# On the simultaneous embedding of uncountably many distinct wild arcs with one wild endpoint in $E^3$, a geometric approach

by

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*To my father, who spoke of reasoning*
*To my mother, who spoke of temperance*
*In all things*

**Abstract.** I study arcs with one wild endpoint, embedded in $E^3$. I define cone neighborhoods and a composition of arcs. I obtain a decomposition theorem, a product theorem for the penetration index, and show that cone neighborhoods of an arc are contained in one another up to penetration index. As corollaries, I conclude that there exist uncountably many wild arcs distinctly embedded in $E^3$, simultaneously embed uncountably many distinct wild arcs and simultaneously embed uncountably many distinct wild discs, each containing a distinct wild arc. This work is all geometric.

Thank you, Robert Edwards, for you listened with caution and sincerity.

In me, there is a deep respect for you, my advisor, Robion Kirby, for you gave me freedom and watched me wander, patiently, when I needed it.

The idea here is to use geometric techniques to distinguish topologically uncountably many wild arcs, with one wild endpoint in $E^3$, given that there exists an arc with penetration index equal to any odd prime natural number. To this end, I define and use cone neighborhoods, which are geometric embedding invariants, to reduce the comparison of two arcs with infinite penetration index to that of comparing arcs with finite penetration index. In this case, having restricted all compositions to being performed with cone neighborhoods which determine the penetration index, we have a product theorem for the penetration index of the composition at our disposal. At this point, a construction is used to embed the cantor set cross an interval, $C\times I$, into $E^3$ so that each element of $C$ determines a distinctly embedded wild arc. The entire construction is "tilted" to embed $(C\times D^2)\times I$ into $E^3$, yielding uncountably many wild discs, each containing a distinct wild arc.

The work is short, so there are no chapters. I depend heavily on pictures for some of the reasoning. Some details, which upon reflection, are easily shown and relatively clear, are left out to avoid confusing the essential issues.

OBJECTIVE. There exists uncountably many distinctly embedded wild arcs in $E^3$, each locally tame except at one end point.

We consider (ambient isotopy classes of) arcs $K \subset E^3$, locally tame except possibly at one end point ($O$ = origin of $E^3$). Assume $K = f(I)$ where $f\colon I \to E^3$ is an embedding $\ni f[0, 1)$ is smooth and $f(1) = O \Rightarrow \exists F\colon D^2 \times I \to E^3 \ni F|_{D^2 \times [0,1)}$ is a smooth embedding $\not\ni O$ and $f(t) = F(0, t)$, $\forall t$... so $F(D^2 \times I)$ is a "disk-bundle" about $K$ whose fiber collapses at $O$; denote $F(D^2 \times I)$ by $C(K)$. Let $\tilde{C}(K)$ be a right circular cone about $[-\text{diameter } C(K), O] \ni C(K) \sim O \subset \operatorname{int} \tilde{C}(K)$.

DEFINITION. A cone neighborhood $N$ of an arc $K \subset E^3$ is the image of a map $F\colon D^2 \times I \to E^3$

1. $F|_{D^2 \times [0, 1]}$ is a tame embedding $\forall t < 1$,
2. $F(D^2 \times 1) = O$,
3. $K \sim O \subset \operatorname{int} F(D^2 \times [0, 1])$

$F(O \times I)$ is called the *core* of $N$.

Note: 1. $C(K)$ and $\tilde{C}(K)$ are examples of cone neighborhoods (see Fig. 1).

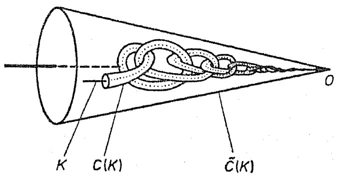


Fig. 1

2. An *initial section* of a cone neighborhood is of the form $F(D^2 \times [0, t_0])$, for some $t_0 < 1$.

DEFINITION. Let $A_0, A_1, \ldots, A_n, \ldots$ be arcs in $E^3$. The composition $A_1 * A_0$ of arcs $A_1$ and $A_0$ is obtained by taking $\varepsilon > 0 \ni B_\varepsilon$ is a ball of radius $\varepsilon$ about $O$ and $f_0(x, t) \in E^3 \sim B_\varepsilon$ if $t \leqslant \frac{1}{2}$, where $f_0\colon D^2 \times I \to E^3$ defines $A_0 = f_0(O \times I)$. Identify $\tilde{C}(f_0(D^2 \times [\frac{1}{2}^+, 1]))$ with $C(A_1)$, keeping $f_0(D^2 \times [0, \frac{1}{2}])$ fixed, and put the composite in a right circular cone: we are wrapping the last "half" of $\tilde{C}(A_0)$ about $A_1$. To form $A_{n+1} * A_n * \ldots * A_1 * A_0$, put a right circular cone, with apex $= O$, about $A_n * \ldots * A_0$; Choose $\varepsilon_n > 0 \ni \varepsilon_n < \frac{1}{2}\min\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}, 1/2^{n-1}\}$. Let $B'_{\varepsilon_n}$ be a tame 3-cell neighborhood of $O$, of radius $< \varepsilon_n$, such that the image of $D^2 \times [0, 1 - 1/2^n]$ (as determined by $A_0$) after having formed $\tilde{C}(A_n * \ldots * A_0)$ lies in $E^3 \sim B'_{\varepsilon_n}$. Let $B_{\varepsilon_n} \subset \operatorname{int} B'_{\varepsilon_n}$ be a tame 3-cell neighborhood of $O$, intersecting $A_n * \ldots * A_0$ tamely at $\partial B_{\varepsilon_n}$, and identify $\tilde{C}(A_n * \ldots * A_0) \cap B_{\varepsilon_n}$ with $C(A_{n+1})$, i.e., wrap $\tilde{C}(A_n * \ldots * A_0) \cap B_{\varepsilon_n}$ around $A_{n+1}$ inside of $B'_{\varepsilon_n}$, keeping $\tilde{C}(A_n * \ldots * A_0) \cap (E^3 \sim B'_{\varepsilon_n})$ fixed. By construction $\ldots * A_{n+1} * A_n * \ldots * A_0$ is well defined because as $n \to \infty$, only the image of $D^2 \times 1$ collapses (see Figures 2 and 3).

