V$ and a $x \in A_n$ such that $z \in B(z^*, 1/k)$, i.e. $z \in C_{\infty} B(z^*, 1/k)$. Thus $z^* \in B(z^*, 1/k) \subset B(z^*, 1/m)$, for all $i < k+1$. This implies, however, that

$$g(z) \cup g(z^*) \subset B(g(z), 1/m) \cap B(g(z^*), 1/k),$$

and hence

$$\emptyset \neq [g(z), B(g(z), 1/m)] \cap [g(z^*), B(g(z^*), 1/k)] \subset U \cap V,$$

which is a contradiction and completes the proof. □

§ 6. Plicky-Ray hyperspaces of spaces of ordinals. In this section we shall consider PR-hyperspaces of spaces of ordinals. By a space of ordinals we mean an arbitrary subspace of some ordinal. From Corollary 3.2 immediately follows

Theorem 6.1 ([BF1], [P3]). If $X$ is a space of ordinals, then $\mathcal{F}[X]$ is paracompact. □

The following theorem, however, sharply contrasts with Theorems 4.4, 4.6 and 5.2 which may suggest that hereditary normality (hereditary paracompactness) of $\mathcal{F}[X]$ is always equivalent to perfect normality (metrizable) of $\mathcal{F}[X]$.

Theorem 6.2. For a space of ordinals $X$ the following conditions are equivalent:
   (i) $\mathcal{F}[X]$ is hereditarily normal;
   (ii) $\mathcal{F}[X]$ is hereditarily paracompact;
   (iii) characters of all non-isolated points of $X$ coincide;
   (iv) $\mathcal{F}[X]$ is $\tau$-metrizable ($\tau$) for some $\tau$.

(1) A regular space $X$ is $\tau$-metrizable (see [H], [5]) if it has a $\tau$-locally finite base (i.e. a base being a union of $\tau$ locally finite families) and if the intersection of less than $\tau$ open subsets of $X$ is open. A space is $\tau$-metrizable if and only if it is metrizable. All $\tau$-metrizable spaces are hereparacompact.

Theorem 6.2 can be generalized in a natural way: the equivalence of conditions (i)-(iv) takes place in every space $X$ such that $\mathcal{F}[X]$ is paracompact and all points have a well-ordered inclusion base of neighbourhoods.
COROLLARY 6.3 (cf. also 4.6). For an ordinal \(\kappa\), the hyperspace \(\mathcal{F}[\kappa]\) is hereditarily normal if and only if \(\kappa \leq \omega_1\).

EXAMPLE 6.4. Consider \(T = \{x: x < \omega_2\text{ and }\varepsilon(x) = \omega_1\}\) as a subspace of \(\omega_2\). By 6.2, \(\mathcal{F}[T]\) is hereditarily paracompact, \(\omega_1\)-metrizable and not perfect. Moreover, \(T\) contains \(\omega_1\) non-isolated points.

EXAMPLE 6.5. Consider \(Y = \{x: \omega < x < \omega_1\}\) as a subspace of \(\omega_1\). Then, \(Y\) is Lindelöf with exactly one non-isolated point, \(\mathcal{F}[Y]\) is hereditarily paracompact by 6.2 and \(\mathcal{F}[Y]\) is not perfect by 1.3.

EXAMPLE 6.6. The space \(Z = X \oplus Y\), where \(X = \omega + 1\), is a Lindelöf subspace of \(\omega_1 + 1\) with exactly two non-isolated points and \(\mathcal{F}[Z]\) is metrizable and countable, \(\mathcal{F}[Y]\) is \(\omega_1\)-metrizable and hereditarily paracompact, but \(\mathcal{F}[Z]\) is not hereditarily normal, by 6.2. Thus, by 1.5, \(\mathcal{F}[X] \times \mathcal{F}[Y]\) is not hereditarily normal (cf. Corollary 3.8).

Proof of Theorem 6.2. The implications (iv) \(\rightarrow\) (ii) and (ii) \(\rightarrow\) (i) are obvious.

(i) \(\rightarrow\) (iii). Suppose that there exist non-isolated points \(x, y \in X\), say \(x < y\), such that \(\chi(x) = \tau = x = \chi(y)\), where \(\chi(x)\) denotes the character of \(x\) in \(X\). Thus there exist sequences \(\{x_n\}_{n<\omega}\) and \(\{y_n\}_{n<\omega}\) such that \(x_n < x < x < y_n < y\), for all \(n < \omega\) and \(n < \omega\), and \(x_n < x < y_n < x\), for all \(n < \omega\) and \(n < \omega\). Of course, both \(\tau\) and \(\chi\) must be regular.