Notice: These cone neighborhoods of $\ldots * \ldots * A_n * \ldots * A_0$ in $E^3$ are contained in one another.



THEOREM 1 (Decomposition of arcs). *A cone neighborhood of an arc $K$ in $E^3$ determines two arcs $K_1$ and $K_2 \subset E^3$ such that $K = K_2 * K_1$.*

Proof. Let $f\colon D^2 \times I \to E^3$ determine a cone-neighborhood of $K$. Let $K_2 = f(O \times I)$, an arc in $E^3$ with "disk bundle" $f(D^2 \times I)$. Let $g\colon D^2 \times I \to E^3$

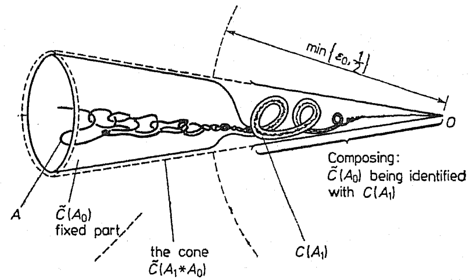


Fig 2. $A_1 * A_0$

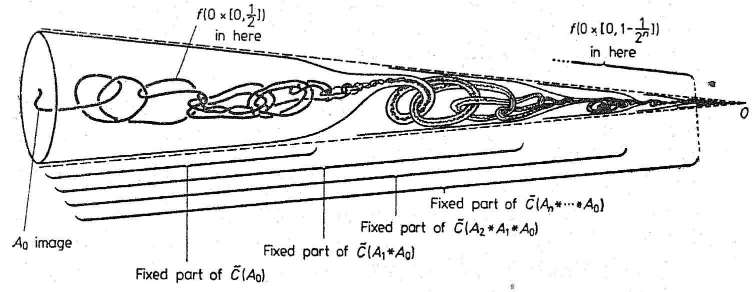


Fig. 3. For the map $f\colon D^2 \times I \to E^3$ defining $\ldots * \ldots * A_{n+1} * A_n * \ldots * A_0$

be a cone-neighborhood of $[-1, 0] \times \{0\} \times \{0\}$, where $g$ carries $D^2 \times I$ onto a right circular cone with apex at $O$. Let $K_1 = g \circ f^{-1}(K)$, then $\tilde{C}(K_1) = g(D^2 \times I)$ and $K = K_2 * K_1$. Q.E.D.

Remark. Time $t$ considerations are irrelevant for finite compositions of arcs. Also, $K_2$ is just the core of the given cone neighborhood of $K$.

OBSERVATIONS. 1. Cone neighborhoods are geometric embedding invariants of an arc $K \subset E^3$.

2. A decomposition of an arc $K$ is a geometric embedding invariant.

DEFINITION. The penetration index $P(K)$ of an arc $K \subset E^3$ is defined by

$$P(K) = \lim_{\varepsilon \to 0} \inf_{\operatorname{diam} B < \varepsilon} \#(\partial B \cap K)$$

where $B$ is a 3-cell neighborhood of $O$ and $\partial B \cap K$ consists of disjoint points.

Note. We may assume $B$ and $\partial B$ are tame and $\#(\partial B \cap K)$ is finite.

Observations. 3. $P(K)$ is a geometric embedding invariant.

4. Given $K, \exists \tilde{C}(K) \ni \forall \varepsilon>0, \exists (D^2, \partial D^2) \subset (\tilde{C}(K), \partial \tilde{C}(K)) \ni \#(D^2 \cap K) = P(K)$, if $P(K)$ is finite, and $0<\operatorname{diam}(D^2 \cup \{O\})<\varepsilon$. Convention: If $P(K)$ is finite, we now assume $\tilde{C}(K)$ determines $P(K)$.

5. By Alford and Ball [1], $\exists$ an arc $K_n \ni P(K_n) = 2n+1$, $\forall n$.

Theorem 2. *Let $K$ and $L$ be arcs in $E^3$ with $P(K)$ and $P(L)$ finite. Then $P(K*L) = P(K)\cdot P(L)$.*

Proof. Let $k = P(K)$ and $l = P(L)$. We show $P(K*L) \leqslant P(K)\cdot P(L)$. Given $\varepsilon>0$, $\exists$ a tame 3-cell neighborhood $B_\varepsilon \subset E^3$ of $O \ni \operatorname{diam} B_\varepsilon<\varepsilon$ and $\#(\partial B_\varepsilon \cap K) = k \Rightarrow \exists k$ disjoint tame disks $D_i \ni P_i \in \operatorname{int} D_i \subset D_i \subset \partial B_\varepsilon$ where $\{P_1, \ldots, P_k\} = \partial B_\varepsilon \cap K$.

Let $h\colon \tilde{C}(L) \to C(K)$ be used in forming $K*L$, so that $h(L) = K*L$. We can assume $h$ is a homeomorphism (modulo initial sections of either "Cone.") $\Rightarrow \exists \delta>0 \ni h(\tilde{C}(L) \cap B_\delta) \subset \operatorname{int} B_\varepsilon$.

$P(L) = l \Rightarrow \exists k$ disjoint tame "meridian" discs $\tilde{D}_i$ of $\tilde{C}(L) \ni$

1. $\tilde{D}_i \subset \operatorname{int} B_\delta$,

2. cone $(\partial \tilde{D}_i, O) \supset$ cone$(\partial \tilde{D}_j, O)$ if $i<j$,

3. $\#(\tilde{D}_i \cap L) = l$, $i = 1, \ldots, k$.

Let $(D_i, D_j)$ be the cylinder of $C(K)$ determined by the meridian discs $D_i$ and $D_j$. We may assume $(D_1, D_k) \supset \ldots \supset (D_{k-1}, D_k)$. Further, we assume $D_1$ occurs "before" $D_k$. We assume the section of $C(K)$ from $D_1$ to $O$ has diameter $<\varepsilon$. $\delta$ may be chosen so that $D_k$ occurs "before" $h(\tilde{D}_1)$ (see Fig. 4a).

Remember. An arc and any cone neighborhood of it are eventually in any neighborhood of $O$. Now, $h(L) \subset \operatorname{int} C(K) \Rightarrow$ we can push $(D_1, h(\tilde{D}_1))$ in along a collar of $C(K)$, keeping $D_1$ and $h(L)$ fixed, resulting in a cylinder $H_1 \ni H_1 \cap \partial C(K) = \partial D_1$. Replace $\partial B_\varepsilon$ by $(\partial B_\varepsilon \cup \partial H_1) \sim \operatorname{int} D_1$. Thus replacing $D_1$ by $D_1' \ni$

$$\#(D_1' \cap h(L)) = l$$

and getting a new tame 3-cell neighborhood $B_\varepsilon'$ of $O \ni \#(\partial B_\varepsilon' \cap K) = k$. We do this for $i = 1, 2, \ldots, k$ (see Fig. 4b). Finally getting, by construction, a tame 3-cell neighborhood of $O$, $\bar{B}_\varepsilon \ni$ 1. $\operatorname{diam} \bar{B}_\varepsilon<\varepsilon$ and 2. $(\# \partial \bar{B}_\varepsilon \cap h(L)) = k\cdot l$. But $K*L = h(L)$. Thus, $P(K*L) \leqslant P(K)\cdot P(L)$.

We now show that $P(K*L) \geqslant P(K)\cdot P(L)$. Let $\delta>0$ be for tame 3-cell neighborhoods $B'$ of $O$

$$\inf_{\operatorname{diam} B'<\delta} \#(\partial B' \cap L) = l > \text{one} \quad \text{and} \quad \inf_{\operatorname{diam} B'<\delta} \#(\partial B' \cap K) = k > \text{one}.$$

Let $B_\delta$ be a ball neighborhood of $O$ of radius $\frac{1}{4}\delta$. Choose a meridian disk $D$ of $C(K)$ which separates $C(K)$ into two components $C_1(K)$ and $C_2(K)$ with $O \in C_2(K)$ and $h^{-1}(C_2(K)) \subset \operatorname{int} B_\delta$. Let $B \subset \operatorname{int} B_\delta$ be a tame 3-cell neighborhood of $O \ni B \cap C(K) \subset C_2(K) \setminus D$ and $\#(\partial B \cap K*L) = P(K*L)$. We may assume $\partial B \cap \partial C(K)$ consists