Suppose that \(\tau < \chi\) and let us put \(Y = A \cup B\), where \(A = \{x\} \cup \{x_n\}_{n<\omega}\), and \(B = \{y\} \cup \{y_n\}_{n<\omega}\). Then \(Y\) is a subspace of \(X\) and \(Y = A \cup B\). From (i) and Proposition 1.5 it follows that the space \(\mathcal{F}[A] \times \mathcal{F}[B]\) is hereditarily normal but this contradicts Katetov’s theorem since \(Y\) is not an intersection of \(\tau\) open subsets of \(\mathcal{F}[B]\) (cf. [17]; Problem 2.7.15).

(iii) \(\rightarrow\) (iv). Suppose that \(\tau\) is such that \(\chi(x) = \tau\) for every non-isolated point \(x \in X\). That the intersection of less than \(\tau\) open subsets of \(\mathcal{F}[X]\) is open follows immediately from the analogous property of \(X\). It suffices to show that \(\mathcal{F}[X]\) has a \(\tau\)-locally finite base. For every non-isolated \(x \in X\) fix a sequence \(\{x_n\}_{n<\omega}\) of points of \(X\) such that \(x_n < x < x\), for all \(n < \omega\) and \(x = \sup\{x_n: n < \omega\}\). Define

\[
U_x(x) = \{(x, x_n): x \text{ is isolated }, \}
\]

\[
\{x_n, x\}, \text{ otherwise ,}
\]

and

\[
\mathcal{V}_x = \{V(F): F \in \mathcal{F}[X]\}, \quad \text{where} \quad V(F) = \{F \cup \{U_x(x): x \in F\}\}.
\]

Let \(\mathcal{B}_x\) be a locally finite open refinement of \(\mathcal{V}_x\), existing by virtue of 6.1. One easily checks that the family \(\mathcal{B} = \bigcup \mathcal{B}_x\) is a base of \(\mathcal{F}[X]\).

§ 7. Iterated Pixley–Roy hyperspaces. In this section we define iterated PR-hyperspaces and examine their properties. Let us define \(\mathcal{F}^1[X] = \mathcal{F}[X] \times \mathcal{F}[X] = \mathcal{F}[\mathcal{F}[X]]\).

THEOREM 7.1. If \(n \geq 2\) then \(\mathcal{F}[X]\) is paracompact.

Proof. From \(n \geq 2\) it follows that \(\mathcal{F}[X] = \mathcal{F}^{n-1}[X]\). Since \(\mathcal{F}^{n-1}[X]\) is a PR-hyperspace it is a union of countably many discrete subspaces \(\{A_n\}_{n<\omega}\), such that \(\bigcap A_n = \emptyset\) is closed for every \(m < \omega\). Now, paracompactness of \(\mathcal{F}^{n}[X]\) follows from Corollary 3.3.

COROLLARY 7.2. If \(X\) is first countable, then \(\mathcal{F}[X]\) is metrizable for every \(n \geq 2\).

Let \(D(t)\) denote a discrete space of cardinality \(t\) and let \(p \in D(t)\). By \(Q(t)\) we shall denote the subspace of \(D(t)\) consisting of those points of \(D(t)\) all coordinates of which, except for finitely many, coincide with \(p\). Clearly, \(Q(t)\) does not depend on the choice of \(p \in D(t)\).

The following theorem is a natural generalization of the well-known fact about the space of rational numbers \(Q\) and has a standard proof.

THEOREM 7.3. Every non-empty \(\sigma\)-discrete metric space, all non-empty open subsets of which have weight (or cardinality) \(\tau\), is homeomorphic to \(Q(t)\).

For the sake of completeness we include a short proof of Theorem 7.3 at the end of this section.

COROLLARY 7.4. The space \(Q\) of the rationals is homeomorphic to \(Q(\omega)\).

COROLLARY 7.5. The space \(Q(t)\) is universal in the class of \(\sigma\)-discrete metric spaces of weight (cardinality) \(\tau\).

Proof. Let \(M\) be a \(\sigma\)-discrete metric space of weight (cardinality) \(\tau\). By 7.3, \(M \times Q(t)\) is homeomorphic to \(Q(t)\) and contains \(M\).