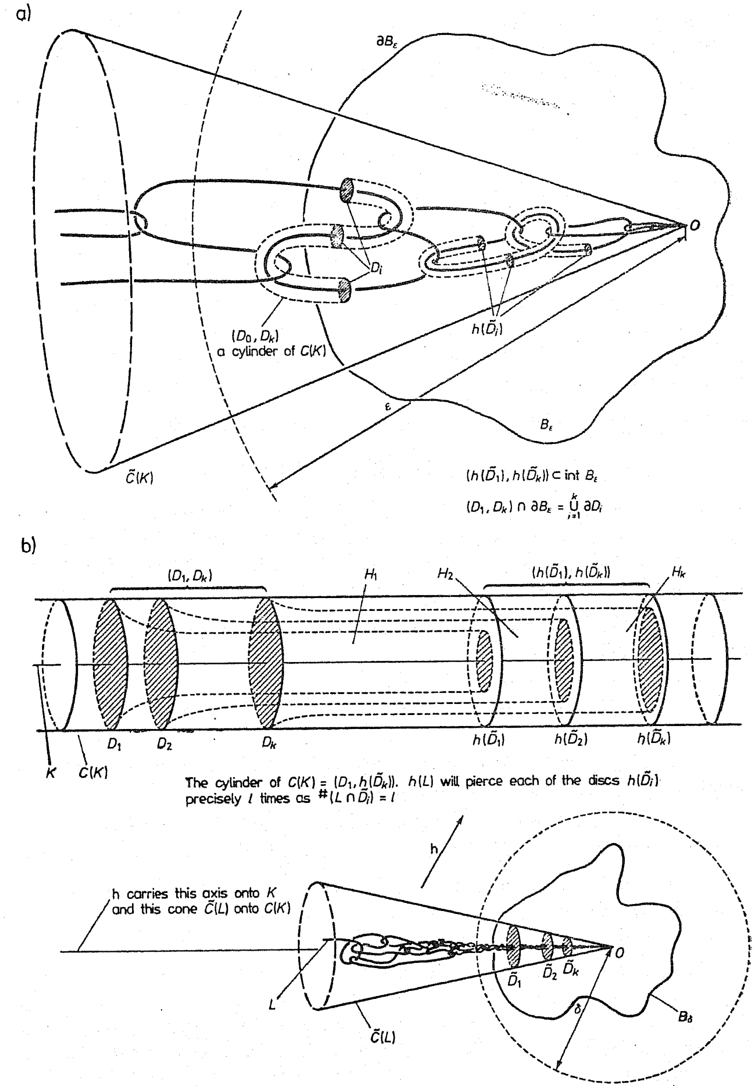


Fig. 4

of meridian circles of $C_2(K)\setminus(D\cup O)$, say $\{S_i\}_{i=1}^{n<\infty}$. (Circles bounding disks on $C_2(K)\setminus(D\cup O)$ are surgered using the innermost circle argument.) Each $S_i$ separates $\partial B$ into 2 components; consider an innermost circle, say $S_0$, of either component, bounding a disk $B_0$ with $\operatorname{int}B_0\cap\partial C(K)=\emptyset$. If $\operatorname{int}B_0\subset\operatorname{int}C_2(K)\Rightarrow h^{-1}(B_0)\subset h^{-1}(C_2(K))$. The join of $h^{-1}(B_0)$ and $(\frac{1}{8}\delta, O, O)$ is tame 3-cell neighborhood of $O$ of diameter less than $\delta\Rightarrow\#(h^{-1}(B_0)\cap L)\geqslant l\Rightarrow\#(B_0\cap K*L)\geqslant l$. Suppose we have $\operatorname{int}B_0\cap C(K)=\emptyset\Rightarrow B_0\cup_{S_0}B_0'$, where $B_0'$ is a meridian disk of $C_2(K)$, is a tame 2-sphere intersecting $K$ once, i.e., $\#((B_0\cup_{S_0}B_0')\cap K)=$ one. But $B_0\cup_{S_0}B_0'$ bounds a tame 3-cell neighborhood $B'$ of $O$ of diameter $<\delta\Rightarrow\#(\partial B'\cap K)\geqslant k>$ one, a contradiction. Thus either component of $\partial B$ determined by any $S_i$ intersects $K*L$ at least $l$ times.

Choose pairwise disjoint meridian disks $\{\tilde{D}_i\}_{i=1}^{n}$ of $h^{-1}(C_2(K)\setminus(D\cup O))$ with $\partial\tilde{D}_i=h^{-1}(S_i)$, and $\tilde{D}_i\cap L=l$ (reindexing the $S_i$'s if necessary). Let $D_i=h(\tilde{D}_i)$, $i=1,\dots,n$. (Compare with Fig. 4b), so $\#(D_i\cap K*L)=l$, $i=1,\dots,n$. Now $\partial B\setminus(\partial B\cap\operatorname{int}C_2(K))\bigcup\limits_{S_1}D_1\bigcup\limits_{S_2}\dots\bigcup\limits_{S_n}D_n$ is the union of a finite collection of tame 2-spheres, at least one of which contains $O$, say $S$. Then

$$\#(S\cap K*L)\leqslant\#(\partial B\cap K*L)$$

by construction. Thus $\#(S\cap K*L)=P(K*L)$. Let $\bar{B}$ be the tame 3-cell neighborhood of $O$ with $\partial\bar{B}=S$. $P(K)=k\Rightarrow\partial\bar{B}\cap C(K)$ consists of at least $k$ meridian disks of $C_2(K)$, say $\{D_i\}_{i=1}^{k'}$, $k'\geqslant k$. But $\#(D_i\cap K*L)=l\Rightarrow P(K*L)\geqslant k\cdot l=P(K)\cdot P(L)$. Hence $P(K*L)=P(K)\cdot P(L)$. Q.E.D.

THEOREM 3. *Let $N_1$ and $N_2$ be two cone neighborhoods of $K$. Then there is a cone neighborhood $N_2'$ of $K$ such that*

1. $N_2'\sim O\subset\operatorname{int}N_1$ *or* $N_1\sim O\subset\operatorname{int}N_2'$,
2. *if $N_2$ decomposes $K$ into $J*L$ and $N_2'$ decomposes $K$ into $J'*L'$, then* $P(L')=P(L)$.

Proof. By general position, we may assume $\partial N_1\cap\partial N_2$ consists of totally tame simple closed intersection curves (except possibly at $O$).

Case 1. $(K\sim O)\cap(\partial N_1\cup\partial N_2)=\emptyset\Rightarrow$ an intersection curve cannot be a median "circle" of $\partial N_1$ $(\partial N_2)$ and bound a disk contractible on $\partial N_2\sim O$ $(\partial N_1\sim O)$.

Warning. We neglect the initial sections of any cone, arc, etc., as our concern is with ambient isotopy classes of arcs only, e.g., contraction over the *base* of a cone is not allowed.

Case 2. Intersection curves which bound discs contractible on both $\partial N_1\sim O$ and $\partial N_2\sim O$ are isolated, with at most finitely many concentric on $\partial N_1$ or $\partial N_2$. Since $K\sim O\subset\operatorname{int}N_2$, they can be removed by a standard procedure, by an ambient isotopy of $E^3$ carrying $N_2$ to $N_2'$ leaving $K$ fixed and $\ni$ $\partial N_1\cap\partial N_2'$ contains no such intersection curves.

Note. $P(\text{core of } N_2')=P(\text{core of } N_2)$.



Case 3. If $N_1$ and $N_2'$ intersect along their boundaries in an infinite number of meridian circles, they determine arcs of the same penetration index, so we can replace $N_2'$ by $N_1$, and then push into a collar of $N_1$ to get $N_2''\ni N_2''\sim O\subset\operatorname{int}N_1$.

Note. Given $\varepsilon>0$, choose a meridian curve $C$ of intersection of $\partial N_1$ and $\partial N_2'\ni$ the components of $\partial N_1$ and $\partial N_2'$ containing $O$ are contained in an $\eta$-ball about $O$. Choose a meridian disc $D$ of $N_1$ $(N_2')$ intersecting $K$ a minimum number of times and $\ni$ the component of $N_1$ $(N_2')$ containing $O$ also is contained in an $\varepsilon$-ball about $O$. Now, use $C$, $D$ and interior collars of $\partial N_1$ and $\partial N_2'$ to construct a meridian disc of $N_2'$ $(N_1)$ in the $\varepsilon$-ball intersecting $K$ in exactly the same points as $D$.

Note. $P(\text{core of } N_2'')=P(\text{core of } N_2')$.

Remark (see Fig. 5). Cases 2 and 3 are reasonable because away from the endpoint $O$ of $K$, i.e. the apex of these cone neighborhoods of $K$, things are tame. Rename $N_2''$ by $N_2$.

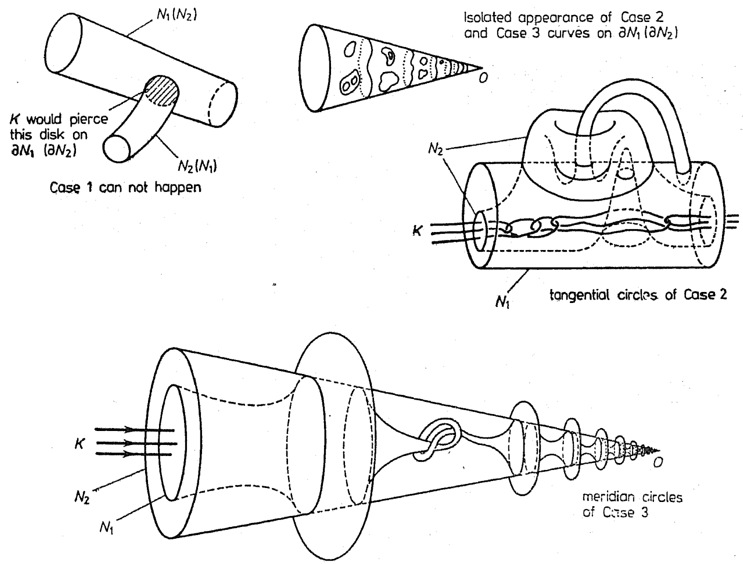


Fig. 5

Case 4. Teardrop intersection curves. Consider intersection curves which contain the apex $O$ of $N_1$ and $N_2$ (see Fig. 6).

Note. Cases 3 and 4 are *essentially* mutually exclusive (modulo initial sections of $N_1$, $N_2$).

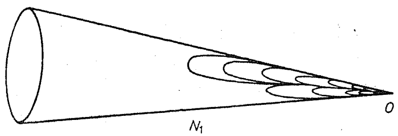


Fig. 6

Let $f\colon D^2\times I\to E^3$ define $N_2$. $g\colon D^2\times I\to E^3$ have image $C$

$$g|_{D^2\times[0,1)} \text{ is 1-1 onto image and } g(D^2\times 1) = 0.$$

We fix $g$ (see Fig. 7).