COROLLARY 7.6. Every metrizable Pixley–Roy hyperspace \(\mathcal{F}[X]\) is a subspace of \(Q(t)\), where \(t = |X|\). Moreover, \(\mathcal{F}[X]\) is homeomorphic to \(Q(t)\) if and only if all non-empty open subsets of \(X\) have cardinality \(\tau\).

Proof. Every first countable Pixley–Roy hyperspace is \(\sigma\)-discrete and clearly \(w(\mathcal{F}[X]) = |X|\). Non-empty open subsets of \(X\) have cardinality \(\tau\) if and only if all non-empty open subsets of \(\mathcal{F}[X]\) have cardinality \(\tau\).

COROLLARY 7.7. If \(M\) is metrizable, then \(\mathcal{F}[M]\) is homeomorphic to \(Q(t)\) if and only if \(M\) is homeomorphic to \(Q(t)\).

Proof. This is an immediate consequence of Theorems 7.3, 7.6, 7.7.

From Corollaries 3.2, 7.4 and 5.2 it follows that \(\mathcal{F}[\mathbb{R}] \cong Q(\omega)\), for all \(n \geq 1\), where \(\mathcal{S}\) is the Sorgenfrey line.

COROLLARY 7.8. If \(X\) is first countable, then \(\mathcal{F}[X]\) is a subspace of \(Q(\omega)\), where \(X = |X|\). Moreover, \(\mathcal{F}[X]\) is homeomorphic to \(Q(t)\) if and only if all non-empty open subsets of \(X\) have cardinality \(\tau\).

From Corollary 7.8 it follows that \(\mathcal{F}[\mathbb{R}] \cong Q(\omega)\), for every \(n \geq 2\), where \(\mathbb{R}\) denotes the real line. The space \(\mathcal{F}[\mathbb{R}]\), although not metrizable, also contains \(Q(\omega)\) homeomorphically [R].

PROBLEM 10. [V.D]. Is the square of \(\mathcal{F}[\mathbb{R}]\) homeomorphic to \(\mathcal{F}[\mathbb{R}]\)?
Proof of Theorem 7.3. Let $M$ be a non-empty metric space all non-empty open subsets of which have weight (or cardinality) $\tau$ and let $M = \bigcup D_\alpha$, where the sets $D_\alpha$ are closed and discrete and $D_0 = \emptyset$. Assume also that $M$ is well-ordered.

Put $\mathcal{P}_0 = \{X\}$, fix $n < \omega$ and suppose that an open partition $\mathcal{P}_n$ of $X$ has been defined in such a way that no two distinct elements of $\bigcup D_\alpha$ belong to the same element of $\mathcal{P}_n$. We shall define $\mathcal{P}_{n+1}$.

Fix $U \in \mathcal{P}_n$ and let $m = \text{min}\{k : U \cap D_k \neq \emptyset\}$ and let $x_0$ be the first element of $D_m$ belonging to $U$. Let $(U_\alpha)_{\alpha < n}$ be a partition of $U$ consisting of non-empty open sets of diameter less than $1/(n+1)$ and such that no two distinct points of $\bigcup D_\alpha$ belong to the same element $U_\alpha$. Additionally we require that $x_0 \in U_0$. One easily sees that such a partition exists. Put $\mathcal{P}_{n+1} = \{U_\alpha : \alpha < n, U \in \mathcal{P}_n\}$.

For every $n \geq 1$ let $f_n : M \to D(\tau)$ be a continuous function defined by

$$f_n(x) = \alpha \quad \text{iff} \quad x \in U_\alpha, \quad \text{for some } U \in \mathcal{P}_{n+1}$$

where $D(\tau)$ is identified with the set $\tau = \{a : a < \tau\}$.

It is routine to check that the diagonal

$$f = \bigcup_{n=1}^{\infty} f_n : M \to D(\tau)$$

of mappings $f_n$ is a homeomorphic embedding of $M$ onto $D(\tau)$. □

Added in proof. A different characterization of normal P-ciay-Ray hyperspaces and other interesting results involving PR-hyperspaces were obtained independently by M. G. Bell ("Hyperspaces of finite subsets", to appear). Second part of Problem 2 has been solved by H. Tanaka ("Paracompactness of Piiay-Ray hyperspaces 1 and 2", to appear) who showed that $\mathcal{F}(\mathcal{U})$ is (hereditarily) paracompact if $\mathcal{F}(\mathcal{U})^0$ is (hereditarily) paracompact for all $m$, $a < \omega$. H. Tanaka also proved the assumption, that $\dim M = 0$ in Lemma 5.7 is redundant.

References