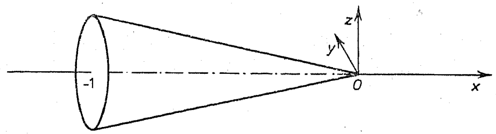


Fig. 7. $\mathrm{im}(g) = C$, a rt. circular cone about $[-1, 0]\times\{0\}\times\{0\}$

Note. $g\circ f^{-1}(N_2) = C$ and $g\circ f^{-1}(K)\sim O\subset \mathrm{int}\, C$ $g\circ f^{-1}|_{N_2}\colon N_2\to C$ is a homeomorphism and $f\circ g^{-1}\colon C\to E^3$ defines and embedding of $C$ into $E^3$ (see Fig. 8).

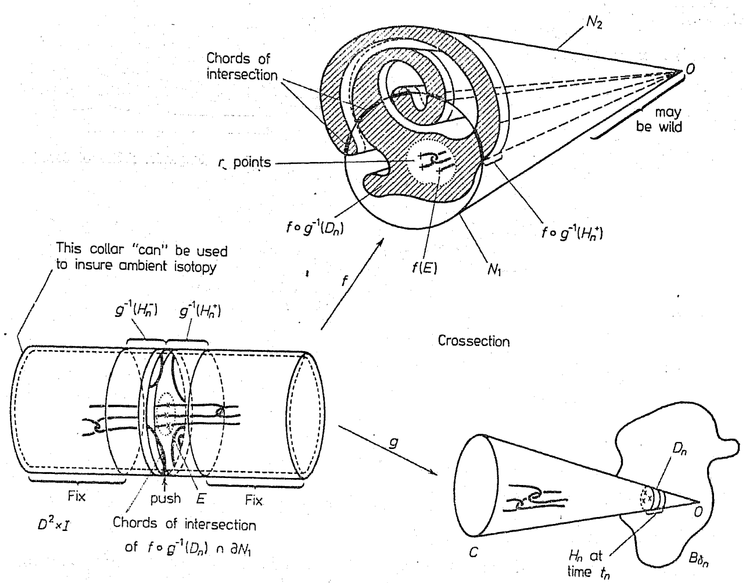


Fig. 8



Given $\varepsilon>0$, let $B_\varepsilon(O)$ be a ball neighborhood of

$$O \Rightarrow \exists\delta>0 \ni f\circ g^{-1}(B_\delta(O)\cap C)\subset B_\varepsilon(O).$$

Let $\delta_n = 1/2^n$, $n = 0, 1, \ldots$ Let $B_n$ be a tame 3-cell neighborhood of $O \ni \partial B_n\cap g\circ f^{-1}(K)$ consists of a minimum number of points, say $r$, and $\mathrm{diam}\, B_n<\delta_n$. Let $D_n = \partial B_n\cap C$, assumed to be tame with time coordinate $t_n$. Choose the $B_n$'s so that $B_{n+1}\subset\mathrm{int}\, B_n$. We may assume a regular neighborhood of $D_n$ intersects $g\circ f^{-1}(K)$ in precisely $r$ disjoint line segments, for all $n$. In particular, there is a bi-collar of $D$ in $C$ with this property, say $H_n\cong D_n\times[t_n-\gamma_n, t_n+\gamma_n]$. Let $H_n^+\cong D_n\times[t_n, t_n+\gamma_n]$ and $H_n^-\cong D_n\times[t_n-\gamma_n, t_n]$, with $H_n^+\cap H_n^- = D_n$, for all $n$. Then $A = f^{-1}(f\circ g^{-1}(D_n)\cap\partial N_1)$ consists of a finite number of disjoint chords of $g^{-1}(D_n)$, by general position.

Connect $r-1$ points of $f^{-1}(f\circ g^{-1}(D_n)\cap K)$ by non-intersecting paths contained interior to these chords in $g^{-1}(D_n)$ to the $r$th point and take a regular neighborhood (a 2-cell) of this $r-1$ frame in $g^{-1}(D_n)$, say $E$. Note $f(E)\subset\mathrm{int}\, N_1$. We may assume that a bicollar of $E$, say $E_n\cong E\times[-1, 1]$ is contained interior to $g^{-1}(H_n)$ with $E_n\cap f^{-1}(K)$ consisting of $r$ disjoint line segments, and $f(E_n)\subset\mathrm{int}\, N_1$. We can now contract a small bicollar of $g^{-1}(D_n)$ into $E_n$ by an isotopy of $D^2\times I$ leaving $D^2\times I\setminus g^{-1}(H_n)$ fixed. Using the normal bundle to $\partial N_2\setminus O$ in $E^3$, we extend the corresponding isotopy of $f\circ g^{-1}(D_n)$ into $f(E)$ to an ambient isotopy of $N_2$ in $E^3$, thus pushing $f\circ g^{-1}(D_n)$ into $\mathrm{int}\, N_1$, leaving $K$ fixed.

Proper choice of the $\gamma_n$'s, allows us to assume $H_n\cap H_m = \emptyset$ if $n\neq m$. So performing this construction for each $n$, we find $f\circ g^{-1}(D_n)\cap\partial N_1 = \emptyset$, $\forall n$. As $n\to\infty$, we are okay because $f(D^2\times 1) = O$ But now there are no more teardop intersections, just the tangential circles of case 2. So, remove the tangential circles and let $N_2'$ be the resulting cone neighborhood of $K$. Then $N_2'\sim O\subset\mathrm{int}\, N_1$, by construction, $P(N_2') = P(N_2)$. Q.E.D.

By Alford and Ball [1] for every odd positive integer $k$, there is an arc $K \ni P(K) = k$. Let $S(i)$ and $T(i)$ be two distinct sequences of primes ($i = 0, 1, \ldots$). For each $i$, let $K_i$ and $L_i$ be arcs $\ni P(K_i) = S(i)$ and $P(L_i) = T(i)$. Form the infinite compositions $\ldots * K_n * K_{n-1} * \ldots * K_1 * K_0 = A$, $\ldots * L_n * L_{n-1} * \ldots * L_1 * L_0 = B$. Let $C_n(D_n)$ be the cone neighborhood of $A(B)$ determined by $K_n(L_n)$. Since the power set of the positive primes is uncountable, it is sufficient to show that $A$ and $B$ are distinctly embedded in $E^3$ to prove the following:

THEOREM 4. *There exist uncountably many distinctly embedded wild arcs in $E^3$, each locally tame except at one endpoint.*

Proof. Suppose there was an ambient isotopy carrying $B$ to $A$. Let $\overline{C}_{r+1}$ be the image of $D_{r+1}$ under this isotopy, where $r$ is such that $T(r)\neq S(i)$ for all $i$. Now $D_{r+1}$ and therefore $\overline{C}(r+1)$ is determined by $\overline{C}(L_r * L_{r-1} * \ldots * L_1 * L_0)$. Let $R = T(r)\cdot T(r-1)\cdot\ldots\cdot T(1)\cdot T(0)$.

Choose $n \ni R<N = S(n)\cdot S(n-1)\ldots S(1)\cdot S(0)$. By Theorem 3 we may assume either $\overline{C}_{r+1}\sim O\subset\mathrm{int}\, C_{n+1}\sim O$ or $C_{n+1}\sim O\subset\mathrm{int}\,\overline{C}_{r+1}\sim O$. By construction, $C_{n+1}$

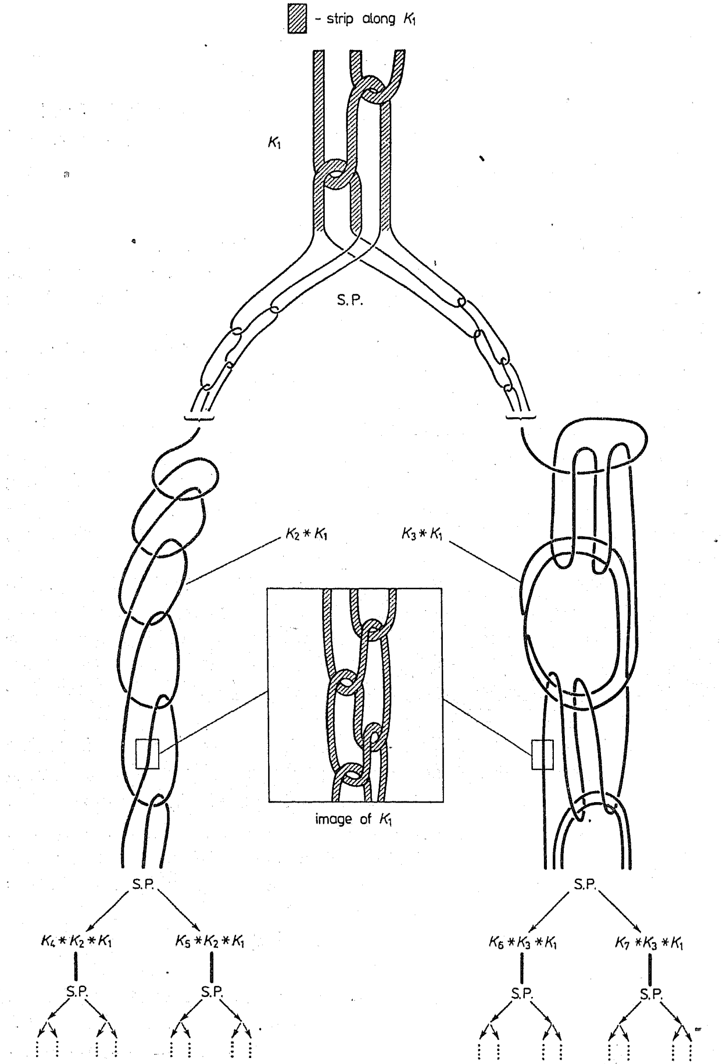


Fig. 9. S. P. — Separation Process: Remove the middle third and separate into two copies of $K_1$; Form $K_2 * K_1$ and $K_3 * K_1$

run along $K_1$ remove the middle third, separate and for $K_2 * K_1$ and $K_3 * K_1$



is the image of $\tilde{C}(K_n * K_{n+1} * \dots * K_1 * K_0)$ under a map $h$ such that $h^{-1}(A) = K_n * K_{n-1} * \dots * K_1 * K_0 \Rightarrow P(h^{-1}(A)) = N$, by Theorem 2. If $\bar{C}_{r+1} \sim O \subset \operatorname{int} C_{n+1} \sim O \Rightarrow h^{-1}(\bar{C}_{r+1})$ is a cone neighborhood of $h^{-1}(A)$, which by Theorem 1 and Theorem 2, decomposes $h^{-1}(A)$ into two factors, one of which has penetration index equal to $R$. Theorem 2 $\Rightarrow$ $R$ divides $N \Rightarrow T(r)$ divides $N$, contradicting our choice of $r$. If $C_{n+1} \sim O \subset \operatorname{int} \bar{C}_{r+1} \sim O$ we similarly show $N$ must divide $R$, contradicting our choice of $n \ni R < N$. Hence $A$ and $B$ are distinctly embedded in $E^3$. Q.E.D.

THEOREM 5. *Uncountably many distinctly embedded wild arcs, each with one wild endpoint, can be simultaneously embedded in* $E^3$.

Proof. Let $\{k_i\}_{i=1}^{\infty}$ be a sequence of distinct odd primes and $\{K_i\}_{i=1}^{\infty}$ wild arcs with $P(K_i) = k_i$. Run a rectangular (▨) strip along $K_1$ (see Fig. 9). Remove the middle third, separate (▨ → ▯▯), and form $K_2 * K_1$ and $K_3 * K_1$ (the separation process S.P.). Let $C$ = Cantor set. As we form $C \times I$ from ▨, removal of the middle third of each substrip, corresponds to an S. P. and formation of compositions with new arcs, yielding this wild tree. Different paths down this tree correspond to different compositions of these arcs, which by Theorem 4, are distinctly embedded in $E^3$. Q.E.D.

THEOREM 6. *Uncountably many wild discs, each containing a distinctly embedded wild arc with one wild endpoint, can be simultaneously embedded in* $E^3$.

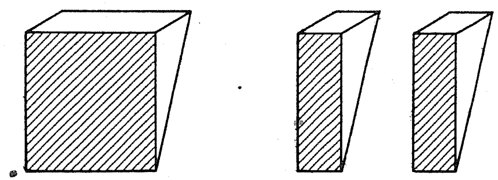

Fig. 10

Proof. Thicken the ▨ -strip used in the construction of Theorem 5 to a wedge (see Fig. 10). Continue the process of Theorem 5, effectively embedding a triangle along each of the arcs of Theorem 5. These triangles are pairwise disjoint by construction. Q.E.D.

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*Accepté par la Rédaction le 23. 7. 1979*

# Some additive properties of sets of real numbers

by

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**Abstract.** Some problems concerning the additive properties of subsets of $\boldsymbol{R}$ are investigated. From a result of G. G. Lorentz in additive number theory, we show that if $P$ is a nonempty perfect subset of $R$, then there is a perfect set $M$ with Lebesgue measure zero so that $P+M = \boldsymbol{R}$. In contrast to this, it is shown that (1) if $S$ is a subset of $\boldsymbol{R}$ is concentrated about a countable set $C$, then $\lambda(S+R) = 0$, for every closed set $P$ with $\lambda(P) = 0$; (2) there are subsets $G_1$ and $G_2$ of $\boldsymbol{R}$ both of which are subspaces of $R$ over the field of rationals such that $G_1 \cap G_2 = \{0\}$, $G_1+G_2 = \boldsymbol{R}$ and $\lambda(G_1) = \lambda(G_2) = 0$. Some other results are obtained under various set theoretical conditions. If $2^{\aleph_0} = \aleph_1$, then there is an uncountable subset $X$ of $R$ concentrated about the rationals such that if $\lambda(G) = 0$, then $\lambda(G+X) = 0$; if $V = L$, then $X$ may be taken to be coanalytic.

P. Erdös and E. Straus conjectured and G. G. Lorentz proved that if $1 \leqslant a_1 < a_2 < \ldots$ is an infinite sequence of integers, then there always is an infinite sequence of integers $1 \leqslant b_1 < b_2 < \ldots$ of density zero so that all but finitely many positive integers are of the form $a_i+b_j$ [1]. In this note we investigate the measure theoretic analogues of this result.

Throughout this paper, the real line will be denoted by $R$. If $A$ and $B$ are subsets of $R$, then $A+B = \{a+b\colon a \in A, b \in B\}$.

THEOREM 1. *Let $P$ be a nonempty perfect subset of $\boldsymbol{R}$. Then there is a perfect set $M$ with Lebesgue measure zero so that $P+M = R$.*

Let us note that it suffices to prove the theorem under the additional assumption that $P \subseteq [0, 1]$. Let us also note that under this assumption it suffices to prove the existence of a closed set $M$ so that $P+M$ contains some closed interval. With this in mind, for each $n$ and $i$, set $I(i, n) = [i/2^n, (i+1)/2^n]$. For each $n$, set

$$A_n = \{i\colon \operatorname{int}(I(i, n)) \cap P \neq \emptyset\}$$

and

$$P_n = \bigcup \{I(i, n)\colon i \in A_n\}.$$

Clearly, $P_1 \supseteq P_2 \supseteq \ldots$ and $\bigcap P_n = P$.

We will prove the following lemma.

LEMMA 2. *There is a sequence of positive integers $m_1 < m_2 < m_3 < \ldots$ and a sequence $\{B_p\}_{p=1}^{\infty}$ of sets of nonnegative integers so that*

1) *for each $p$, $B_p \subset [1, 2^{m_p+1})$,*